Exam Questions – Chapter 9 Differentiation (A2)

Q1.

Prove, from first principles, that the derivative of x^3 is $3x^2$

(4)

(Total for question = 4 marks)

Q2.

Given that θ is measured in radians, prove, from first principles, that

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for cos ($A \pm B$) and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

(Total for question = 5 marks)

Q3.

The curve C has equation

$y = \frac{1}{2}x^3 - 9x^{\frac{3}{2}} + \frac{8}{x} + 30, \qquad x > 0$

(a) Find $\frac{dy}{dx}$.

(4)

(b) Show that the point $P(4,-8)$ lies on C.

(2)

(c) Find an equation of the normal to C at the point P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(6)

(Total 12 marks)

Q4.

$$f'(x) = \frac{(3 - x^2)^2}{x^2}, \quad x \neq 0$$

(a) Show that $f'(x) = 9x^{-2} + A + Bx^2$,

where A and B are constants to be found.

(3)

(b) Find $f''(x)$.

(2)

Given that the point $(-3, 10)$ lies on the curve with equation $y = f(x)$,

(c) find $f(x)$.

(5)

(Total 10 marks)

Q5.

$$h(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}, \qquad x \geq 0$$

(a) Show that $h(x) = \dfrac{2x}{x^2+5}$

(4)

(b) Hence, or otherwise, find $h'(x)$ in its simplest form.

(3)

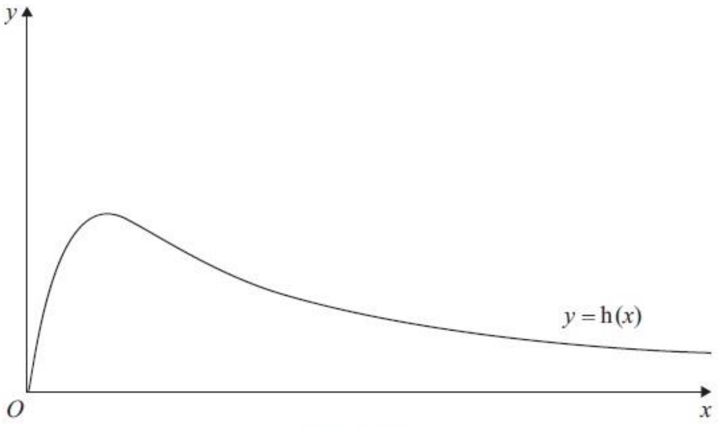

Figure 2

Figure 2 shows a graph of the curve with equation $y = h(x)$.

(c) Calculate the range of $h(x)$.

(5)

(Total 12 marks)

Q6.

$$f(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6}, \qquad x > 2, x \in \mathbb{R}$$

(a) Given that

$$\frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \frac{B}{x-2}$$

find the values of the constants A and B.

(4)

(b) Hence or otherwise, using calculus, find an equation of the normal to the curve with equation $y = f(x)$ at the point where $x = 3$

(5)

(Total for question = 9 marks)

Q7.

The point P lies on the curve with equation

$$x = (4y - \sin 2y)^2$$

Given that P has (x, y) coordinates $\left(p, \frac{\pi}{2}\right)$, where p is a constant,

(a) find the exact value of p.

(1)

The tangent to the curve at P cuts the y-axis at the point A.

(b) Use calculus to find the coordinates of A.

(6)

(Total for question = 7 marks)

Q8.

(i) Given $y = 2x(x^2 - 1)^5$, show that

(a) $\frac{dy}{dx} = g(x)(x^2 - 1)^4$ where $g(x)$ is a function to be determined.

(4)

(b) Hence find the set of values of x for which $\frac{dy}{dx} \geq 0$

(2)

(ii) Given

$x = \ln(\sec 2y)$, $\quad 0 < y < \frac{\pi}{4}$

find $\frac{dy}{dx}$ as a function of x in its simplest form.

(4)

(Total for question = 10 marks)

Q9.

$$f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \qquad x \neq \pm 3,\ x \neq -\frac{1}{2}$$

(a) Show that

$$f(x) = \frac{5}{(2x+1)(x+3)}$$

(5)

The curve C has equation $y = f(x)$. The point $P\left(-1, -\frac{5}{2}\right)$ lies on C.

(b) Find an equation of the normal to C at P.

(8)

(Total 13 marks)

Q10.

The function f is defined by

$$f(x) = \frac{2x-3}{x^2+4} \qquad x \in \mathbb{R}$$

(a) Show that

$$f'(x) = \frac{ax^2 + bx + c}{\left(x^2+4\right)^2}$$

where a, b and c are constants to be found.

(3)

(b) Hence, using algebra, find the values of x for which f is decreasing.

You must show each step in your working.

(3)

(Total for question = 6 marks)

Q11.

The curve C has parametric equations

$x = 3t - 4,\ y = 5 - \dfrac{6}{t}, \quad t > 0$

(a) Find $\dfrac{dy}{dx}$ in terms of t

(2)

The point P lies on C where $t = \dfrac{1}{2}$

(b) Find the equation of the tangent to C at the point P. Give your answer in the form $y = px + q$, where p and q are integers to be determined.

(3)

(c) Show that the cartesian equation for C can be written in the form

$y = \dfrac{ax + b}{x + 4}, \quad x > -4$

where a and b are integers to be determined.

(3)

(Total for question = 8 marks)

Q12.

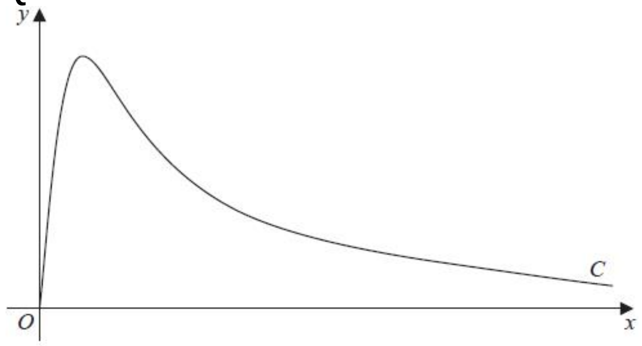

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4\tan t, \qquad y = 5\sqrt{3}\sin 2t, \qquad 0 \leqslant t < \frac{\pi}{2}$$

The point P lies on C and has coordinates $\left(4\sqrt{3}, \dfrac{15}{2}\right)$.

(a) Find the exact value of $\dfrac{dy}{dx}$ at the point P.

Give your answer as a simplified surd.

(4)

The point Q lies on the curve C, where $\frac{dy}{dx} = 0$

(b) Find the exact coordinates of the point Q.

(2)

(Total for question = 6 marks)

Q13.

The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is the normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.

You must show clearly how you obtained your answers.

(6)

(Total for question = 13 marks)

Q14.

A solid glass cylinder, which is used in an expensive laser amplifier, has a volume of 75 π cm^3.
The cost of polishing the surface area of this glass cylinder is £2 per cm^2 for the curved surface area and £3 per cm^2 for the circular top and base areas.

Given that the radius of the cylinder is r cm,

(a) show that the cost of the polishing, £C, is given by

$$C = 6\pi r^2 + \frac{300\pi}{r}$$

(4)

(b) Use calculus to find the minimum cost of the polishing, giving your answer to the nearest pound.

(5)

(c) Justify that the answer that you have obtained in part (b) is a minimum.

(1)

(Total for question = 10 marks)

Q9.

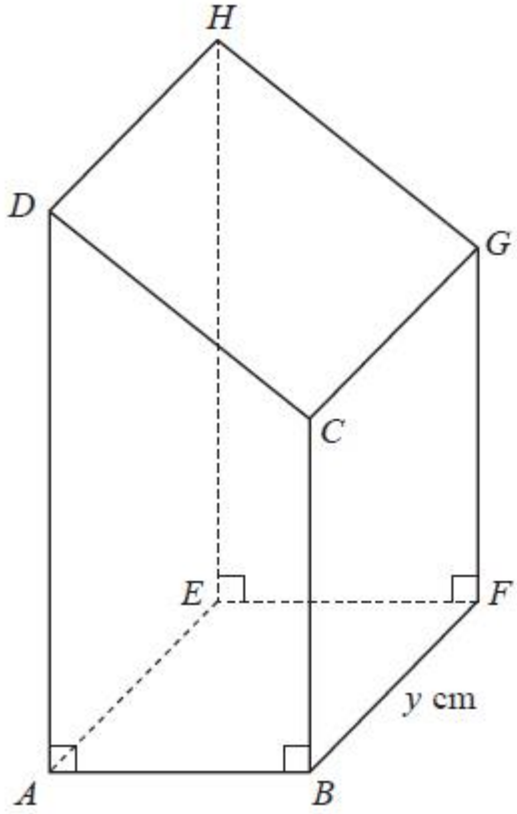

Figure 4

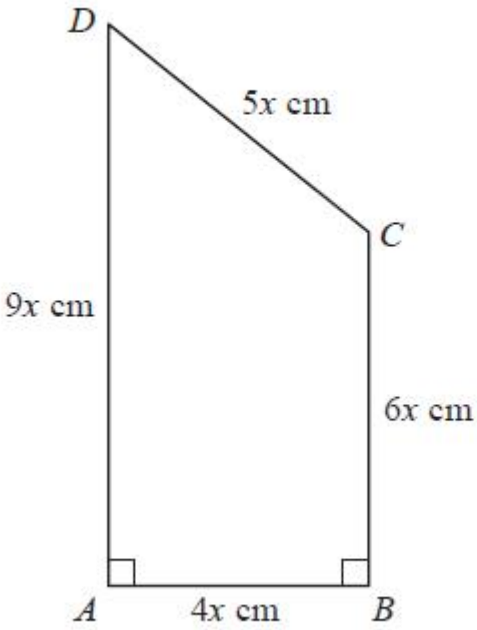

Figure 5

Figure 4 shows a closed letter box *ABFEHGCD*, which is made to be attached to a wall of a house.

The letter box is a right prism of length y cm as shown in Figure 4. The base *ABFE* of the prism is a rectangle. The total surface area of the six faces of the prism is S cm^2.

The cross section *ABCD* of the letter box is a trapezium with edges of lengths $DA = 9x$ cm, $AB = 4x$ cm, $BC = 6x$ cm and $CD = 5x$ cm as shown in Figure 5.
The angle $DAB = 90°$ and the angle $ABC = 90°$.

The volume of the letter box is 9600 cm^3.

(a) Show that

$$y = \frac{320}{x^2}$$

(2)

(b) Hence show that the surface area of the letter box, S cm^2, is given by

$$S = 60x^2 + \frac{7680}{x}$$

(4)

(c) Use calculus to find the minimum value of S.

(6)

(d) Justify, by further differentiation, that the value of S you have found is a minimum.

(2)

(Total 14 marks)

Q10.

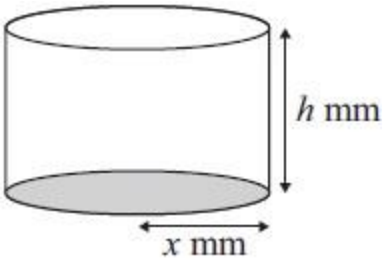

Figure 3

A manufacturer produces pain relieving tablets. Each tablet is in the shape of a solid circular cylinder with base radius x mm and height h mm, as shown in Figure 3.

Given that the volume of each tablet has to be 60 mm^3,

(a) express h in terms of x,

(1)

(b) show that the surface area, A mm^2, of a tablet is given by $A = 2\pi x^2 + \dfrac{120}{x}$

(3)

The manufacturer needs to minimise the surface area A mm^2, of a tablet.

(c) Use calculus to find the value of x for which A is a minimum.

(5)

(d) Calculate the minimum value of A, giving your answer to the nearest integer.

(2)

(e) Show that this value of A is a minimum.

(2)

(Total 13 marks)

Q14.

The curve C has equation

$$4x^2 - y^3 - 4xy + 2^y = 0$$

The point P with coordinates $(-2, 4)$ lies on C.

(a) Find the exact value of $\frac{dy}{dx}$ at the point P.

(6)

The normal to C at P meets the y-axis at the point A.

(b) Find the y coordinate of A, giving your answer in the form $p + q\ln 2$, where p and q are constants to be determined.

(3)

(Total for question = 9 marks)

Q15.

The curve C has equation

$$x^2 - 3xy - 4y^2 + 64 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

(b) Find the coordinates of the points on C where $\frac{dy}{dx} = 0$

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

(Total for question = 11 marks)

Q17.

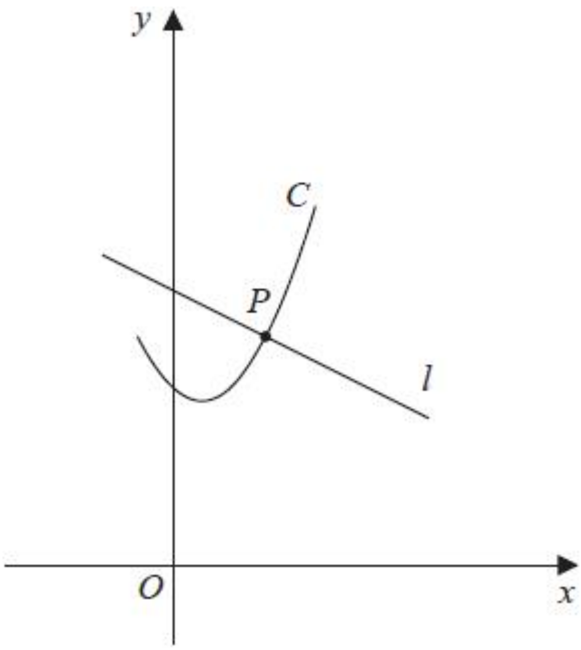

Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$x = 2\tan t + 1$ $\qquad y = 2\sec^2 t + 3$ $\qquad \frac{\pi}{4} \leq t \leq \frac{\pi}{3}$

The line *l* is the normal to *C* at the point *P* where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for *l* is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

(b) Show that all points on C satisfy the equation

$$y = \frac{1}{2}(x - 1)^2 + 5$$

(2)

The straight line with equation

$y = -\frac{1}{2}x + k$ $\qquad$ where *k* is a constant

intersects *C* at two distinct points.

(c) Find the range of possible values for k.

(5)

(Total for question = 12 marks)

Q18.

A curve C has parametric equations

$$x = 4t + 3, \quad y = 4t + 8 + \frac{5}{2t}, \quad t \neq 0$$

(a) Find the value of $\frac{dy}{dx}$ at the point on C where $t = 2$, giving your answer as a fraction in its simplest form.

(3)

(b) Show that the cartesian equation of the curve C can be written in the form

$$y = \frac{x^2 + ax + b}{x - 3}, \quad x \neq 3$$

where a and b are integers to be determined.

(3)

(Total for question = 6 marks)

Q22.

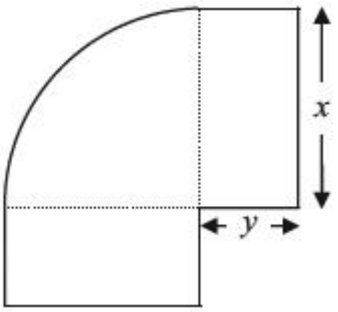

Figure 3

Figure 3 shows a flowerbed. Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to y metres.

Given that the area of the flowerbed is 4 m^2,

(a) show that

$$y = \frac{16 - \pi x^2}{8x}$$

(3)

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation

$$P = \frac{8}{x} + 2x$$

(3)

(c) Use calculus to find the minimum value of P.

(5)

(d) Find the width of each rectangle when the perimeter is a minimum. Give your answer to the nearest centimetre.

(2)

(Total 13 marks)