Particle Decays and Stability

1 Understanding the Problem

Imagine a system of unstable nuclei close together. When one of the nuclei decays, it produces some number of additional neutrons that each have a chance to hit another nucleus. If the particle "hits" another nucleus, the second nucleus decays. We are simplifying the problem by assuming that subatomic particles can actually hit each other.

1.1 Viewed as a differential equation

The particles will decay naturally with some rate. And as decays happen, secondary decays will result with some probability. The differential equation governing the total number of particles is

$$\frac{dN}{dt} = -rn + k\frac{dN}{dt}$$

where n is the current number of particles, r is the natural decay rate, and k is the probability that a decay leads to another decay. Solving for $\frac{dn}{dt}$ algebraically gives

$$\frac{dN}{dt} = -rN\frac{1}{1-k}$$

(note that this analysis is only valid as long as $k < 1$. If $k > 1$ several of the signs change)

1.2 Viewed as an infinite series

Imagine we have 1 decay. That decay has a probability k of leading to a second decay, which has a probability k of leading to a third decay... Then the total number of decays that result from a single natural decay is (on average)

$$1(1 + k(1 + k(1 + \ldots..$$

which yields the series

$$1 + k + k^2 + k^3 + \ldots = \sum_{n=0}^{\infty} k^n$$

for $k < 1$ this converges to

$$\frac{1}{1-k}$$

as before.

1.3 But what is k?

The parameter k can be thought of as the product of the number of neutrons that are produced per reaction (η) multiplied by the chance that each neutron will hit another nucleus (P). So we have

$$k = \eta P$$

Since $0 < P < 1$, if $\eta = 1$, so that only a single neutron is produced, we have global stability (A.S.) If, however, $\eta > 1$ as it is with most particle decays, then the series $1 + k + k^2 + k^3 + \ldots = \sum_{n=0}^{\infty} k^n$ diverges and we have instability.
Getting event further down the rabbit hole, we can get (from dimensional analysis and logic) that

$$P \sim \begin{cases} R\sigma n & 0 \le R\sigma n \le 1 \\ 1 & R\sigma n > 1 \end{cases}$$

Intuitively, the reason for the piecewise definition is that the neutron either hits something, or it does not. We can't have a single neutron hit more than one nucleus!