Function Algebra

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Adding, Subtracting, and Multiplying ... Oh my!

- $f + g(x) = f(x) + g(x)$
- $f - g(x) = f(x) - g(x)$
- $fg(x) = f(x)g(x)$

The domain of $f + g$, $f - g$, and fg is

$$\text{dom}(f) \cap \text{dom}(g)$$

Example

Find $f+g, f-g, fg$ and find their domains.

Let $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{x-1}$.

- $f+g(x)=\sqrt{x}+\dfrac{1}{x-1}$
- $f-g(x)=\sqrt{x}-\dfrac{1}{x-1}$
- $fg(x)=(\sqrt{x})\left(\dfrac{1}{x-1}\right)$

Since $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x-1}$, then $\mathrm{dom}(f) = [0, \infty)$ and $\mathrm{dom}(g) = (-\infty, 1) \cup (1, \infty)$

$$\begin{aligned}
\mathrm{dom}(fg) &= \mathrm{dom}(f+g) \\
&= \mathrm{dom}(f-g) \\
&= \mathrm{dom}(f) \cap \mathrm{dom}(g) \\
&= [0, \infty) \cap \{x \mid x \neq 1\} \\
&= [0, 1) \cup (1, \infty)
\end{aligned}$$

Division of Functions

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

The domain of $\frac{f}{g}(x)$ is the intersection of the domain of f and the domain of g minus where $g(x) = 0$.

$$(\mathrm{dom}(f) \cap \mathrm{dom}(g)) \setminus \{x \mid g(x) \neq 0\}$$

Example

Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x-1}$.

Find $\dfrac{f}{g}(x)$ and the domain $\mathrm{dom}(\dfrac{f}{g})$. Solution:

$$\frac{f}{g}(x) = \frac{\sqrt{x}}{\frac{1}{x-1}}$$
$$= (\sqrt{x})(x-1)$$

Solution (continued)

To compute the domain of $\frac{f}{g}(x)$, we start with the intersection of the domains, then take away the zeros of $g(x)$.
$\text{dom}(f) \cap \text{dom}(g) = [0, 1) \cup (1, \infty)$.
We must take away the values of x for which

$$\begin{aligned} g(x) &= 0 \\ \frac{1}{x-1} &= 0 \\ (x+1)\frac{1}{x-1} &= 0(x-1) \\ 1 &= 0 \end{aligned}$$

This means there are no x-values for which $g(x) = 0$.
Therefore, the domain of $\frac{f}{g}(x) = [0, 1) \cup (1, \infty)$.

Composition

If $f(x) = 3x + 1$ and $g(x) = x^2$, we TRANSFORMED the graph of f by plugging $g(x)$ into the function f.

$$f \circ g(x) = f(g(x)) = f(x^2) = 3(x^2) + 1 = 3x^2 + 1$$

We call this method of combining functions *composition* of functions.

What is a function?

From our previous example, $g(x) = x^2$.
This means that anything that goes into the parentheses of g gets squared.
For example

$$g(\clubsuit) = (\clubsuit)^2$$

$$g(\smiley) = (\smiley)^2$$

$$g(\underline{\hspace{4cm}}) = (\underline{\hspace{4cm}})^2$$

So, therefore,

$$g(f(x)) = g(3x+1) = (3x+1)^2$$

Composition is not Commutative

Note that from our previous examples, we computed that

$$f \circ g(x) = f(g(x)) = 3x^2 + 1$$

and

$$g \circ f(x) = g(f(x)) = (3x + 1)^2 = 9x^2 + 6x + 1$$

These are very different functions.

Composition with numbers

If

$$f \circ g(x) = f(g(x)) = 3x^2 + 1$$

$$g \circ f(x) = g(f(x)) = (3x + 1)^2 = 9x^2 + 6x + 1$$

Then to find $f(g(2))$ and $g(f(2))$, you just need to replace x with 2.

$$f \circ g(2) = f(g(2)) = 3(2)^2 + 1 = 13$$

$$g \circ f(2) = g(f(2)) = (3(2) + 1)^2 == 49$$

You can check these answers with the back of the book.

You try

If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$, then compute

- $f(g(1))$
- $g(f(1))$

Solving it numerically

Recall: $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$
To compute $f(g(1))$, we first compute $g(1) = \sqrt{1+1} = \sqrt{2}$.
Therefore

$$\begin{aligned} f(g(1)) &= f(\sqrt{2}) \\ &= (\sqrt{2})^2 - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

Your turn! Compute $g(f(1))$ on your whiteboard

If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$, then compute

$$g(f(1))$$

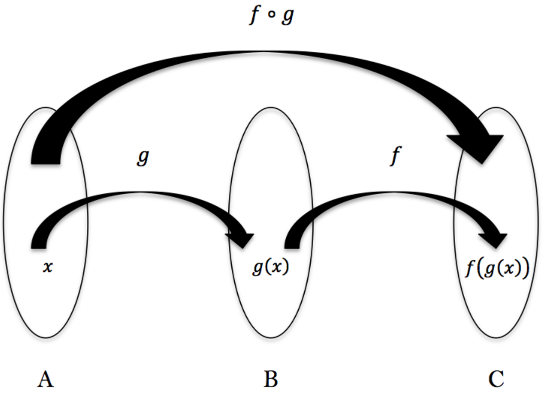
$g(f(1))$

If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$, then compute

$$g(f(1)) = g(0) = \sqrt{1} = 1$$

The domain of $f \circ g$

The domain of $f \circ g$ is the set of all x-values such that $g(x)$ exists, and also that $g(x)$ is in the domain of f.

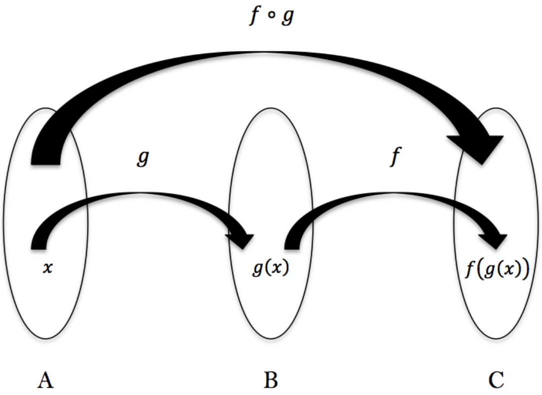


Example p. 298 # 68

Find the domain of $f \circ g$, if

$$f(x) = \frac{5}{x+4} \text{ and } g(x) = \frac{1}{x}$$

Find the domain of f first.



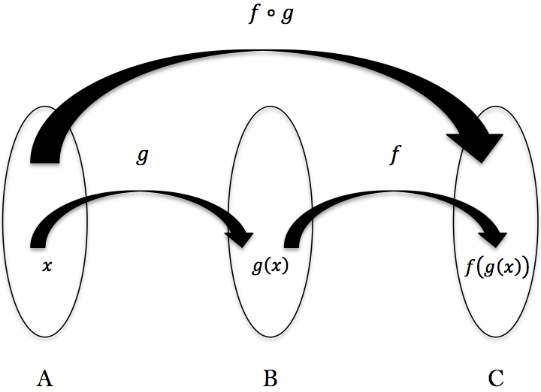
Example p. 298 # 68

Find the domain of f if

$$f(x) = \frac{5}{x+4}$$

is

$$\operatorname{dom}(f) = \{x \mid x \neq 4\}$$



Example p. 298 # 68

Now we need to find the x such that $g(x)$ is in the domain of f

$$g(x) = \frac{1}{x}$$

and

$$\frac{1}{x} \in \mathrm{dom}(f) = \{x \mid x \neq 4\}$$

which means that we need

$$\frac{1}{x} \neq 4$$

so

$$x \neq \frac{1}{4}$$

Which means the domain of $f \circ g$ is $\{x \in \mathbb{R} \mid x \neq 0, -\frac{1}{4}\}$

But that was so confusing! Isn't there an easier way to do it?

YES! Don't simplify!

$$f \circ g(x) = f(g(x)) = f(\frac{1}{x}) = \frac{5}{\frac{1}{x} + 4}$$

To find the domain, we need to make sure none of the denominators are 0.

$$x \neq 0$$

and

$$\frac{1}{x} + 4 \neq 0$$

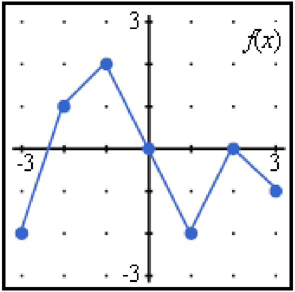
so

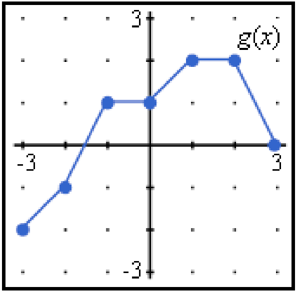
$$\frac{1}{x} \neq -4$$

$$x \neq -\frac{1}{4}$$

Graphic Computation

Use the graphs of f and g below to find the indicated values.

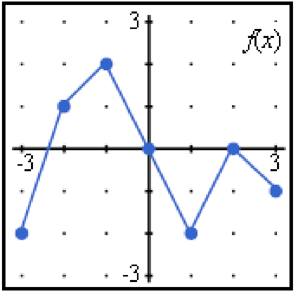


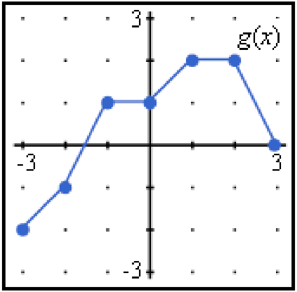


- $f \circ g(-1) =$
- $g \circ f(-1) =$
- $f + g(-3) =$
- $fg(3) =$
- $\dfrac{f}{g}(1) =$

Solutions to Graphic Computation

Use the graphs of f and g below to find the indicated values.





- $f \circ g(-1) = f(g(-1)) = f(1) = -2$
- $g \circ f(-1) = g(f(-1)) = g(2) = 2$
- $f + g(-3) = f(-3) + g(-3) = -2 + -2 = -4$
- $fg(3) = f(3)g(3) = (-1)(0) = 0$
- $\dfrac{f}{g}(1) = \dfrac{f(1)}{g(1)} = \dfrac{-2}{2} = -1$