# BACHELOR OF COMPUTER APPLICATIONS (BCA)

**CSB-1113 (Mathematics - I)**

**Block 3: Limit, Continuity and Differentiability of Function**

## Unit 11: Differentiable Functions

### Structure



## 11.1 INTRODUCTION

The concept of differentiability in calculus is a fundamental aspect that elucidates the smoothness and rate of change of a function with respect to its input variable. In mathematics, differentiability of a function at a specific point describes its ability to have a well-defined derivative at that point, signifying how the function behaves locally around that location.

Understanding differentiability is crucial in analysing the behaviour of functions, particularly in calculus and its applications across various disciplines. It provides insights into the function's instantaneous rate of change, aiding in the comprehension of motion, growth, optimization, and various other phenomena.

Differentiability goes hand in hand with continuity, but not all continuous functions are necessarily differentiable. A function is differentiable at a point if it is continuous at that point and has a well-defined tangent (derivative) at that location. This concept forms the foundation for the study of derivatives, which are integral in understanding rates of change, optimization problems, and the behaviour of functions in a local vicinity.

Exploring the principles of differentiability enables us to delve deeper into the intricacies of functions, paving the way for advanced calculus techniques, including optimization, curve sketching, related rates problems, and the fundamental theorem of calculus. Moreover, differentiability acts as a vital tool for modelling real-world phenomena in physics, engineering, economics, and other scientific fields, providing a precise understanding of how quantities change with respect to each other.

## 11.2 OBJECTIVES

Studying differentiability serves various objectives that deepen one's understanding of functions and their behaviours. Here are key objectives when studying differentiability:

1. **Understanding Rates of Change:** Differentiability provides a precise framework to comprehend how a function's output changes concerning its input at a specific point. This understanding is foundational in physics, engineering, and economics, where rates of change are crucial.
2. **Derivatives and Tangent Lines:** Explore the relationship between derivatives and tangent lines. Studying differentiability allows us to determine the slope of the tangent line to a curve at a given point, offering insights into the local behaviour of functions.
3. **Local Behaviour of Functions:** Analyse the local behaviour of functions through differentiability. Functions that are differentiable at a point exhibit smooth and continuous behaviour around that point.
4. **Optimization Techniques:** Differentiability aids in optimization problems by identifying critical points, where the derivative is zero or undefined, to determine maximum or minimum values of functions.
5. **Related Rates Problems:** Solve problems involving related rates, where differentiability helps establish relationships between changing quantities, enabling the derivation of one quantity concerning another.
6. **Curve Sketching:** Utilize differentiability to sketch accurate and detailed graphs of functions by understanding critical points, slopes, concavity, and inflection points.
7. **Fundamental Theorem of Calculus:** Grasp the Fundamental Theorem of Calculus, which relies on the concept of differentiability to link integration and differentiation.
8. **Applications in Science and Engineering:** Apply differentiability in modelling real-world phenomena, such as motion, growth, decay, and various physical and engineering problems that involve changes in quantities over time.
9. **Advanced Calculus Concepts:** Lay the groundwork for more advanced calculus concepts like Taylor series, limits, continuity, and multivariable calculus that build upon the principles of differentiability.
10. **Enhanced Problem-Solving Skills:** Develop strong analytical and problem-solving skills by working with functions in various contexts, interpreting their behaviours, and applying differentiability to solve complex problems.

## 11.3 DEFINITION OF DIFFERENTIABILITY

Let $I = [a, b]$ be an interval and $f : I \to \mathbb{R}$is a function.

**i.** Let $p$ be an interior point of $I$.

$f$ is said to be differentiable at $p$ if $\lim\limits_{x \to p} \frac{f(x)-f(p)}{x-p}$ exists and finite.

If the limit be $l$, $l$is said to be the derivative of $f$ at $p$ and is denoted by $f'(p)$. Since $p$ is an interior point of the domain of $f$, in order that $\lim\limits_{x \to p} \frac{f(x)-f(p)}{x-p}$ may exist, both the limits $\lim\limits_{x \to p^+} \frac{f(x)-f(p)}{x-p}$ and $\lim\limits_{x \to p^-} \frac{f(x)-f(p)}{x-p}$ should exist and should be equal.

$\lim\limits_{x \to p^+} \frac{f(x)-f(p)}{x-p}$ is said be right hand derivative of $f$ at $p$ and it is denoted by $Rf'(p)$.

$\lim\limits_{x \to p^-} \frac{f(x)-f(p)}{x-p}$ is said be left hand derivative of $f$ at $p$ and it is denoted by $Lf'(p)$.

**ii.** Let $p$ be the left end point $a$.

$f$ is said to be differentiable at $a$ if $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists and finite. If this limit is $l$, $l$is said to be the derivative of $f$ at $a$ and is denoted by $f'(a)$.

**iii.** Let $p$ be the right end point $b$.

$f$ is said to be differentiable at $b$ if $\lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ exists and finite. If this limit is $l$, $l$is said to be the derivative of $f$ at $b$ and is denoted by $f'(b)$.

## 11.4 BASIC PROPERTIES OF DIFFERENTIABLE FUNCTIONS

***Theorem 11.4.1*** Let $I$ be an interval and a function $f : I \to \mathbb{R}$ be differentiable at a point $p \in I$. Then $f$ is continuous at$p$.

Proof: We have $f(x) - f(p) = \frac{f(x)-f(p)}{x-p}.(x-p), \quad \forall\, x \neq p \in I.$

Since $f$ is differentiable at $p$, $\lim_{x \to p} \frac{f(x)-f(p)}{x-p}$ exists and finite and equal to $f'(p)$. Therefore

$$\lim_{x \to p}[f(x) - f(p)] = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}.(x-p) = f'(p).0 = 0.$$

Therefore $\lim_{x \to p} f(x) = f(p)$, this shows that $f$ is continuous at $p$.

**Note:** The converse of the above theorem is not true.

For example, consider the function $f(x) = |x|, x \in \mathbb{R}$. Clearly this function is continuous at $x = 0$. Let us check differentiability of this function at $x = 0$.

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to o} \frac{f(0+h) - f(0)}{h} = \lim_{h \to o} \frac{h}{h} = 1$$

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to o} \frac{f(0-h) - f(0)}{h} = \lim_{h \to o} \frac{-h}{h} = -1$$

Since $Rf'(0) \neq Lf'(0)$, therefore $f$ is not differentiable at 0.

***Theorem 11.4.2*** Let two functions $f$ and $g$ are differentiable at a point $p$, then their sum $f + g$, difference $f - g$, product $fg$, quotient $\frac{f}{g}$(where $g(p) \neq 0$), and the scalar multiplication $cf$ are also differentiable at $p$, and

i. $(f+g)' = f' + g'$

ii. $(fg)' = f'g + fg'$

iii. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

iv. $(cf)' = cf'$, where $c$ is a real constant.

***Theorem 11.4.3*** **(Chain Rule of Differentiability)** Let $f: I \to \mathbb{R}$ and $g: J \to \mathbb{R}$ are function such that $f(I) \subset J$,where$I$ and $J$ are some open intervals of real line containing $p$ and $q = f(p)$ respectively. If $f$ is differentiable at $p$ and $g$ is differentiable at $q$, then the composition $(gof)(x) = g(f(x))$ is differentiable at $p$ and

$$(gof)'(x) = g'(q)f'(p) = (g'of)(p)f'(p) \dots\dots\dots\dots(A)$$

Also, if $f$ is differentiable at $I$ and $g$ is differentiable at $J$, then the composition $(gof)$ is differentiable on $I$ and equation $(A)$ holds for all $p \in I$.

## 11.5 SOME EXAMPLES

***Example 11.5.1*** Check the differentiability and find the derivative (if exists) of the function $f(x) = k, x \in \mathbb{R}$, and $k$ is a fixed real number.

**Solution.** Let $p$ be an arbitrary real number then for $x \neq p$, we have

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} \frac{k - k}{x - p} = 0.$$

Thus, $f$ is differentiable $\forall\, p \in R$ and $f'(x) = 0, x \in \mathbb{R}$.

***Example 11.5.2*** Check the differentiability and find the derivative (if exists) of the function $f(x) = x^2, x \in \mathbb{R}$.

**Solution:** Let $p$ be an arbitrary real number then for $x \neq p$, we have

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} \frac{x^2 - p^2}{x - p} = \lim_{x \to p}(x + p) = 2p.$$

That is, $f'(p) = 2p, \forall\, p \in \mathbb{R}$. The derivative of the function is defined by $f'(x) = 2x, x \in \mathbb{R}$.

***Example 11.5.3*** Check the continuity and differentiability of the function $f(x) = x^2 \sin\frac{1}{x}$ at $x = 0$.

**Solution:** Clearly $f(0) = 0$, let us check the limit

$$R.H.L. = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0}(0 + h)^2 \sin\frac{1}{(0 + h)} = 0$$

$$L.H.L. = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0}(0 - h)^2 \sin\frac{1}{(0 - h)} = 0$$

$$\Rightarrow R.H.L. = L.H.L.$$

Hence limit exists also $\lim_{x \to 0} f(x) = f(0)$, therefore given function is continuous at $x = 0$.

**Differentiability at $x = 0$**

$$Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^2 \sin\frac{1}{(0+h)} - 0}{h} = \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{(0 - h)^2 \sin\frac{1}{(0-h)} - 0}{-h} = \lim_{h \to 0}(-h) \sin\frac{1}{-h} = 0$$

$$\Rightarrow Rf'(0) = Lf'(0)$$

Hence the given function is differentiable at 0 and the value of the derivative is 0.

***Example 11.5.4*** If $f(x) = \begin{cases} x - 1 & when\ x > 1 \\ 0 & when\ x = 1 \\ 1 - x & when\ x < 1 \end{cases}$, show that $f(x)$ is continuous but not differentiable at $x = 1$.

**Solution: Continuity at $x = 1$.**

Clearly $f(x)$ is defined at $x = 1$ and $f(1) = 0$. Now

$$R.H.L. = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}(1+h) - 1 = 0$$

$$L.H.L. = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 1 - (1-h) = 0$$

$$\Rightarrow R.H.L. = L.H.L.$$

Hence limit exists also $\lim_{x\to 1} f(x) = f(1) = 0$, therefore given function is continuous at $x = 0$.

**Differentiability at $x = 1$.**

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[(1+h) - 1] - 0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{[1 - (1-h)] - 0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$\Rightarrow Rf'(1) \neq Lf'(1).$$

Therefore, the given function is not differentiable at $x = 1$.

## 11.6 LET US SUM UP

A function $f(x)$ is differentiable at a point $x = a$ if its derivative exists at that point. Geometrically, it means the function has a well-defined tangent line at that point.

All differentiable functions are continuous, but not all continuous functions are differentiable. There are various rules like the power rule, product rule, quotient rule, and chain rule help in finding derivatives of different functions efficiently.

**Applications**:

- **Optimization**: Using derivatives to find maximum or minimum points of functions.
- **Rate of Change**: Describing how quantities change with respect to one another.
- **Physics and Sciences**: Modelling motion, growth, and decay in various natural phenomena.

## 11.7UNITENDEXCERCISE

**11.7.1** Let $f(x) = \begin{cases} 1+x, & x \leq 2 \\ 5-x, & x > 2 \end{cases}$, discuss the continuity and differentiability of $f(x)$ at $x = 2$.

**11.7.2** If $f(x) = \begin{cases} x & when\ x < 1 \\ 2-x & when\ 1 \leq x < 2 \\ -2+3x-x^2 & when\ x > 2 \end{cases}$, discuss the continuity and differentiability at $x = 1$ and $x = 2$.

**11.7.3** Show that the function $f(x) = \begin{cases} x\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at the point $x = 0$ but not differentiable at that point.

**11.7.4** Show that the function $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is differentiable at $x = 0$ but $f'$ is not continuous at $x = 0$.

**11.7.5** Let $f(x) = \begin{cases} 1 & when\ x < 0 \\ \cos x & when\ 0 \le x < \pi \\ -1 & when\ x > \pi \end{cases}$, discuss the continuity and differentiability at $x = 0$ and $x = \pi$.

## 11.8 REFERENCES AND SUGGESTED READINGS


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