MANGALAYATAN UNIVERSITY
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BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 3: Limit, Continuity and Differentiability of Function

Unit 9: Limit of a Function and Algebra of Limits

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9.1 INTRODUCTION

The concept of the limit of a function is fundamental in calculus, offering a precise way to understand the behaviour of a function as it approaches a certain value. In mathematics, the limit of a function at a particular point describes the behaviour of the function's values as its input gets arbitrarily close to that point. This notion plays a pivotal role in understanding continuity, derivatives, and integration, forming the bedrock of calculus and its applications across various fields such as physics, engineering, economics, and more. Understanding limits enables us to analyse functions behaviour at specific points and provides a gateway to exploring the fundamental principles of change and continuity in mathematics.

Epsilon-delta proofs define a limit to lie within an interval as small as possible. This guarantees that the function stays close to the undefined point, even though the point is never reached by the function.

9.2 OBJECTIVES

After going through this unit, you should be able to

- Learn and understand the$\epsilon - \delta$ definition of limit.
- Apply the $\epsilon - \delta$ definition.
- Learn basic results related to limits.
- Understand the algebra of limits.

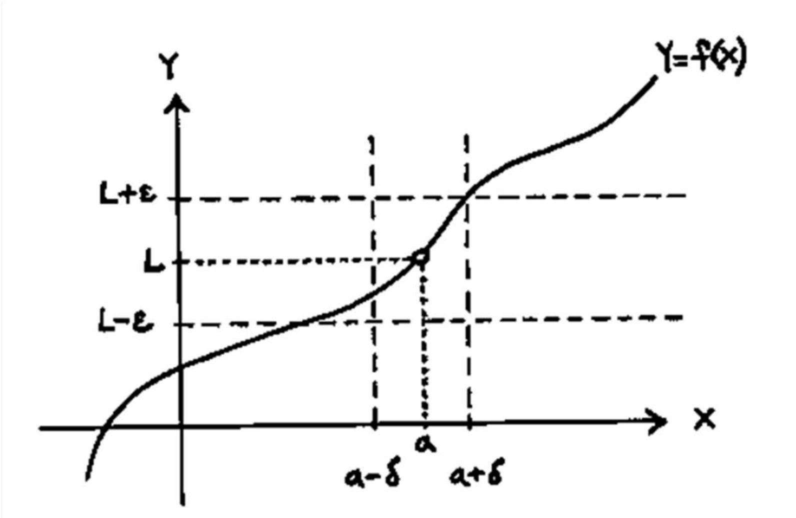
9.3 $\epsilon - \delta$ DEFINITION OF THE LIMIT

A real valued function $f(x)$ is said to have limit l at a point $x = a$ if

$\forall\, \epsilon > 0\ \exists$ a $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

Then we write $\lim\limits_{x \to a} f(x) = l$.

i.e., we can set $f(x)$ as close as we want to l by making x sufficiently close to a.



9.4 SOME EXAMPLES BASED ON $\epsilon - \delta$ DEFINITION OF THE LIMIT

Example 9.4.1 Show that $\lim\limits_{x \to 3} x^2 = 9$.

Solution: Here $f(x) = x^2, a = 3,\ \ l = 9$. Let $\epsilon > 0$.

$|f(x) - 9| = |x^2 - 9| = |x - 3||x + 3| < \epsilon$ for all x with $|x - 3| < \delta$.

With $\delta \leq 1, |x - 3| < \delta \leq 1$ gives $2 < x < 4$, therefore $5 < x + 3 < 7$. So

$$|x^2 - 9| < 7|x - 3|.$$

Thus, if we choose $\delta = \min\{1, \frac{\epsilon}{7}\}$, then for any given $\epsilon > 0$,

$$|x - 3| < \delta\ \Rightarrow |x^2 - 9| < \epsilon.$$

Hence $\lim\limits_{x \to 3} x^2 = 9$.

Example 9.4.2 Show that $\lim\limits_{x \to a} x^2 = a^2$.

Solution: As in previous example, with $\delta \leq 1, |x - a| = |x + a - 2a| < \delta \leq 1$ gives $|x + a| < 1 + 2|a|$, and so

$$|x^2 - a^2| = |x - a||x + a| < (1 + 2|a|)|x - a|.$$

Thus, if we choose $\delta = \min\{1, \frac{\epsilon}{(1+2|a|)}\}$, then for any given $\epsilon > 0$,

$$|x - a| < \delta\ \Rightarrow |x^2 - a^2| < \epsilon.$$

Hence $\lim\limits_{x \to a} x^2 = a^2$. (Note that $\delta = \delta(\epsilon, a)$).

Example 9.4.3 Show that $\lim\limits_{x \to a} \sin x = \sin a$.

Solution: We know that $|\sin x| \le |x|, \forall\, x \in \mathbb{R}$. Now

$$|\sin x - \sin a| = 2\left|\sin\left(\frac{x-a}{2}\right)\right|\left|\cos\left(\frac{x+a}{2}\right)\right| \le 2\left|\sin\left(\frac{x-a}{2}\right)\right| \le |x-a|.$$

Thus, if we choose $\delta = \epsilon$, then for any given $\epsilon > 0$,

$$|x-a| < \delta \;\Rightarrow |\sin x - \sin a| < \epsilon.$$

Hence $\lim_{x\to a} \sin x = \sin a$.

Example 9.4.4 Show that $\lim_{x\to 0} \sqrt{x} = 0,\ x \ge 0$.

Solution: Here $f(x) = \sqrt{x}, a = 0, l = 0$. Let $\epsilon > 0$.

For $x \ge 0, |f(x) - 0| = \sqrt{x} < \epsilon \Rightarrow 0 < x < \epsilon^2$. So if we chose $\delta = \epsilon^2$ then

$$0 < |x-0| < \delta \Rightarrow \left|\sqrt{x} - 0\right| < \epsilon.$$

Hence $\lim_{x\to 0} \sqrt{x} = 0,\ x \ge 0$.

9.5 SOME IMPORTANT RESULTS REGARDING LIMIT

If the limit of a function $f(x)$ exist at $x = a$ then it is unique.

Proof: Let, if possible, there are two real numbers l_1, l_2 such that $l_1 \neq l_2$ and

$\lim_{x\to a} f(x) = l_1$ also $\lim_{x\to a} f(x) = l_2$, then for any given $\epsilon > 0, \exists\ \delta_1, \delta_2$ such that

$|f(x) - l_1| < \epsilon$ whenever $0 < |x-a| < \delta_1$ and

$|f(x) - l_2| < \epsilon$ whenever $0 < |x-a| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ then

$$|l_1 - l_2| = |(f(x) - l_2) - (f(x) - l_1)| \le |f(x) - l_2| + |f(x) - l_1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ is at our choice therefore the above inequality forces that $l_1 = l_2$.

9.6 ALGEBRA OF LIMITS

Let $f(x)$ and $g(x)$ are two real valued functions such that

$\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ and k be any constant then

a) $\lim_{x\to a}(f(x) \pm g(x)) = l \pm m$

b) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = l\,m$

c) $\lim_{x\to a} \frac{f(x)}{g(x)} = l/m$, provided $m \neq 0$.

d) $\lim_{x\to a} kf(x) = kl$

LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem: Let $p(x)$ and $q(x)$ be two polynomial functions. Let a be any real number. Then,

$$\lim_{x\to a} p(x) = p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

Proof: Consider the polynomial $p(x) = a_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Then

$$\lim_{x\to a} p(x) = \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0\ ) = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a)$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\ \neq\ 0$, then

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

Example: Evaluate $\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}$

Solution: Here $f(x) = \frac{p(x)}{q(x)}$, where $p(x) = 2x^2 - 3x + 1$ and $q(x) = 5x + 4$ also $q(3) \neq 0$. Thus

$$\mathit{lim}_{x\to 3}\frac{2x^2 - 3x + 1}{5x + 4} = \frac{2.3^2 - 3.3 + 1}{5.3 + 4} = \frac{10}{19}.$$

Calculating a limit when $\frac{f(x)}{g(x)}$ has the Indeterminate Form $\frac{0}{0}$.

- First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
- We then need to find a function that is equal to $h(x) = \frac{f(x)}{g(x)}$ for all $x\ \neq\ a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
  - If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
  - If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
  - If $\frac{f(x)}{g(x)}$ is a complex fraction, we begin by simplifying it.
- Last, we apply the limit laws.

Evaluating a Limit by Factoring and Canceling

Example: Evaluate $\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}$.

Solution: $\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3} = \lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x\to 3}\frac{x}{2x+1} = \frac{3}{7}$.

Evaluating a Limit by Multiplying Conjugate

Example: Evaluate $\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}$.

Solution: $\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1} = \lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}.\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1} = \lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)} = \lim_{x\to -1}\frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$

Evaluating a Limit When the Limit Laws do not apply

Example: Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

Solution: Both $\frac{1}{x}$ and $\frac{5}{x(x-5)}$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right) = \lim_{x\to 0}\frac{x-5+5}{x(x-5)} = \lim_{x\to 0}\frac{x}{x(x-5)} = \lim_{x\to 0}\frac{1}{x-5} = -\frac{1}{5}.$$

9.7 LET US SUM UP

So, in this first unit we have learned about the $\epsilon - \delta$ definition of the limit also some numerical examples studied. Finding limit of function at a point is the procedure to study the behavior of the function in the neighborhood of that point. There are some points that must be noted.

a) We calculate the limit of a function to understand the behaviour of a function as it approaches a certain value.

b) In the definition we seen that there exists a delta such that $0 < |x-a| < \delta$, it means that function need not be defined at the point $x = a$.

c) The value of δ depends on both ϵ and the point at which we find the limit.

9.8 UNIT END EXERCISE

By using $\epsilon - \delta$ definition solve the following problems.

9.8.1 Show that $\lim_{x\to 2} f(x) = 4$, where $f(x) = \frac{x^2-4}{x-2}, x \neq 2$.

9.8.2 Show that $\lim_{x\to 2} f(x) = 4$,

Where $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 12 & x = 2. \end{cases}$

9.8.3 Show that $\lim_{x\to a} \cos x = \cos a$.

9.9 REFERENCES AND SUGGESTED READINGS

1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc., 2002.

2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.

3. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd.

4. Gorakh Prasad, Differential Calculus, Pothishala Pvt. Ltd., Allahabad.

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