BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 4: Integration

Unit 14: Method of Integration by Substitution

Structure

14.1 INTRODUCTION

In the realm of calculus, integration serves as a powerful tool for finding areas under curves, determining volumes of irregular shapes, and solving various real-world problems. Among the arsenal of integration techniques, the method of integration by substitution stands out as a versatile and fundamental approach. This unit serves as an entry point into understanding the principles and applications of integration by substitution.

14.2OBJECTIVES

After completing this unit, students will be able to

1. Understand the concept of substitution and its role in integration.
2. Learn how to identify appropriate substitutions for different types of integrals.
3. Master the technique of integration by substitution through step-by-step examples.
4. Apply integration by substitution to solve a variety of integrals involving trigonometric, exponential, and other functions.
5. Explore advanced applications of integration by substitution in calculus and beyond.

14.3METHOD OF INTEGRATION BY SUBSTITUTION

If we have to evaluate an integral of the type $\int f\{\phi(x)\} \cdot \phi'(x)dx$ then we put $\phi(x) = t$ and $\phi'(x)dx = dt$. With this substitution, the integrand becomes easily integrable.

Substitution Method I: When the integrand is of the form $f(ax + b)$, we put $(ax + b) = t$ and $dx = \frac{1}{a}dt$.

Substitution Method II: When the integrand is of the form $x^{n-1} \cdot f(x^n)$, we put $x^n = t$ and $nx^{n-1}dx = dt$.

Substitution Method III: When the integrand is of the form $\{f(x)\}^n \cdot f'(x)$, we put $f(x) = t$ and $f'(x)dx = dt$.

Substitution Method IV: When the integrand is of the form $\frac{f'(x)}{f(x)}$, we put $f(x) = t$ and $f'(x)dx = dt$.

Example 1: $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,$ where $n \neq -1$.

Solution: Putting $ax + b = t$, we get $adx = dt$ or $dx = \frac{1}{a}dt$.

$$\therefore \int (ax+b)^n dx = \frac{1}{a}\int t^n dt = \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C.$$

Example 2: $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

Solution: Put $(ax+b) = t$ so that $dx = \frac{1}{a}dt$.

$$\begin{aligned} \therefore \int \cos(ax+b)dx &= \frac{1}{a}\int \cos\, t dt \\ &= \frac{1}{a}\sin\, t + C \\ &= \frac{1}{a}\sin(ax+b) + C. \end{aligned}$$

Example 3: $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$

Solution: Put $(ax+b) = t$ so that $dx = \frac{1}{a}dt$.

$$\begin{aligned} \therefore \int \operatorname{cosec}^2(ax+b)dx &= \frac{1}{a}\int \operatorname{cosec}^2\, t dt \\ &= -\frac{1}{a}\cot\, t + C = -\frac{1}{a}\cot(ax+b) + C. \end{aligned}$$

Example 4: Evaluate: $\int (3x+5)^7 dx$

Solution: Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3}dt$.

$$\therefore \int (3x+5)^7 dx = \frac{1}{3}\int t^7 dt = \frac{1}{3} \cdot \frac{t^8}{8} + C = \frac{(3x+5)^8}{24} + C.$$

Example 5: Evaluate $\int (4-9x)^5 dx$.

Solution: Put $(4-9x) = t$ so that $-9dx = dt$ or $dx = -\frac{1}{9}dt$.

$$\therefore \int (4-9x)^5 dx = -\frac{1}{9}\int t^5 dt = -\frac{1}{9} \cdot \frac{t^6}{6} + C = \frac{-(4-9x)^6}{54} + C.$$

Example 6: Evaluate $\int \frac{1}{(2-3x)^4}dx$

Solution: Put $(2-3x) = t$ so that $-3dx = dt$ or $dx = -\frac{1}{3}dt$.

$$\therefore \int \frac{1}{(2-3x)^4}dx = -\frac{1}{3}\int \frac{1}{t^4}dt = -\frac{1}{3}\cdot\frac{1}{(-3t^3)} + C = \frac{1}{9(2-3x)^3} + C.$$

Example 7: Evaluate $\int \sqrt{ax+b}dx$

Solution: Put $(ax+b) = t$ so that $adx = dt$.

$$\therefore \int \sqrt{ax+b}dt = \frac{1}{a}\int \sqrt{t}dt = \frac{2}{3a}t^{3/2} + C = \frac{2(ax+b)^{3/2}}{3a} + C.$$

Example 8: Evaluate $\int \cos\ 2xdx$.

Solution: Put $2x = t$ so that $2dx = dt$ or $dx = \frac{1}{2}dt$.

$$\therefore \int \cos\ 2xdx = \frac{1}{2}\int \cos\ tdt = \frac{1}{2}\sin\ t + C = \frac{1}{2}\sin\ 2x + C.$$

Example 9: Evaluate $\int e^{(5x+3)}dx$

Solution: Put $(5x+3) = t$ so that $5dx = dt$ or $dx = \frac{1}{5}dt$.

$$\therefore \int e^{(5x+3)}dx = \frac{1}{5}\int e^t dt = \frac{1}{5}\cdot e^t + C = \frac{1}{5}e^{(5x+3)} + C.$$

Example 10: Evaluate $\int \sec^2(3x+5)dx$

Solution: Put $(3x+5) = t$ so that $3dx = dt$ or $dx = \frac{1}{3}dt$.

$$\therefore \int \sec^2(3x+5)dx = \frac{1}{3}\int \sec^2\ tdt = \frac{1}{3}\tan\ t + C$$
$$= \frac{1}{3}\tan(3x+5) + C.$$

Example 11: Evaluate $\int \sin^3\ xdx$

Solution: We know that $\sin\ 3x = 3\sin\ x - 4\sin^3\ x$.

$$\therefore \quad \sin^3\ x = \frac{1}{4}(3\sin\ x - \sin\ 3x).$$

So, $\int \sin^3\ xdx = \int \left(\frac{3}{4}\sin\ x - \frac{1}{4}\sin\ 3x\right)dx$

$$= \frac{3}{4}\int \sin\ xdx - \frac{1}{4}\int \sin\ 3xdx$$
$$= \frac{3}{4}(-\cos\ x) - \frac{1}{4}\cdot\frac{(-\cos\ 3x)}{3} + C$$
$$= -\frac{3}{4}\cos\ x + \frac{\cos\ 3x}{12} + C.$$

Example 12: Evaluate $\int \frac{\text{lo}}{x}dx$

Solution: Put $\log\ x = t$ so that $\frac{1}{x}dx = dt$.

$$\therefore \int \frac{\log\ x}{x}dx = \int tdt = \frac{1}{2}t^2 + C = \frac{1}{2}(\log\ x)^2 + C.$$

Example 13: Evaluate$\int \frac{\sec^2(\log x)}{x} dx$

Solution: Put $\log x = t$ so that $\frac{1}{x} dx = dt$.

$$\therefore \int \frac{\sec^2(\log x)}{x} dx = \int \sec^2 t dt = \tan t + C = \tan(\log x) + C.$$

Example 14: Evaluate$\int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx$

Solution: Put $\tan^{-1} x = t$ so that $\frac{1}{(1+x^2)} dx = dt$.

$$\therefore \int \frac{e^{\tan^{-1} x}}{(1+x^2)} dx = \int e^t dt = e^t + C = e^{\tan^{-1} x} + C.$$

Example 15: Evaluate $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$

Solution: Put $\sqrt{x} = t$ so that $\frac{1}{2} x^{-1/2} dx = dt$ or $\frac{1}{\sqrt{x}} dx = 2dt$.

$$\therefore \int \frac{\sin \sqrt{x}}{\sqrt{x}} dx = 2\int \sin t dt$$
$$= 2(-\cos t) + C = -2\cos t + C = -2\cos \sqrt{x} + C.$$

Example 16: Evaluate $\int \cos^3 x \sin x \, dx$

Solution: Put $\cos x = t$ so that $\sin x dx = -dt$.

$$\therefore \int \cos^3 x \sin x dx = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{1}{4}\cos^4 x + C.$$

Example 17: Evaluate $\int (\sqrt{\sin x})\cos x \, dx$

(ii) Put $\sin x = t$ so that $\cos x dx = dt$.

$$\therefore \int (\sqrt{\sin x})\cos x dx = \int \sqrt{t} dt = \frac{2}{3} t^{3/2} + C = \frac{2}{3}(\sin x)^{3/2} + C.$$

Example 18: Evaluate $\int \frac{\operatorname{cosec}^2 x}{(1+\cot x)} dx$

Solution: Put $(1 + \cot x) = t$ so that $-\operatorname{cosec}^2 x dx = dt$.

$$\therefore \int \frac{\operatorname{cosec}^2 x}{(1+\cot x)} dx = -\int \frac{1}{t} dt$$
$$= -\log t + C = -\log |(1+\cot x)| + C.$$

Example 19: Evaluate$\int \frac{\sin x}{(3+4\text{co} \quad x)^2} dx$

Solution: Put $(3 + 4\cos x) = t$ so that $-4\sin x dx = dt$.

$$\therefore \int \frac{\sin x}{(3+4\cos x)^2} dx = -\frac{1}{4}\int \frac{1}{t^2} dt = \frac{1}{4t} + C = \frac{1}{4(3+4\cos x)} + C.$$

Example 20: Evaluate $\int \frac{2x}{(2x+1)^2} dx$

Solution: Put $(2x+1)=t$ so that $2x=(t-1)$ and $dx=\frac{1}{2}dt$.

$$\begin{aligned}\therefore \int \frac{2x}{(2x+1)^2}dx &= \frac{1}{2}\int \frac{(t-1)}{t^2}dt \\ &= \frac{1}{2}\int \frac{1}{t}dt-\frac{1}{2}\int \frac{1}{t^2}dt=\frac{1}{2}\log\ |t|+\frac{1}{2t}+C \\ &= \frac{1}{2}\log\ |(2x+1)|+\frac{1}{2(2x+1)}+C.\end{aligned}$$

Example 21: Evaluate$\int \frac{(2+3x)}{(3-2x)}dx$

Solution: Put $(3-2x)=t$ so that $x=\left(\frac{3-t}{2}\right)$ and $dx=-\frac{1}{2}dt$.

$$\begin{aligned}\therefore \int \frac{(2+3x)}{(3-2x)}dt &= -\frac{1}{2}\int \frac{\left[2+\left(\frac{9-3t}{2}\right)\right]}{t}dt=-\frac{1}{4}\int \frac{(13-3t)}{t}dt \\ &= \frac{-13}{4}\int \frac{1}{t}dt+\frac{3}{4}\int dt=-\frac{13}{4}\log\ |t|+\frac{3}{4}t+C \\ &= -\frac{13}{4}\log\ |(3-2x)|+\frac{3}{4}(3-2x)+C.\end{aligned}$$

Example 22: Evaluate$\int \frac{3x^2}{(1+x^6)}dx$

Solution: Put $x^3=t$ so that $3x^2dx=dt$.

$$\therefore \int \frac{3x^2}{(1+x^6)}dx=\int \frac{dt}{(1+t^2)}=\tan^{-1}\ t+C=\tan^{-1}\ x^3+C.$$

Example 23: Evaluate $\int \frac{x^3}{(x^2+1)^3}dx$

Solution: Put $(x^2+1)=t$ so that $x^2=(t-1)$ and $xdx=\frac{1}{2}dt$.

$$\begin{aligned}\therefore \int \frac{x^3}{(x^2+1)^3}dx &= \int \frac{x^2\cdot x}{(x^2+1)^3}dx \\ &= \frac{1}{2}\int \frac{(t-1)}{t^3}dt=\frac{1}{2}\int \frac{1}{t^2}dt-\frac{1}{2}\int \frac{1}{t^3}dt \\ &= \frac{-1}{2t}+\frac{1}{4t^2}+C=\frac{-1}{2(x^2+1)}+\frac{1}{4(x^2+1)^2}+C \\ &= \frac{-(1+2x^2)}{4(x^2+1)^2}+C.\end{aligned}$$

Example 24: $\frac{x^8}{(1-x^3)^{1/3}}dx$

Solution: Put $(1-x^3)=t$ so that $x^3=(1-t)$ and $x^2dx=-\frac{1}{3}dt$.

$$\therefore \int \frac{x^8}{(1-x^3)^{1/3}}dx = \int \frac{x^6 \cdot x^2}{(1-x^3)^{1/3}}dx$$
$$= -\frac{1}{3}\int \frac{(1-t)^2}{t^{1/3}}dt = -\frac{1}{3}\int \frac{(1+t^2-2t)}{t^{1/3}}dt$$
$$= -\frac{1}{3}\int t^{-1/3}dt - \frac{1}{3}\int t^{5/3}dt + \frac{2}{3}\int t^{2/3}dt$$
$$= -\frac{1}{2}t^{2/3} - \frac{1}{8}t^{8/3} + \frac{2}{5}t^{5/3} + C$$
$$= -\frac{1}{2}(1-x^3)^{2/3} - \frac{1}{8}(1-x^3)^{8/3}$$
$$+\frac{2}{5}(1-x^3)^{5/3} + C.$$

Example 25: Evaluate $\int \frac{dx}{x\cdot\sqrt{x^6-1}}$.

Solution: Put $x^3 = t$ so that $3x^2dx = dt$ or $x^2dx = \frac{1}{3}dt$.

$$\therefore \int \frac{dx}{x\cdot\sqrt{x^6-1}} = \int \frac{x^2}{x^3\cdot\sqrt{x^6-1}}dx$$

[multiplying numerator and denominator by x^2]

$$= \frac{1}{3}\int \frac{1}{t\sqrt{t^2-1}}dt = \frac{1}{3}\sec^{-1} t + C = \frac{1}{3}\sec^{-1} x^3 + C.$$

Example 26: Evaluate $\int \frac{1}{(\sqrt{x}+x)}dx$

Solution: $\int \frac{1}{(\sqrt{x}+x)}dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$

Now, put $(1+\sqrt{x}) = t$ so that $\frac{1}{\sqrt{x}}dx = 2dt$.

$$\therefore \int \frac{1}{(\sqrt{x}+x)}dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$$
$$= 2\int \frac{1}{t}dt = 2\log|t| + C = 2\log|(1+\sqrt{x})| + C.$$

Example 27: Evaluate $\int \frac{(x-1)}{\sqrt{x+4}}dx$

Solution: Put $(x+4) = t^2$ so that $x = (t^2-4)$ and $dx = 2tdt$.

$$\therefore \int \frac{(x-1)}{\sqrt{x+4}}dx = 2\int \frac{(t^2-5)t}{t}dt$$
$$= 2\int t^2dt - 10\int dt = \frac{2t^3}{3} - 10t + C$$
$$= \frac{2}{3}(x+4)^{3/2} - 10(x+4)^{1/2} + C.$$

Example 28: Evaluate $\int x\sqrt{x+2}dx$

Solution: Put $(x+2) = t^2$ so that $x = (t^2-2)$ and $dx = 2tdt$.

$$\therefore \int x\sqrt{x+2}\,dx = \int (t^2-2)2t^2\,dt = 2\int t^4\,dt - 4\int t^2\,dt$$
$$= \frac{2t^5}{5} - \frac{4t^3}{3} + C = \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C.$$

Example 29: Evaluate $\int (4x+2)\sqrt{x^2+x+1}\,dx$

Solution: Put $(x^2+x+1) = t$ so that $(2x+1)dx = dt$.

$$\therefore \int (4x+2)\left(\sqrt{x^2+x+1}\right)dx = 2\int \sqrt{t}\,dt$$
$$= \frac{4}{3}t^{3/2} + C = \frac{4}{3}(x^2+x+1)^{3/2} + C.$$

Example 30: Evaluate $\int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}dx$

Solution: Put $(2x^2+3x+1) = t$ so that $(4x+3)dx = dt$.

$$\therefore \int \frac{(4x+3)}{\sqrt{2x^2+3x+1}}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{2x^2+3x+1} + C.$$

Example 31: Evaluate $\int \frac{(2x+5)}{(x^2+5x+9)}dx$

Solution: Put $(x^2+5x+9) = t$ so that $(2x+5)dx = dt$.

$$\therefore \int \frac{(2x+5)}{(x^2+5x+9)}dx = \int \frac{1}{t}dt = \log|t| + C$$
$$= \log|(x^2+5x+9)| + C.$$

Example 32: Evaluate $\int \frac{(6x-7)}{(3x^2-7x+5)^2}dx$

Solution: Put $(3x^2-7x+5) = t$ so that $(6x-7)dx = dt$.

$$\therefore \int \frac{(6x-7)}{(3x^2-7x+5)^2}dx = \int \frac{1}{t^2}dt = -\frac{1}{t} + C = \frac{-1}{(3x^2-7x+5)} + C.$$

Example 33: Evaluate $\int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}dx$

Solution: Put $(\cos x + \sin x) = t$ so that $(\cos x - \sin x)dx = dt$.

$$\therefore \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)}dx = \int \frac{1}{t}dt$$
$$= \log|t| + C = \log|(\cos x + \sin x)| + C.$$

Example 34: Evaluate $\int \frac{\sec x}{\log(\sec x + \tan x)}dx$

Solution: Put $\log(\sec x + \tan x) = t$.
Then, on differentiation, we get

$$\frac{1}{(\sec\ x+\tan\ x)}\cdot(\sec\ x\tan\ x+\sec^2\ x)dx=dt$$

$$\text{or}\quad \sec\ xdx=dt$$

$$\therefore \int \frac{\sec\ x}{\log(\sec\ x+\tan\ x)}dx=\int \frac{1}{t}dt$$

$$=\log\ |t|+C=\log\ |\log(\sec\ x+\tan\ x)|+C.$$

Example 35: Evaluate $\int \frac{\sin\ 2x}{(a^2\sin^2\ x+b^2\cos^2\ x)}dx$.

Solution: Put $(a^2\sin^2\ x+b^2\cos^2\ x)=t$ so that

$$2(a^2-b^2)\sin\ x\cos\ xdx=dt \Leftrightarrow \sin\ 2xdx=\frac{dt}{(a^2-b^2)}.$$

$$\therefore\quad I=\int \frac{\sin\ 2x}{(a^2\sin^2\ x+b^2\cos^2\ x)}dx=\int\frac{dt}{(a^2-b^2)t}$$

$$=\frac{1}{(a^2-b^2)}\log\ |t|+C$$

$$=\frac{1}{(a^2-b^2)}\log(a^2\sin^2\ x+b^2\cos^2\ x)+C.$$

14.4 LET US SUM UP

So, in this first unit we have learned about the

1. the concept of substitution and its role in integration.
2. how to identify appropriate substitutions for different types of integrals.
3. the technique of integration by substitution through step-by-step examples.
4. integration by substitution to solve a variety of integrals involving trigonometric, exponential, and other functions.
5. advanced applications of integration by substitution in calculus and beyond.

14.5 UNIT END EXERCISE

14.5.1 Evaluate $\int \frac{1}{c^2+b^2x^2}dx$.

14.5.2 Evaluate $\int e^x\cos e^x\ dx$.

14.5.3 Evaluate $\int \sin^2 x\cos x\ dx$.

14.5.4 Evaluate $\int \frac{4x^3}{1+x^8}dx$.

14.5.5 Evaluate $-\int \frac{\cos^{-1}x}{\sqrt{(1-x^2)}}\ dx$.

14.5.6 Evaluate $\int 20^{5x}\ dx$.

14.5.7 Evaluate $\int \frac{cosec\ x^2}{1+\cot}dx$

14.5.8 Evaluate $\int \frac{1}{(1+x^2)\tan^{-1}x}dx$.

14.5.9 Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

14.5.10 Evaluate $\int \frac{\sin x \cos x}{a \cos^2 x + b\, \mathrm{si}\ ^2 x} dx$

14.6 REFERENCES AND SUGGESTED READINGS

1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc., 2002.
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4. Shanti Narayan, Integral Calculus, S. Chand, New Delhi.