BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 1: Introduction to Sets Functions and Numbers

Unit 4: Linear Diophantine Equations

Structure

4.1 INTRODUCTION

Diophantine equations are named after the ancient Greek mathematician Diophantus, who lived in the 3rd century AD. Diophantine equations are equations that require integer solutions only. In other words, the solutions to these equations must be whole numbers, not fractions or decimals.

The history of Diophantine equations dates back to ancient times, with the Babylonians and Egyptians working on problems of this nature. However, it was Diophantus who made significant contributions to the study of these equations. He wrote a book called "Arithmetica," which contained numerous problems that could be solved using Diophantine equations.

One of the most famous examples of a Diophantine equation is "Fermat's Last Theorem," which states that there are no integer solutions to the equation $x^n + y^n = z^n$ for $n > 2$. The theorem was first conjectured by Pierre de Fermat in 1637, but it was not proven until 1995 by Andrew Wiles.

Over the centuries, mathematicians have continued to work on Diophantine equations, and many important results have been obtained. For example, in the 19th century, mathematicians like Carl Friedrich Gauss and Ernst Eduard Kummer made significant contributions to the theory of Diophantine equations.

Today, Diophantine equations continue to be an active area of research in mathematics. The development of computer algorithms has greatly facilitated the study of these equations, allowing mathematicians to solve more complex problems than ever before.

4.2 OBJECTIVES

After going through this unit, students will be able to;

- Define Linear Diophantine Equation
- Decide whether a Linear Diophantine Equation is solvable or not
- Find the general solution of a Linear Diophantine Equation

4.3 LINEAR DIOPHANTINE EQUATION

The most general form of a linear Diophantine equation is

$$ax + by = c$$

where x and y are unknown, which has to be determined only in terms of integers. $(x_0,\ y_0)$ is said to a solution of $ax + by = c$ if $x = x_0, y = y_0$ satisfy it.

Theorem 1: The Linear Diophantine Equation Theorem states that a linear Diophantine equation $ax + by = c$ has integer solutions if and only if $gcd(a, b)$ divides c.

Proof: To prove this theorem, we need to show two things:

i. If the equation $ax + by = c$ has integer solutions, then $gcd(a, b)$ divides c.
ii. If $gcd(a, b)$ divides c, then the equation $ax + by = c$ has integer solutions.

Suppose that the equation $ax + by = c$ has integer solutions x and y. Let $d = gcd(a, b)$. Then, we can write $a = dm$and $b = dn$ for some integers m and n. Substituting these values into the equation gives us $dmx + dny = c.$ Dividing both sides by d, we get $mx + ny = \frac{c}{d}$.Since m and n are integers, the left-hand side of last equation is an integer, and so is $\frac{c}{d}$.

Therefore, d divides c, as required.

Conversely, suppose that $gcd(a, b)$ divides c. Let $d = gcd(a, b)$. Then, we can write $c = td, t$ is an integer. Since $d = gcd(a, b)$ then there exists integers x_0 and y_0 such that

$$ax_0 + by_0 = d.$$

Multiplying both sides of (5.1) by $\frac{c}{d}$ we get

$$\frac{c}{d}ax_0 + \frac{c}{d}by_0 = \frac{c}{d}d$$

Or

$$c = \left(\frac{c}{d}x_0\right) + b\left(\frac{c}{d}y_0\right) = ax + by$$

Therefore, the original equation ax + by = c has integer solutions, as required.

Therefore, we have shown that a linear Diophantine equation of the form ax + by = c has integer solutions if and only if gcd(a, b) divides c.

Theorem 2: The general solution of linearDiophantine equation of the form $ax + by = c$ is given by $x^* = x_0 - \frac{b}{d}t,\ y^* = y_0 + \frac{a}{d}t$ is $(x_0,\ y_0)$ is one known solution and $\gcd(a, b) = d$.

Proof: Let$(x_0,\ y_0)$ is one known solution of

$$ax + by = c \qquad (1)$$

then we have

$$ax_0 + by_0 = c$$

Now let $(x^*,\ y^*)$ be the general solution of (1). Then we have

$$ax^* + by^* = c$$

Which gives

$$a(x^* - x_0) + b(y^* - y_0) = 0$$

$$\Rightarrow a(x^* - x_0) = -b(y^* - y_0)$$

Since $\gcd(a, b) = d$ therefore there exists integer r_1, r_2 such that $a = r_1 d,\ \ b = r_2 d$. Putting these values in the last equation we get

$$r_1 d(x^* - x_0) = -r_2 d(y^* - y_0)$$

$$\Rightarrow r_1(x^* - x_0) = -r_2(y^* - y_0)$$

$$\Rightarrow r_1 \,/\, -r_2(y^* - y_0)$$

$$\Rightarrow r_1 \,/\, (y^* - y_0)$$

Therefore, $y^* = y_0 + tr_1$ for some integer t.

$$\Rightarrow y^* = y_0 + \frac{a}{d}t.$$

From $r_1(x^* - x_0) = -r_2(y^* - y_0)$ and $y^* = y_0 + tr_1$ we get

$$r_1(x^* - x_0) = -r_2 r_1 t$$

$$\Rightarrow (x^* - x_0) = -r_2 t$$

$$\Rightarrow x^* = x_0 - r_2 t$$

$$\Rightarrow x^* = x_0 - \frac{b}{d}t.$$

Thus $x^* = x_0 - \frac{b}{d}t,\ y^* = y_0 + \frac{a}{d}t$ is the general solution of (1).

Example 1: Find the general solution of the linearDiophantine equation $170x - 455y = 625$.

Solution: First, we apply the Euclidean algorithm to find the $gcd\ (170{,}455)$, we have

$$455 = 170.2 + 115$$

$$170 = 115.1 + 55$$

$$115 = 55.2 + 5$$

$$55 = 5.11 + 0$$

This gives$gcd\ (170{,}455) = 5$.

Since $5/625 = gcd\ (170{,}455)$, therefore $170x - 455y = 625$ has a solution.

We see that $(x_0, y_0) = (1, -1)$ is a particular solution of the equation $170x - 455y = 625$.

Therefore, its general solution by using last theorem is given by

$x^* = 1 - \left(-\frac{455}{5}\right)t = 1 + 91t, t$ is an integer and

$y^* = -1 + \left(\frac{170}{5}\right)t = -1 + 34t, t$ is an integer.

Example 2: Check weather $112x + 70y = 168$ hove a solution or not

Solution: First we find gcd of 112 and 70, we have

$$\begin{aligned} 112 &= 70 \times 1 + 42 \\ 70 &= 42 \times 1 + 28 \\ 42 &= 28 \times 1 + 14 \\ 28 &= 14 \times 2 + 0 \end{aligned}$$

Here $r_4 = 0 \Rightarrow r_3 = 14$ be required solution
Thus

$$(112,70) = 14$$

Now as

$14 \mid 168 \Rightarrow$ Given equation is solvable.

Example 3: Find the general solution of $311x - 112y = 73$.

Solution: Given Diophantine linear equation is

$$311x - 112y = 73 \tag{1}$$

And
$(311, 112) = 1 = d$

Now as$d \mid 73 \Rightarrow eq^n(1)$ has a solution.
Now from equation (1), we have

$$y = \frac{311x - 73}{112} \tag{2}$$

Now we shall find ged of 311 and 73 with respect to 112.
and

$$311 = 112 \times 3 - 25$$
$$73 = 112 \times 1 - 39$$

Then from eq^n(2) and above result, we have

$$y \quad = \frac{(112 \times 3 - 25) \times -(112 \times 1 - 39)}{112}$$
$$= (3x - 1) + \left(\frac{39 - 25x}{112}\right)$$

Let

$$u = \frac{39 - 25x}{112}$$

or

$$112u + 25x = 39 \tag{3}$$

$$x = \frac{39 - 112u}{25} \tag{4}$$

Now find gad of 112 and 39 with respect to 25.
So

$$112 = 25 \times 4 + 12$$
$$39 = 25 \times 2 - 11$$

Then from equation (4) and above results;

$$x = \frac{(25 \times 2 - 11) - (25 \times 4 + 12)u}{25}$$

$$= (2 - 4u) - \frac{(11 + 12u)}{25}$$

$$25v + 12u = -11^2 \tag{5}$$

$$u = \frac{-11 - 25v}{12} \tag{6}$$

Now find gcd of 11 and 25 with respect to 12, we have

$$25 = 12 \times 2 + 1$$
$$11 = 12 \times 1 - 1$$

Then from equation (7) and above results, we have

$$u = \frac{-(12 \times 1 - 1) - (12 \times 2 + 1)v}{12} = -1 - 2v + \left(\frac{1-u}{12}\right)$$

Let$\frac{1-v}{12}$ = w or

$$12w + v = 1 \tag{7}$$

Now to put $w = 0$ in above equation (8) as u has unity coefficient, we have

$$v = 1.$$

Now from eq^n (6) $u = -3$, also from eq^n (4)$x = 15$, and from $eq^n(4)$,$y = 41$.

So, one solution of eq^n (1) is

$x_0 = 15$ and $y_0 = 41$

Then general solution of $eq^n(1)$ is $\left(x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t\right)$

i.e. $\left(15 + \frac{112}{1}t, 41 - \frac{311}{1}.t\right) = (15 + 112t, 41 - 311t)$

where t is an integer.

Example 4: Find the general solution of $13x + 21y = 1791$.

Solution: The general Diophantine equation is

$$13x + 21y = 1791 \tag{1}$$

Here

$$(13,21) = 1 = d \tag{2}$$

Andd| 1791 ⇒ equation (1) is solvable

From equation (1), we have

$$x = \frac{1791 - 21y}{13} \tag{3}$$

Now find gcd of 1791 and 21 with respect to 13.

$$21 = 13 \times 2 - 5$$
$$1791 = 13 \times 138 - 3$$

Then from eq^n(3) and above results, we have

$$x = \frac{(13 \times 138 - 3) - (13 \times 2 - 5)y}{13}$$
$$= (138 - 2y) + \left(\frac{5y - 3}{13}\right)$$

Let

$$\frac{5y - 3}{13} = v \tag{4}$$

or

$$13v - 5y = -3$$

or

$$y = \frac{13v + 3}{5} \tag{5}$$

$$= \frac{(5 \times 3 - 2)v + (5 \times 1 - 2)}{5},$$

[By finding gcd as we have done above]

$$= (3v + 1) - \frac{(2v + 2)}{5}$$

$$\begin{aligned} Let \quad w &= -\frac{(2v + 2)}{5} \\ \Rightarrow 5w + 2v &= -2 \\ \Rightarrow v &= \frac{-5w - 2}{2} = \frac{-(2 \times 2 + 1)w - 2}{2} \\ &= (-1 - 2w) - \frac{w}{2} \end{aligned}$$

In this equation one coefficient is unity.

So putting$w = 0 \quad \Rightarrow \quad v = -1$

Then from eq^n (5), $y = -2$, and from eq^n (3)

$$\text{x} = \frac{1791 + 42}{13} = 141$$

Therefore, the general solution of equation (1) is

$$\left(x_0 + \frac{b}{d}t, y_0 - \frac{c}{d}t\right)$$

i.e., $(141 + 21t, -2 - 13t)$ where $t \in I$

Example 5: Find the general solution of $70x + 112y = 168$

Solution: The given equation is

$$70x + 112 = 168 \tag{1}$$

First of all find gcd of 70 and 112, we have

$$\begin{aligned} 112 &= 70 \times 1 + 42 \\ 70 &= 42 \times 1 + 28 \\ 42 &= 28 \times 1 + 14 \\ 28 &= 14 \times 2 + 0 \end{aligned}$$

So

$$\text{god } (70,112) = 14$$

Now, 14|168. Therefore, equation (1) has a solution.

Now, on dividing (1) by 14 , we have

$$5x + 8y = 12$$

We see that $x = -4$ and $y = 4$ satisfy the above equation.

So $(-4,4)$ is the solution of given equation.

Thus

$$x_1 = x_0 - \frac{b}{d}t = -4 - \frac{112}{14}t = -4 - 8t$$
$$y_1 = y_0 + \frac{a}{d}t = 4 + \frac{70}{14}t = 4 + 5t$$

Thus, the general solution is $(-4 - 8t, 4 + 5t)$.

Example 6: Determine all the solutions in the integers of the following Diophantine equation, $56x + 72y = 40$.

Solution: Find gad of 56 and 72, we have

$$\begin{aligned} 72 &= 56 \times 1 + 16 \\ 56 &= 16 \times 3 + 8 \\ 16 &= 8 \times 2 + 0 \end{aligned}$$

Hence gcd $(56,72) = 8$

Now,

$$\begin{aligned} 8 \quad &= 56 - 3 \times 16 \\ &= 56 - 3 \times (72 - 56 \times 1) \\ &= 4 \times 56 - 3 \times 72 \\ 40 \quad &= 20 \times 56 + 72 \times (-15) \end{aligned}$$

$\Rightarrow$

$x = 20$ and $y = -15$ is a solution of $56x + 72y = 40$.

Then the general solution of given equation is

$$\begin{aligned} (x_1, y_1) \quad &= \left(x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t\right) \\ &= \left(20 + \frac{72}{8}t, -15 - \frac{56}{8}t\right) \\ &= (20 + 9t, -15 - 7t) \end{aligned}$$

for any integer t.

Example 7: Find all the solutions of $10x - 7y = 17$.

Solution: Here $a = 10, b = -7$ and $c = 17$

Now gcd $(a, b) =$ gcd $(10, -7) = 1 = d$ which divides c.

Now taking $x_0 = 1$ and $y_0 = -1$ as solution of given Diophantine equation, so that

$$10x_0 - 7y_0 = 17$$

Thus, the general solution of given equation is

$$\begin{aligned} (x_1, y_1) \quad &= \left(x_0 + \frac{b}{d}t, y_0 - \frac{a}{d}t\right) \\ &= \left(1 + \frac{(-7)}{1}t, -1 - \frac{10}{1}t\right) \\ &= (1 - 7t, -1 - 10t) \vee t \in I \end{aligned}$$

Example 8: Find the positive solution of the equation.

$$11x + 5y = 79$$

Solution: Given linear Diophantine equation is

$$11x + 5y = 79 \tag{1}$$

Here

$$\gcd(11,5) = 1 \text{ and } 1/79$$

$\Rightarrow$ The given equation is solvable.

From equation (1),

$$y = \frac{79 - 11x}{5} = \frac{(5 \times 16 - 1) - (5 \times 2 + 1)x}{5}$$

$$= (16 - 2x) - \frac{(x+1)}{5}$$

Let $-\frac{(x+1)}{5} = w$

$$\Rightarrow x + 5w = -1 \tag{2}$$

Here one coefficient is unity

Putting$w = 0$, we have $x = -1$

From equation (1), $y = 18$

Thus $(x, y) = (-1,18)$ be a solution of equation (1). Then the general solution is given by

$$x = -1 + 5t, y = 18 - 11t$$

Now the required solution is to be positive.

Therefore, we have to put those value of t for which x and y are positive.

so putting $t = 1$, we have

$$x = -1 + 5 \times 1 = 4 \text{ and } y = 18 - 11 \times 1 = 7$$

For $t = 2$ and $> 2, y$ will be negative.

Therefore the only positive solution of given problem is

$$x = 4 \text{ and } y = 7.$$

Example 9: Find the general solution of $x + by = c$

Solution: Here $a = 1, b = b, c = c$

and

$$\gcd(1, b) = 1$$

Then by taking

$$y = 0 \Rightarrow x = c[\because x + b = c]$$

Then $(c, 0)$ be the solution of given Diophantine equation. So the general solution be $(x_0 + bt, y_0 - at)$

i.e.,

$$(c + bt, -t) \forall t \in I.$$

4.4 LET US SUM UP

So, in this unit we have learned the following concepts

- Linear Diophantine Equation
- Condition for solution of a Linear Diophantine Equation
- General solution of a Linear Diophantine Equation

4.5 UNIT END EXERCISE

4.5.1 Check whether the following Diophantine equations are solvable or not

(i) $15x + 30y = 40$.

(ii) $9x + 18y = 15$.

(iii) $10x + 16y = 43$.

4.5.2 Determine all the integral solutions of the following Diophantine equation $56x + 72y = 40$.

4.5.3 Determine all the integral solutions of the following Diophantine equation $172x + 20y = 100$.

4.6 REFERENCES AND SUGGESTED READINGS

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II. Rosen, K. H. (2011). *Elementary number theory*. London: Pearson Education.

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