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BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 2: Introduction to Matrix Theory

Unit 5: Matrices: Types and Basic Properties

Structure

5.1 INTRODUCTION

Welcome to the unit on Matrices - Types and Basic Properties! In this unit, we will explore the fundamental concepts surrounding matrices, their various types, and the essential properties that govern their behaviour. Matrices are foundational mathematical objects that find applications in diverse fields ranging from mathematics and physics to computer science, engineering, and economics.

Why Study Matrices? Matrices provide a structured way to represent and manipulate complex data and relationships. Understanding matrices is crucial for solving systems of linear equations, analyzing transformations, and processing data efficiently in numerous real-world applications.

5.2OBJECTIVES

After going through this unit, students will be able to;

- Define matrix and understand related terms
- Differentiate between various types of matrices
- Do addition of matrices

5.3 MATRIX: DEFINITION AND TYPES

Matrix: A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns) is called a matrix of order m by n, written as an $m \times n$ matrix.

Such an array is enclosed by [ ] or ( ).

Each of the mn numbers constituting the matrix is called an element or an entry of the matrix.

Usually, we denote a matrix by a capital letter.

The plural of matrix is matrices.

Example:

(i) $A = \begin{bmatrix} 3 & 5 & -4 \\ 0 & 1 & 9 \end{bmatrix}$ is a matrix, having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements.

(ii) $B = \begin{bmatrix} 9 & 4 & \sqrt{2} & -1 \\ 1 & 8 & -3 & 2 \\ 6 & 0 & 5 & 7 \end{bmatrix}$ is a matrix, having 3 rows and 4 columns. Its order is 3×4 and it has 12 elements.

How to Describe a Matrix

In order to locate the position of a particular element of a matrix, we have to specify the number of the row and that of the column in which the element occurs.

An element occurring in the ith row and jth column of a matrix A will be called the (i, j) th element of A, to be denoted by a_{ij}.

In general, an $m \times n$ matrix A may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}.$$

Example 1: Consider the matrix $A = \begin{bmatrix} 3 & -2 & 5 \\ 6 & 9 & 1 \end{bmatrix}$.

Clearly, the element in the 1st row and 2nd column is -2. So, we write $a_{12} = -2$.

Similarly, $a_{11} = 3; a_{12} = -2; a_{13} = 5; a_{21} = 6; a_{22} = 9$ and $a_{23} = 1$.

Example 2: Construct a 3×2 matrix whose elements are given by $a_{ij} = (i + 2j)$.

Solution: A 3×2 matrix has 3 rows and 2 columns.

In general, a 3×2 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2}$$

Thus $a_{ij} = (i + 2j)$ for $i = 1,2,3$ and $j = 1,2$

$$\therefore \quad a_{11} = (1 + 2 \times 1) = 3; a_{12} = (1 + 2 \times 2) = 5;$$
$$a_{21} = (2 + 2 \times 1) = 4; a_{22} = (2 + 2 \times 2) = 6;$$
$$a_{31} = (3 + 2 \times 1) = 5; a_{32} = (3 + 2 \times 2) = 7.$$

Hence, $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}_{3 \times 2}$.

Example 3: Construct a 2×3 matrix whose elements are given by $a_{ij} = \frac{1}{2}|5i - 3j|$.

Solution: A 2×3 matrix has 2 rows and 3 columns.

In general, a 2×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{3 \times 2}$$

Thus, $a_{ij} = \frac{1}{2}|5i - 3j|$ where $i = 1,2$ and $j = 1,2,3$.

$$\therefore \quad a_{11} = \frac{1}{2}|5 \times 1 - 3 \times 1| = \frac{1}{2} \cdot |2| = \frac{1}{2} \times 2 = 1;$$

$$a_{12} = \frac{1}{2}|5 \times 1 - 3 \times 2| = \frac{1}{2} \cdot |5 - 6| = \frac{1}{2} \cdot |-1| = \frac{1}{2} \times 1 = \frac{1}{2};$$

$$a_{13} = \frac{1}{2}|5 \times 1 - 3 \times 3| = \frac{1}{2} \cdot |5 - 9| = \frac{1}{2} \cdot |-4| = \frac{1}{2} \times 4 = 2;$$

$$a_{21} = \frac{1}{2}|5 \times 2 - 3 \times 1| = \frac{1}{2} \cdot |10 - 3| = \frac{1}{2} \cdot |7| = \frac{1}{2} \times 7 = \frac{7}{2};$$

$$a_{22} = \frac{1}{2}|5 \times 2 - 3 \times 2| = \frac{1}{2} \cdot |10 - 6| = \frac{1}{2} \cdot |4| = \frac{1}{2} \times 4 = 2;$$

$$a_{23} = \frac{1}{2}|5 \times 2 - 3 \times 3| = \frac{1}{2} \cdot |10 - 9| = \frac{1}{2} \cdot |1| = \frac{1}{2} \times 1 = \frac{1}{2}.$$

$$\text{Hence, } A = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ \frac{7}{2} & 2 & \frac{1}{2} \end{bmatrix}.$$

Example 4: If a matrix has 12 elements, what are the possible orders it can have?

Solution: We know that a matrix of order $m \times n$ has mn elements.

Hence, all possible orders of a matrix having 12 elements are $(12 \times 1), (1 \times 12), (6 \times 2), (2 \times 6), (4 \times 3)$ and (3×4).

Types of Matrices

Row Matrix: A matrix having only one row is known as a row matrix or a row vector.

Examples:

(i) $A = [5 \quad 18]$ is a row matrix of order 1×2.

Column Matrix: A matrix having only one column is known as a column matrix or a column vector.

Examples:

(i) $A = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column matrix of order 3×1.

(ii) $B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is a column matrix of order 2×1.

Zero or Null Matrix: A matrix each of whose elements is zero is called a zero matrix or a null matrix.

Examples: The matrices [0], [0 0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of order $(1 \times 1), (1 \times 2), (2 \times 2)$ and (2×3) respectively.

Square Matrix: A matrix having the same number of rows and columns is called a square matrix.

A matrix of order $(n \times n)$ is called a square matrix of order n or an n-rowed square matrix.

A matrix of order $m \times n$, where $m \neq n$, is called a rectangular matrix.

Examples:

(i) The matrix $\begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$ is a 2-rowed square matrix.

(ii) The matrix $\begin{bmatrix} 5 & 3 & 6 \\ 7 & \sqrt{2} & -4 \\ -9 & \frac{1}{3} & 0 \end{bmatrix}$ is a 3-rowed square matrix.

Diagonal Elements of a Matrix: Let $A = [a_{ij}]_{m\times n}$ be an $m \times n$ matrix. Then, the elements a_{ij} for which $i = j$, are called the diagonal elements of A.

Thus, the diagonal elements of $A = [a_{ij}]_{m\times n}$ are $a_{11}, a_{22}, a_{33}, a_{44}$, etc.

The line along which the diagonal elements lie is called the diagonal of the matrix.

Example Let $A = \begin{bmatrix} 3 & 2 & -1 \\ \sqrt{5} & \frac{5}{8} & 7 \\ 6 & -4 & \sqrt{2} \end{bmatrix}$

Then, the diagonal elements of A are:

$$a_{11} = 3, a_{22} = \frac{5}{8}, a_{33} = \sqrt{2}.$$

Diagonal Matrix: A square matrix in which every nondiagonal element is zero is called a diagonal matrix.

If $A = [a_{ij}]_{n\times n}$ be a diagonal matrix then $a_{ij} = 0$ when $i \neq j$ and we write it as

$$A = \text{diag } [a_{11}, a_{22}, a_{33}, \dots, a_{nn}].$$

Example Let $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then, A is a diagonal matrix.

We may write it as, $A = \text{diag } [6,4,-2]$.

Scalar Matrix: A square matrix in which every nondiagonal element is zero and all diagonal elements are equal is known as a scalar matrix.

Examples

(i) $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ is a scalar matrix of order 2.

(ii) $B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ is a scalar matrix of order 3.

Unit Matrix: A square matrix in which every nondiagonal element is 0 and every diagonal element is 1 is called a unit matrix or an identity matrix.

Thus, a square matrix $[a_{ij}]_{n\times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 \text{ when } i \neq j, \\ 1 \text{ when } i = j. \end{cases}$$

A unit matrix of order n will be denoted by I_n or simply by I.

Examples

(i) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

(ii) $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Comparable Matrices: Two matrices A and B are said to be comparable if they are of the same order, i.e., they have the same number of rows and the same number of columns.

Example $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 & 0 \\ 1 & 4 & -9 \end{bmatrix}$ are comparable matrices, each being of order (2×3).

Equal Matrices: Two matrices A and B are said to be equal, written as $A = B$, if they are of the same order and their corresponding elements are equal.

Example 1: Find x, y, z when $\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix}$.

Solution: Since the corresponding elements of equal matrices are equal, we have

$$\begin{bmatrix} 5 & 3 \\ x & 7 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 7 \end{bmatrix} \Leftrightarrow x = 1, y = 5 \text{ and } z = 3.$$

Example 2: Find x, y, z, w when $\begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$.

Solution: We know that in equal matrices, the corresponding elements are equal.

$$\therefore \quad \begin{bmatrix} x - y & 2x + z \\ 2x - y & 3z + w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$$\Leftrightarrow x - y = -1, 2x - y = 0, 2x + z = 5 \text{ and } 3z + w = 13.$$

Solving the first two equations, we get $x = 1$ and $y = 2$.
Putting $x = 1$ in $2x + z = 5$, we get $z = 3$.
Putting $z = 3$ in $3z + w = 13$, we get $w = 4$.
$\therefore x = 1, y = 2, z = 3$ and $w = 4$.

Example 3: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Why?

Solution: Since the given null matrices are not comparable, they are not equal.

5.4 ADDITION OF MATRICES

Let A and B be two comparable matrices, each of order $(m \times n)$. Then, their sum $(A + B)$ is a matrix of order $(m \times n)$, obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ then

$$A + B = [a_{ij} + b_{ij}]_{m \times n}$$

Remark: For two matrices A and B, the sum $(A + B)$ exists only when A and B are comparable.

Example 1: If $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ then A and B are matrices of order 2×2 and 2×3 respectively.

So, A and B are not comparable.

Hence, $A + B$ is not defined.

Example 2: Let $A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 2 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$.
Clearly, each one of A and B is a 2×3 matrix.
So, A and B are comparable matrices.
$\therefore A + B$ is defined.

Now, $A + B = \begin{bmatrix} 5 + 4 & 0 + (-3) & -2 + (-6) \\ 3 + (-1) & 2 + 0 & -7 + 4 \end{bmatrix}$

$$= \begin{bmatrix} 9 & -3 & -8 \\ 2 & 2 & -3 \end{bmatrix}.$$

Some Results on Addition of Matrices

Theorem 1: Matrix addition is commutative, i.e., $A + B = B + A$ for all comparable matrices A and B.

Proof: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then,

$$\begin{aligned} A + B &= [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} \\ &= [a_{ij} + b_{ij}]_{m\times n} \quad [\text{ by the definition of addition of matrices }] \\ &= [b_{ij} + a_{ij}]_{m\times n} \quad [\because \text{ addition of numbers is commutative }] \\ &= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n} = B + A. \end{aligned}$$

Hence, $A + B = B + A$.

Theorem 2: Matrix addition is commutative,

i.e, $(A + B) + C = A + (B + C)$ for all comparable matrices A, B, C.

Proof: Let $A = [a_{ij}]_{m\times n}, B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$. Then,

$$\begin{aligned} (A + B) + C &= \left([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}\right) + [c_{ij}]_{m\times n} \\ &= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n} \\ &= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n} \\ &= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n} \end{aligned}$$

$[\because$ addition of numbers is associative $]$

$$\begin{aligned} &= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n} \\ &= [a_{ij}]_{m\times n} + \left([b_{ij}] + [c_{ij}]_{m\times n}\right) = A + (B + C). \end{aligned}$$

Hence, $(A + B) + C = A + (B + C)$.

Theorem 3: If A is an $m \times n$ matrix and O is an $m \times n$ null matrix then

$$A + O = O + A = A.$$

Proof: Let $A = [a_{ij}]_{m\times n}$ and $O = [b_{ij}]_{m\times n}$,

where $b_{ij} = 0$ for all suffixes i and j.

Then,

$$\begin{aligned} A + O &= [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n} \\ &= [a_{ij} + 0]_{m\times n} \quad [\because \quad b_{ij} = 0] \\ &= [a_{ij}]_{m\times n} = A. \\ A + O &= A. \end{aligned}$$

$\therefore \quad A + O = A$.

Similarly, $O + A = A$.

Hence, $A + O = O + A = A$.

Remark: The null matrix O of order $m \times n$ is the additive identity in the set of all $m \times n$ matrices.

Example 4: Let $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then verify that

$$A + O = O + A = A.$$

Solution: Clearly, each one of A and O is a matrix of order (2×3).

So, $(A + O)$ and $(O + A)$ are both defined.

Now,
$$\begin{aligned} A + O &= \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3+0 & 5+0 & 4+0 \\ 1+0 & 2+0 & -3+0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A. \end{aligned}$$

And,

$$\begin{aligned} O + A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 0+3 & 0+5 & 0+4 \\ 0+1 & 0+2 & 0+(-3) \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 2 & -3 \end{bmatrix} = A. \end{aligned}$$

Hence, $A + O = O + A = A$.

Negative of a Matrix: Let $A = [a_{ij}]_{m\times n}$. Then, the negative of A is the matrix $(-A) = [-a_{ij}]_{m\times n}$, obtained by replacing each element of A with i its corresponding additive inverse. $(-A)$ is called the additive inverse of A.

Example 5: If $A = \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$, find $(-A)$ and verify that

$$A + (-A) = (-A) + A = 0.$$

Solution: Clearly, we have:

$$(-A) = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix}$$

$$\begin{aligned} A + (-A) &= \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix} + \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3+(-3) & -2+2 & 0+0 \\ -5+5 & 7+(-7) & \sqrt{2}+(-\sqrt{2}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O. \end{aligned}$$

And, $(-A) + A = \begin{bmatrix} -3 & 2 & 0 \\ 5 & -7 & -\sqrt{2} \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 \\ -5 & 7 & \sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} -3+3 & 2+(-2) & 0+0 \\ 5+(-5) & -7+7 & -\sqrt{2}+\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $A + (-A) = (-A) + A = O$.

Solved Examples

Example 1: Let $A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix}$. Verify that $A + B = B + A$.

Solution: Here, A is a 2×3 matrix and B is a 2×3 matrix. So, A and B are comparable.

Therefore, $(A + B)$ and $(B + A)$ both exist and each is a 2×3 matrix.

Now,

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2+4 & 3+(-2) & 5+3 \\ -1+2 & 0+6 & 4+(-1) \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}. \end{aligned}$$

And,

$$B + A = \begin{bmatrix} 4 & -2 & 3 \\ 2 & 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & -2+3 & 3+5 \\ 2+(-1) & 6+0 & (-1)+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 1 & 6 & 3 \end{bmatrix}.$$

Hence, $A + B = B + A$.

Example 2: Let $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$.

Verify that $(A + B) + C = A + (B + C)$.

Solution: Clearly, each one of the matrices A, B, C is a (3×2) matrix. So, $(A + B) + C$ and $A + (B + C)$ are both defined and each one is a 3×2 matrix.

Now, $(A + B) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1+3 & -2+1 \\ 5+0 & 4+2 \\ 3+(-3) & 0+5 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix}.$$

$$\therefore \quad (A + B) + C = \begin{bmatrix} 4 & -1 \\ 5 & 6 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & -1+3 \\ 5+(-2) & 6+2 \\ 0+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

Also, $(B + C) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 2 \\ 1 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 3+4 & 1+3 \\ 0+(-2) & 2+2 \\ -3+1 & 5+6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 1 & -2 \\ 5 & 4 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -2 & 4 \\ -2 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1+7 & -2+4 \\ 5+(-2) & 4+4 \\ 3+(-2) & 0+11 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}.$$

Hence, $(A + B) + C = A + (B + C)$.

Example 3: Find the additive inverse of the matrix $A = \begin{bmatrix} 2 & -5 & 0 \\ 4 & 3 & -1 \end{bmatrix}$.

Solution: Clearly, the additive inverse of the given matrix A is the matrix $-A$, given by

$$-A = \begin{bmatrix} -2 & -(-5) & 0 \\ -4 & -3 & -(-1) \end{bmatrix} = \begin{bmatrix} -2 & 5 & 0 \\ -4 & -3 & 1 \end{bmatrix}.$$

Subtraction of Matrices: If A and B are two comparable matrices then we define

$$(A - B) = A + (-B).$$

Example 4: If $A = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 7 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 8 & -4 \end{bmatrix}$, find $(A - B)$.

Solution: We have, $(-B) = \begin{bmatrix} -1 & -2 & 3 \\ -4 & -8 & 4 \end{bmatrix}$.

$$\begin{aligned}\therefore (A-B) &= A+(-B)\\ &= \begin{bmatrix}2 & -3 & 1\\ 0 & 7 & -9\end{bmatrix}+\begin{bmatrix}-1 & -2 & 3\\ -4 & -8 & 4\end{bmatrix}\\ &= \begin{bmatrix}2+(-1) & -3+(-2) & 1+3\\ 0+(-4) & 7+(-8) & -9+4\end{bmatrix}=\begin{bmatrix}1 & -5 & 4\\ -4 & -1 & -5\end{bmatrix}.\end{aligned}$$

Hence, $(A-B)=\begin{bmatrix}1 & -5 & 4\\ -4 & -1 & -5\end{bmatrix}$.

Example 5: If $\begin{bmatrix}a+4 & 3b\\ 8 & -6\end{bmatrix}=\begin{bmatrix}2a+2 & b+2\\ 8 & a-8b\end{bmatrix}$, then find the value of $(a-2b)$.

Solution: Comparing the corresponding elements of given equal matrices, we have $a+4=2a+2, 3b=b+2$ and $a-8b=-6$.

From these equations, we get $a=2$ and $b=1$.

$$\therefore (a-2b)=(2-2\times 1)=(2-2)=0.$$

Example 6: If $\begin{bmatrix}9 & -1 & 4\\ -2 & 1 & 3\end{bmatrix}=A+\begin{bmatrix}1 & 2 & -1\\ 0 & 4 & 9\end{bmatrix}$, then find the matrix A.

Solution: Clearly, we have

$$\begin{aligned}A &= \begin{bmatrix}9 & -1 & 4\\ -2 & 1 & 3\end{bmatrix}-\begin{bmatrix}1 & 2 & -1\\ 0 & 4 & 9\end{bmatrix}\\ &= \begin{bmatrix}9-1 & -1-2 & 4+1\\ -2-0 & 1-4 & 3-9\end{bmatrix}=\begin{bmatrix}8 & -3 & 5\\ -2 & -3 & -6\end{bmatrix}.\end{aligned}$$

Scalar Multiplication: Let A be a given matrix and k be a number. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k, to be denoted by kA.

If A is an $(m\times n)$ matrix then kA is also an $(m\times n)$ matrix.
If $A=[a_{ij}]_{m\times n}$ then $kA=[k\cdot a_{ij}]_{m\times n}$.

Example 7: If $A=\begin{bmatrix}5 & 4 & -2\\ 6 & -1 & 7\end{bmatrix}$, find (i) $3A$
(ii) $\frac{1}{2}A$
(iii) $-2A$.

Solution: We have:
(i) $3A=\begin{bmatrix}3\cdot 5 & 3\cdot 4 & 3\cdot(-2)\\ 3\cdot 6 & 3\cdot(-1) & 3\cdot 7\end{bmatrix}=\begin{bmatrix}15 & 12 & -6\\ 18 & -3 & 21\end{bmatrix}$.

(ii) $\frac{1}{2}A=\begin{bmatrix}\frac{1}{2}\cdot 5 & \frac{1}{2}\cdot 4 & \frac{1}{2}\cdot(-2)\\ \frac{1}{2}\cdot 6 & \frac{1}{2}\cdot(-1) & \frac{1}{2}\cdot 7\end{bmatrix}=\begin{bmatrix}\frac{5}{2} & 2 & -1\\ 3 & -\frac{1}{2} & \frac{7}{2}\end{bmatrix}$.

(iii) $-2A=(-2)\cdot A=\begin{bmatrix}(-2)\cdot 5 & (-2)\cdot 4 & (-2)\cdot(-2)\\ (-2)\cdot 6 & (-2)\cdot(-1) & (-2)\cdot 7\end{bmatrix}$

$$=\begin{bmatrix}-10 & -8 & 4\\ -12 & 2 & -14\end{bmatrix}.$$

Properties of Scalar Multiplication

Theorem 1: If A and B are two matrices of the same order and k is a scalar then prove that $k(A+B)=kA+kB$.

PROOF Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$. Then,

$$\begin{aligned} k(A+B) &= k\cdot\left([a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}\right) \\ &= k\cdot[a_{ij}+b_{ij}]_{m\times n} \quad \text{[by definition of addition of matrices]} \\ &= [k(a_{ij}+b_{ij})]_{m\times n} \quad \text{[by definition of scalar multiplication]} \\ &= [k\cdot a_{ij}+k\cdot b_{ij}]_{m\times n} \quad \text{[by distributive law]} \\ &= [k\cdot a_{ij}]_{m\times n}+[k\cdot b_{ij}]_{m\times n} \\ &= kA+kB. \end{aligned}$$

Hence, $k(A+B)=kA+kB$.

Theorem 2: If A is any matrix and k_1, k_2 are any scalars then prove that
(i) $(k_1+k_2)A=k_1A+k_2A$
(ii) $k_1(k_2A)=(k_1k_2)A$.

Proof: Let $A=[a_{ij}]_{m\times n}$. Then,
(i)

$$\begin{aligned} (k_1+k_2)A &= (k_1+k_2)\cdot[a_{ij}]_{m\times n} \\ &= [(k_1+k_2)\cdot a_{ij}]_{m\times n} \text{ [by definition of scalar multiplication]} \\ &= [k_1\cdot a_{ij}+k_2\cdot a_{ij}]_{m\times n} \quad \text{[by distributive law]} \\ &= [k_1\cdot a_{ij}]_{m\times n}+[k_2\cdot a_{ij}]_{m\times n} \end{aligned}$$

[by definition of addition of matrices]

$$= k_1A+k_2A.$$

Hence, $(k_1+k_2)A=k_1A+k_2A$.
(ii)

$$\begin{aligned} k_1(k_2A) &= k_1[k_2\cdot a_{ij}]_{m\times n} = [k_1(k_2\cdot a_{ij})]_{m\times n} \\ &= [(k_1k_2)\cdot a_{ij}]_{m\times n} \end{aligned}$$

[by associativity of multiplication in numbers]

$$= (k_1k_2)\cdot[a_{ij}]_{m\times n} = (k_1k_2)\text{ A}.$$

Hence, $k_1(k_2A)=(k_1k_2)A$.

Solved Examples

Example 1: If $A=\begin{bmatrix}3 & 5\\ 7 & -9\end{bmatrix}$ and $B=\begin{bmatrix}6 & -4\\ 2 & 3\end{bmatrix}$, find $(4A-3B)$.

Solution: $(4A-3B)=4A+(-3B)$.

Now, $4A=\begin{bmatrix}4\cdot 3 & 4\cdot 5\\ 4\cdot 7 & 4\cdot(-9)\end{bmatrix}=\begin{bmatrix}12 & 20\\ 28 & -36\end{bmatrix}$.

And, $-3B=(-3)\cdot B=\begin{bmatrix}(-3)\cdot 6 & (-3)\cdot(-4)\\ (-3)\cdot 2 & (-3)\cdot 3\end{bmatrix}=\begin{bmatrix}-18 & 12\\ -6 & -9\end{bmatrix}$.

$$\begin{aligned} \therefore\quad 4A-3B &= 4A+(-3B) \\ &= \begin{bmatrix}12 & 20\\ 28 & -36\end{bmatrix}+\begin{bmatrix}-18 & 12\\ -6 & -9\end{bmatrix} \\ &= \begin{bmatrix}12+(-18) & 20+12\\ 28+(-6) & -36+(-9)\end{bmatrix}=\begin{bmatrix}-6 & 32\\ 22 & -45\end{bmatrix}. \end{aligned}$$

Hence, $(4A-3B)=\begin{bmatrix}-6 & 32\\ 22 & -45\end{bmatrix}$.

Example 2: Let $A = \text{diag}\,[3, -5, 7]$ and $B = \text{diag}\,[-1, 2, 4]$.

Find (i) $(A + B)$ (ii) $(A - B)$ (iii) $-5A$ (iv) $(2A + 3B)$.

Solution: We have

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$\therefore$ (i) $A + B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 11 \end{bmatrix}.$

(ii) $(A - B) = A + (-B)$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(iii) $-5A = (-5) \cdot A = (-5) \cdot \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -35 \end{bmatrix}.$

(iv) $2A + 3B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 26 \end{bmatrix}.$

Example 3: Simplify $\cos\theta \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \cdot \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$

Solution: We have

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 4: If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, find $(x - y)$.

Solution: We have

$$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6+1 & 8+y \\ 10+0 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow 2x + 1 = 5 \text{ and } 8 + y = 0$$

$$\Rightarrow x = 2 \textit{ and } \; y = -8.$$

$$\Rightarrow \quad (x - y) = 2 - (-8) = 10.$$

Example 5: Find a matrix X, if $X + \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}.$

Solution: Let $A = \begin{bmatrix} 4 & 6 \\ -3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix}.$

Then, the given matrix equation is $X + A = B$.

$$\begin{aligned} \therefore X + A = B \Rightarrow X &= B + (-A) \\ &= \begin{bmatrix} 3 & -6 \\ 5 & -8 \end{bmatrix} + \begin{bmatrix} -4 & -6 \\ 3 & -7 \end{bmatrix} \\ &= \begin{bmatrix} 3 + (-4) & -6 + (-6) \\ 5 + 3 & -8 + (-7) \end{bmatrix} = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}. \end{aligned}$$

Hence, $X = \begin{bmatrix} -1 & -12 \\ 8 & -15 \end{bmatrix}$.

Example 6: Find a matrix X such that $2A + B + X = O$, where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ and

$$B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}.$$

Solution: We have

$$\begin{aligned} 2A + B + X &= O \Rightarrow X = -(2A + B). \\ \text{Now, } (2A + B) &= 2 \cdot \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -2+3 & 4+(-2) \\ 6+1 & 8+5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}. \\ \therefore \quad X = -(2A + B) &= \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}. \end{aligned}$$

Example 7: Find matrices X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution: Adding the given matrices, we get

$$(X + Y) + (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 5+3 & 2+6 \\ 0+0 & 9+(-1) \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2} \cdot \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}.$$

On subtracting the given matrices, we get

$$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad 2Y = \begin{bmatrix} 5-3 & 2-6 \\ 0-0 & 9-(-1) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

$$\Rightarrow \quad Y = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}.$$

Hence, $X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$.

5.5 LET US SUM UP

So, in this first unit we have learned about;

- Matrix and their types
- When and how matrices can be added
- Matrix equation with addition and subtraction

5.6 UNIT END EXERCISE

1. If $A = \begin{bmatrix} 4 & -2 & 6 & 1 \\ 7 & 5 & 8 & -3 \\ 9 & 8 & 4 & 3 \end{bmatrix}$ then write

(i) the number of rows in A,

(ii) the number of columns in A,

(iii) the order of the matrix A,

(iv) the number of all entries in A,

(v) the elements $a_{23}, a_{31}, a_{14}, a_{33}, a_{22}$ of A.

2. Write the order of each of the following matrices:

(i) $A = \begin{bmatrix} 1 & 5 & 4 & -2 \\ 0 & 9 & -1 & 5 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 6 & -6 \\ \frac{1}{2} & \frac{3}{4} \\ 9 & -1 \end{bmatrix}$

(iii) $C = [7 \quad -12 \quad 5 \quad 0]$

(iv) $D = [8 \quad -2]$

(v) $E = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

(vi) $F = [8]$

3. If a matrix has 12 elements, what are the possible orders it can have?

4. Find all possible orders of matrices having 7 elements.

5. Construct a 3×2 matrix whose elements are given by $a_{ij} = (2i + j)$.

6. Construct a 4×3 matrix whose elements are given by $a_{ij} = \frac{i+j}{j}$.

7. Construct a 2×2 matrix whose elements are $a_{ij} = \frac{(i+3j)^2}{2}$.

8. Construct a 2×3 matrix whose elements are $a_{ij} = \frac{(i-j)^2}{2}$.

9. Construct a 3×4 matrix whose elements are given by $a_{ij} = \frac{1}{2}|-3i + j|$.

10. If $A = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -2 \\ 4 & -3 & 1 \end{bmatrix}$, verify that $(A + B) = (B + A)$.

11. If $A = \begin{bmatrix} 3 & 5 \\ 4 & 0 \\ 6 & -1 \end{bmatrix}, B = \begin{bmatrix} -3 & -3 \\ 3 & 2 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ 3 & -4 \\ 1 & 6 \end{bmatrix}$, verify that $(A + B) + C = A + (B + C)$.

12. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 4 \\ 5 & -3 & 2 \end{bmatrix}$, find $(2A - 3B)$.

13. Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 5 \\ 3 & 4 \end{bmatrix}$. Find:

(i) $A + 2B$

(ii) $B - 4C$

(iii) $A - 2B + 3C$

14. Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & -1 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -5 & 1 \\ -4 & 0 & 6 \end{bmatrix}$. Compute $5A - 3B + 4C$.

15. If $5A = \begin{bmatrix} 5 & 10 & -15 \\ 25 & 20 & 10 \\ 10 & 0 & -5 \end{bmatrix}$, find

16. Find matrices A and B, if

$$A + B = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 4 & -6 \\ 7 & 3 & 8 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} -5 & -4 & 8 \\ 13 & 2 & 0 \\ -1 & 7 & 4 \end{bmatrix}.$$

17. Find matrix X, if $\begin{bmatrix} 3 & 5 & -9 \\ -1 & 4 & -7 \end{bmatrix} + X = \begin{bmatrix} 6 & 2 & 3 \\ 4 & 8 & 6 \end{bmatrix}$.

18. If $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ -7 & 3 \\ 6 & 4 \end{bmatrix}$, find a matrix C such that $A + B - C = O$.

19. Find the matrix X such that $2A - B + X = O$, where $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$.

20. If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, find a matrix C such that $(A + B + C)$ is a zero matrix.

21. If $A =$ diag $[2, -5, 9]$, $B =$ diag $[-3, 7, 14]$ and $C =$ diag $[4, -6, 3]$, find diagonal of

(i) $A - 2B$

(ii) $B + C - 2A$

(iii) $2A + B - 4C$.

22. Find the values of x and y, when

(i) $\begin{bmatrix} x + y \\ x - y \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 2x + 6 & 7 \\ 0 & 3y - 7 \end{bmatrix} = \begin{bmatrix} x - 3 & 7 \\ 0 & 5 \end{bmatrix}$

(iii) $2\begin{bmatrix} x & 5 \\ 7 & y - 3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

5.7 REFERENCES AND SUGGESTED READINGS

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