BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 2: Introduction to Matrix Theory

Unit 7: Rank of the Matrix

Structure

7.1 INTRODUCTION

Welcome to the unit on the Rank of Matrix! In this module, we will delve into one of the fundamental concepts in linear algebra that plays a pivotal role in various mathematical and computational applications.

Understanding the Significance:

The rank of a matrix is a measure of its essential characteristics. It provides crucial insights into the properties of the matrix and its solutions. Whether in solving systems of linear equations, analysing transformations, or investigating the independence of vectors, the rank serves as a cornerstone concept.

7.2 OBJECTIVES

After completing this unit, students will be able to

- Define submatrix and minor of a matrix
- Understand the definition of rank
- Calculate rank of matrix with the help of elementary transformation

7.3 SUB-MATRIX, MINOR AND RANK

Sub-Matrix: A matrix, which is obtained by deleting some rows or some columns or both of matrix A, is called a sub-matrix of A. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 0 & 11 \end{bmatrix}, \text{ then}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 6 & 7 \\ 10 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 0 & 11 \end{bmatrix}, \text{ etc. are all sub-}$$

Matrices of the matrix A.

Particular Case. The matrix A is a sub-matrix of itself.

5.2. Minor of a Matrix

If any r rows and any r columns from an $m \times n$ matrix A are retained and the remaining $(m-r)$ rows and $(n-r)$ columns deleted, then the determinant of the remaining $r \times r$ sub-matrix of A is called a minor of A of order r. For example,

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}_{5\times 4}$, then

(i) The elements $a_{11}, a_{12}, a_{21}, a_{32}, a_{44}$, etc. are minors of A of order 1.

(i) The elements $a_{11}, a_{12}, a_{21}, a_{32}, a_{44}$, etc. are minors of A of order 1.

(ii) The determinants $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{13} & a_{14} \\ a_{53} & a_{54} \end{vmatrix}$, etc. are minors of A of order 2 .

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{vmatrix}, \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{52} & a_{53} & a_{54} \end{vmatrix}$, etc. are minors of A of order 3.

(iv)

$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{vmatrix},$ etc. are minors of A of order 4 .

Note. There cannot be any minor of order higher than 4 in the above example.

5.3. Rank of a Matrix

A natural number r is called the rank of the matrix A if

(i) There exists at least one non-zero minor of order r.
(ii) Every minor of order $(r+1)$, if any, vanishes. The rank of the matrix A is denoted by $\rho(A)$ or rank (A).

Note 1. The rank of a zero (null) matrix of any order is taken to be zero. Hence,

$\rho(A) = 0$ when A is a zero matrix.

Note 2. $\rho(A) \geq$ when $A \neq 0$.

Note 3. $\rho|A| \leq r$ when every minor of order $(r+1)$ vanishes.

Note 4. $\rho(A) \geq r$ when there exists a non-zero minor of order r.

Note 5. $\rho(A) \leq m$ when A is a $m \times n$ matrix such that $m \leq n$.

Note 6. $\rho(A) \leq n$ when A is a $m \times n$ matrix such that $n \leq m$.

Note 7. $\rho(A) = \rho(A)$ where A' is the transpose of A.

Note 8 . Rank of a matrix, whose every element is 1 , is 1 .

Note 9. If A is a square matrix of order n such that $|A| \neq 0$, then $\rho(A) = n$.

Note 10. If I_n is an identity matrix of order n, then $|I_n| = 1 \neq 0$ and therefore $\rho(I_n) = n$.

Note 11. If A is a diagonal matrix of order n with nonzero diagonal elements, then $|A| \neq 0$ and therefore $\rho(A) = n$.

Note 12. The property (ii) of the number r in the definition of rank of a matrix implies that every minor of order $(r+2)$ vanishes since every minor of order $(r+2)$ can be expressed as the sum of the multiples of minors of order $(r+1)$. It is simple to visualise that every minor of order higher than r will vanish. Thus, the rank of a matrix is the largest order of a non-vanishing minor of a matrix.

5.4. Nullity of a Matrix

Let A be a square matrix of order n. Then $n-\rho(A)$ is called the nullity of the matrix A and is denoted by $N(A)$. We know that the rank of a non-singular square matrix of order n is n and therefore its nullity is equal to zero.

Theorem 1. The rank of a matrix is equal to the rank of the transposed matrix.

Proof. We know that the transpose of a matrix is obtained by changing rows into columns and columns into rows. Also, by a property of determinants, we know that the value of a determinant remains unchanged if its rows are changed into columns and columns into rows. So, by transposing a matrix, the values of its minors do not change. Hence, $\rho(A)=\rho(A)$.

Theorem 2. $\rho(A^*)=\rho(A)$

Proof. We have proved in Theorem 1 that the rank of the transpose of a matrix is the same as the rank of the original matrix. So, to prove the theorem under consideration, what remains to prove for us is that the rank of the conjugate of a matrix is the same as the rank of the original matrix.

If A is real, then the result is trivial. If A is not real, then let us assume that the value of a minor of A with complex elements is $p+iq$. Then the value of the corresponding minor of the conjugate matrix of A is $p-iq$.

If $p+iq=0$ then $p-iq=0$
If $p+iq\neq 0$ then $p-iq\neq 0$
Hence,

$$\rho(\bar{A})=\rho(A)$$

Also, $\rho(A)=\rho(A)$
In view of Eqs. (5.3) and (5.4), we have

$$\rho(A^*)=\rho(A)$$

Note. $\rho(A^*)=\rho(A^*)=\rho(A)$.

Theorem 3. If A is a non-zero column matrix and B is a non-zero row matrix, then show that $\rho(AB)=1$.

Proof. Let $L=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ be a non-zero column matrix. Let $B=|b_{11}b_{12}\ldots\ldots b_{1n}|$ be a non-zero row matrix.
$\because$ A and B are both non-zero matrices.
$\therefore$ AB is also a non-zero matrix.
$\Rightarrow$ AB has at least one non-zero element.
$\Rightarrow$ AB will have at least one non-zero minor of order 1 .

$$\therefore \rho(AB)\geq 1$$

Now,

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots.. & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \dots.. & a_{21}b_{1n} \\ \dots.. & \dots.. & \dots.. & \dots.. \\ a_{n1}b_{11} & a_{n1}b_{12} & \dots.. & a_{n1}b_{1n} \end{bmatrix}$$

From the form of AB, it is clear that every minor of order 2 is zero.

Example 1: Find the rank of the matrix A, where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} \quad |\text{Operating } R_3 - R_1$$

$$= 0 \quad 1 \because R_1 \text{ and } R_3 \text{ are identical}$$

$$\therefore \rho(A) \neq 3$$

A minor of order 2 is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

$$\therefore \rho(A) = 2$$

Example 2: Find the rank of the matrix A, where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= 1(10 - 6) + 2(12 - 8) + 3(4 - 10)$$

$$= 4 + 8 - 18$$

$$= 6 \neq 0$$

$$\therefore \quad \rho(A) = 3$$

Example 3: Find the rank of the matrix A, where:

$$A = \begin{bmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 8 & 1 & 8 \end{vmatrix}$$
$$= 8\begin{vmatrix} 0 & 8 & 1 \\ 0 & 1 & 8 \\ 8 & 1 & 8 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 8 \\ 0 & 0 & 1 \\ 0 & 8 & 1 \end{vmatrix}$$
$$= 8[8(64-1)] - [1(0-8)]$$
$$= 4032 + 8$$
$$= 4040 \neq 0$$
$$\therefore \quad \rho(A) = 4$$

Example 4: Find the rank of the matrix A, where:

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$$
$$= 3\begin{vmatrix} 6 & 1 & 1 & 8 \\ 4 & 2 & 2 & -1 \\ 10 & 3 & 3 & 7 \\ 16 & 4 & 4 & 15 \end{vmatrix} \quad \text{Taking 3 common from } c_3$$
$$= 3(0)$$
$$= 0$$
$$\therefore \quad \rho(A) \neq 4$$

It is easy to see that all the minors of order 3 are zero-valued.

$$\therefore \rho(A) \neq 3$$

There is a minor $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix}$ of order 2 such that $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$.

Hence, $\rho(A) = 2$

Example 5: Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 1 & 3 & 3 & 11 \end{bmatrix}$$

Solution:∵ The matrix A is or order 3×4

$$\therefore \rho(A) \leq 3$$

A minor of order 3 is

$$\begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 1 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 4 & -6 \\ 0 & 4 & 0 \end{vmatrix} \quad \Bigg| \quad \text{Operating } R_2 - R_1, R_3 - R_1$$

Example 1: Find the rank of the following matrix using elementary transformations:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(-2)$

$$A - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Operating $R_{32}(-1)$

$$A - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The single minor of order 3 is zero.
A minor of order 2 is

$$\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$$

$$\therefore \rho(A) = 2$$

employing elementary transformations.

Example 2: Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ employing elementary transformations.

Solution: We have

$$A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

Operating $R_{21}(-1), R_{31}(-1)$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 3 & -1 & -4 & 0 \\ -6 & 2 & 8 & 0 \end{bmatrix}$$

Operating $R_{32}(2)$

$$A - \begin{bmatrix} 2 & 3 & 4 & -1 \\ 3 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All the third order minors are zero but the second order

$$\text{minor } \begin{vmatrix} 4 & -1 \\ -4 & 0 \end{vmatrix} = -4 \neq 0$$

$$\therefore \rho(A) = 2$$

7.4 LET US SUM UP

So, in this first unit we have learned

- submatrix and minor of a matrix
- the definition of rank
- Calculate rank of matrix with the help of minor and elementary transformation

7.5UNIT END EXERCISE

7.5.1 Find the rank of the following matrices

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 6 & -2 & -4 \\ -1 & 0 & 4 \\ 1 & -2 & 5 \end{bmatrix}$$

7.5.2 If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ find the rank of A.

7.5.3 If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$ find the rank of A by using elementary transformation.

7.5.4 If$B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & 2 \\ 13 & 4 & 2 \end{bmatrix}$find the rank of B by using elementary transformation.

7.6 REFERENCES AND SUGGESTED READINGS

1. Richard Bronson. Schaum's Outline of Matrix Operations. Tata McGraw Hill.
2. Hari Kishan. A Textbook of Matrices, Atlantic Publishers.
3. Shanti Narayan, P K Mittal. A Textbook of Matrices, S. Chand.
4. R S Aggarwal, Senior Secondary School Mathematics, Bharti Bhawan.

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