BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 3: Limit, Continuity and Differentiability of Function

Unit 12: Successive Differentiation and Leibnitz Theorem

Structure

12.1 INTRODUCTION

Successive differentiation refers to the process of taking derivatives of a function multiple times. When you differentiate a function once, you find its first derivative, representing its rate of change. Successive or higher-order differentiation involves finding subsequent derivatives beyond the first, revealing information about the function's behaviour, curvature, and higher-level rates of change.

For instance, if $f(x)$ represents a function, its first derivative, often denoted as $f'(x)$ or $\frac{df}{dx}$, describes the rate at which $f(x)$ is changing with respect to x. Successive differentiation involves taking the second derivative $f''(x)$ or $\frac{d^2f}{dx^2}$, the third derivative, and so on.

Each successive derivative can provide insights into different aspects of the function. For example:

- The first derivative indicates the slope or rate of change of the function.
- The second derivative describes the curvature of the function. For instance, a positive second derivative suggests a function is concave up, while a negative second derivative implies concavity downwards.
- Higher-order derivatives provide information about the rate at which previous rates of change are changing, often related to concepts in physics, engineering, economics, and other fields.

Mathematically, you can perform successive differentiation by applying differentiation rules repeatedly, using techniques like the product rule, quotient rule, and chain rule.

Successive differentiation finds applications in various fields, including physics (to describe motion and forces), engineering (for analysing circuits, signal processing, etc.), economics (to model rates of change in economic variables), and many other scientific disciplines where understanding rates of change is crucial.

12.2 OBJECTIVES

After completing this unit, you will be able to

- Understand the concept of Successive Differentiation
- Calculate the nth derivative of functions
- Apply the concept of Successive Differentiation to various problems
- Remember and use of Leibnitz theorem

12.3 STANDARD FORMULAE

a) $\frac{d}{dx}(x^n) = nx^{n-1}$

b) $\frac{d}{dx}(c) = 0$ (where c is a constant)

c) $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right)$

d) $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

e) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$

f) $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right) \cdot g'(x)$

g) $\frac{d}{dx}(e^x) = e^x$

h) $\frac{d}{dx}(\log_b(x)) = \frac{1}{x \cdot \ln(b)}$ (where b is the base of the logarithm)

i) $\frac{d}{dx}(\sin(x)) = \cos(x)$

j) $\frac{d}{dx}(\cos(x)) = -\sin(x)$

k) $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

l) $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

m) $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

n) $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$

o) $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$

p) $\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$

q) $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

r) $\frac{d}{dx}(\sinh(x)) = \cosh(x)$

s) $\frac{d}{dx}(\cosh(x)) = \sinh(x)$

t) $\frac{d}{dx}(\tanh(x)) = \text{sech}^2(x)$

u) $\frac{d}{dx}\left(\text{csch}(x)\right) = -\text{csch}(x) \cdot \coth(x)$

v) $\frac{d}{dx}\left(\text{sech}(x)\right) = -\text{sech}(x) \cdot \tanh(x)$

w) $\frac{d}{dx}(\coth(x)) = -\text{csch}^2(x)$

x) $\frac{d}{dx}\left(\text{arcsinh}(x)\right) = \frac{1}{\sqrt{x^2+1}}$

y) $\frac{d}{dx}\left(\text{arccosh}(x)\right) = \frac{1}{\sqrt{x^2-1}}$

z) $\frac{d}{dx}\left(\text{arctanh}(x)\right) = \frac{1}{1-x^2}$

12.4 n^{th} ORDER DERIVATIVES

The n^{th} order derivative of a function $y = f(x)$, denoted as $f^{(n)}(x)$or $\frac{d^n}{dx^n}f(x)$, represents the derivative of the function obtained by successively differentiating it n times.

Mathematically, if $f(x)$ is a function, its nth order derivative is calculated as:

$$f^{(n)}(x) = \frac{d^n}{dx^n}f(x)$$

Other notations of n^{th} derivatives are $D^n y$ and y_n or y^n

This process involves applying the differentiation operator n times to $f(x)$.

For example, the second derivative $f''(x)$ is the result of applying the differentiation operator twice to $f(x)$, and similarly, the n^{th} derivative involves applying the operator n times.

Higher order derivatives are used to analyse more intricate properties of functions, such as curvature, higher rates of change, and behaviour at specific points.

Calculation of n^{th} derivative

n^{th}Derivative of $y = x^m$

The function $y = x^m$ has the following derivatives with respect to x:

$$y' = \frac{dy}{dx} = mx^{m-1}$$

$$y'' = \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$y''' = \frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$$

$$y^{(4)} = \frac{d^4y}{dx^4} = m(m-1)(m-2)(m-3)x^{m-4}$$

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$$y^{(n)} = \frac{d^ny}{dx^n} = m(m-1)(m-2)\ldots(m-n+1)x^{m-n}$$

n^{th}Derivative of $y = e^{ax}$

$$y = e^{ax}$$
$$y' = ae^{ax}$$
$$y'' = a^2e^{ax}$$
$$y''' = a^3e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$y^{(n)} = a^ne^{ax}$$

n^{th} Derivative of $y = a^{mx}$

$$y = a^{mx}$$
$$y' = \frac{dy}{dx} = (\ln(a)) \cdot m \cdot a^{mx}$$
$$y'' = \frac{d^2y}{dx^2} = (\ln(a))^2 \cdot m^2 \cdot a^{mx}$$
$$y''' = \frac{d^3y}{dx^3} = (\ln(a))^3 \cdot m^3 \cdot a^{mx}$$
$$y^{(4)} = \frac{d^4y}{dx^4} = (\ln(a))^4 \cdot m^4 \cdot a^{mx}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y^{(n)} = \frac{d^ny}{dx^n} = (\ln(a))^n \cdot m^n \cdot a^{mx}$$

n^{th} Derivative of $y = (ax + b)^{-1}$

$$y = (ax + b)^{-1}$$
$$y' = \frac{dy}{dx} = -\frac{a}{(ax + b)^2}$$
$$y'' = \frac{d^2y}{dx^2} = \frac{2a^2}{(ax + b)^3}$$
$$y''' = \frac{d^3y}{dx^3} = -\frac{6a^3}{(ax + b)^4}$$
$$y^{(4)} = \frac{d^4y}{dx^4} = \frac{24a^4}{(ax + b)^5}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y^{(n)} = \frac{d^ny}{dx^n} = (-1)^n \frac{n! \cdot a^n}{(ax + b)^{n+1}}$$

n^{th} Derivative of $y = (ax + b)^m$

$$y = (ax + b)^m$$
$$y' = \frac{dy}{dx} = ma(ax + b)^{m-1}$$
$$y'' = \frac{d^2y}{dx^2} = ma^3(m - 1)(ax + b)^{m-2}$$

$$y''' = \frac{d^3y}{dx^3} = ma^3(m-1)(m-2)(ax+b)^{m-3}$$

$$y^{(4)} = \frac{d^4y}{dx^4} = ma^4(m-1)(m-2)(m-3)(ax+b)^{m-4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y^{(n)} = \frac{d^ny}{dx^n} = ma^4(m-1)(m-2)\ldots(m-n+1)(ax+b)^{m-n}$$

n^{th} Derivative of $y = \log(ax+b)$

$$y = \log(ax+b)$$

$$y' = \frac{dy}{dx} = \frac{a}{ax+b}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{a^2}{(ax+b)^2}$$

$$y''' = \frac{d^3y}{dx^3} = \frac{2a^3}{(ax+b)^3}$$

$$y^{(4)} = \frac{d^4y}{dx^4} = -\frac{6a^4}{(ax+b)^4}$$

$$y^{(n)} = \frac{d^ny}{dx^n} = (-1)^{n-1}\frac{(n-1)!\cdot a^n}{(ax+b)^n}$$

n^{th} Derivative of $y = \sin(ax+b)$

$$y = \sin(ax+b)$$

$$y' = \frac{dy}{dx} = a\cdot\cos(ax+b)$$

$$y'' = \frac{d^2y}{dx^2} = -a^2\cdot\sin(ax+b)$$

$$y''' = \frac{d^3y}{dx^3} = -a^3\cdot\cos(ax+b)$$

$$y^{(4)} = \frac{d^4y}{dx^4} = a^4\cdot\sin(ax+b)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y^{(n)} = \frac{d^ny}{dx^n} = a^n\cdot\sin\left(ax+b+\frac{n\pi}{2}\right)$$

n^{th} Derivative of $y = cos(ax+b)$

$$y = \cos(ax+b)$$

$$y' = \frac{dy}{dx} = -a\cdot\sin(ax+b)$$

$$y'' = \frac{d^2y}{dx^2} = -a^2\cdot\cos(ax+b)$$

$$y''' = \frac{d^3y}{dx^3} = a^3\cdot\sin(ax+b)$$

$$y^{(4)} = \frac{d^4y}{dx^4} = a^4 \cdot \cos(ax+b)$$

....

$$y^{(n)} = \frac{d^ny}{dx^n} = a^n \cdot \cos\left(ax+b+\frac{n\pi}{2}\right)$$

n^{th} Derivative of $y = e^{ax}\, sin\,(bx+c)$

$y = e^{ax}\sin(bx+c)$

$y_1 = \ a\,e^{ax}\sin(bx+c) + \ e^{ax}b\cos(bx+c)$

$= e^{ax}\ [a\sin(bx+c) + \ b\cos(bx+c)]$

$= e^{ax}\ [\,r\ cos\alpha\sin(bx+c) + \ r\ sin\alpha\cos(bx+c)]$

Putting $a = r\ cos\alpha,\ \ b = r\ sin\alpha$

$= e^{ax}\ \ r\ \ \sin(bx+c+\alpha)$

Similarly, $y_2\ = e^{ax}\ \ r^2\ \ \sin(bx+c+2\alpha)$

⋮

$y_n\ = e^{ax}\ \ r^n\ \sin(bx+c+n\alpha)$ where $r^2 = \ a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

$\therefore y_n\ = e^{ax}\ \ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Similarly, if $y = e^{ax}\cos\,(ax+b)$

$y_n = e^{ax}\ \ r^n\ \cos\,(bx+c+n\alpha)$

$= e^{ax}\ \ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

12.5 EXAMPLES

Example 1: Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$

$\therefore y_n = \frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$

$\Rightarrow\ y_n\ = (-1)^{n+1}\ n!\left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$

Example 2: Find the n^{th} derivative of $\sin 6x\cos 4x$

Solution: Let $y\ = \ \sin 6x\cos 4x$

$= \frac{1}{2}\,(\sin 10\,x + \cos 2\,x)$

$\therefore y_n = \ \frac{1}{2}\left[10^n\sin\left(10x+\frac{n\pi}{2}\right) + 2^n\cos\left(2x+\frac{n\pi}{2}\right)\right]$

Example 3: Find n^{th} derivative of sin^2xcos^3x

Solution: Let y = sin^2xcos^3x

$= sin^2xcos^2x\ cosx$

$= \frac{1}{4} sin^2 2x\ cosx\ = \frac{1}{8}(1 - cos4x)cosx$

$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$

$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$

$= \frac{1}{16}(2cosx - cos3x - cos5x)$

$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

Example 4: Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$

$= \left(\frac{1}{2} 2\ sin^2x\right)^2$

$= \frac{1}{4}((1 - \cos 2x)^2$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$

$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$

$= \frac{3}{8} - \frac{1}{2} cos2x + \frac{1}{8} cos4x$

$\therefore y_n = -\frac{1}{2} 2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} 4^n cos\left(4x + \frac{n\pi}{2}\right)$

Example 5: Find the n^{th} derivative of $e^{3x}\cos x\ \sin^2 2x$

Solution: Let $y = e^{3x}\cos x\ \sin^2 2x$

Now $\cos x\ \sin^2 2x\ = \frac{1}{2}\ (\cos x - \cos x \cos 4x)$

$\because \sin^2 2x\ = \frac{1}{2}(1 - \cos 4x)$

$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x\ + \cos 3x\right)$

$\Rightarrow\ y\ = e^{3x}\cos x\ \sin^2 2x = \frac{1}{2} e^{3x} \cos x - \frac{1}{4} e^{3x} \cos 5x - \frac{1}{4} e^{3x} \cos 3x$

$\therefore y_n = \frac{1}{2} e^{3x}\ (9+1)^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\ (9+25)^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$

$-\frac{1}{4} e^{3x}\ (9+9)^{\frac{n}{2}}\ \cos\left(3x + n\tan^{-1}\frac{3}{3}\right)$

$= \frac{1}{2} e^{3x}\ 10^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\ 34^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$

$-\frac{1}{4} e^{3x}\ 18^{\frac{n}{2}}\ \cos(3x + n\tan^{-1} 1)$

Example 6: If $y\ = \sin ax\ + \cos ax$, prove that $y_n\ = a^n[1 + (-1)^n \sin 2ax\ ]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$\therefore y_n = a^n \left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$

$= a^n \left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$

$=a^n \left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$

$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$

$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\ ]^{\frac{1}{2}}$

$= a^n[1 + (-1)^n \sin 2ax\ ]^{\frac{1}{2}} \because \cos n\pi = (-1)^n$ and $\sin n\pi = 0$

Example 7: Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$

$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$

$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$

Differentiating above $(n-1)$ times w.r.t.x, we get

$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$

$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}]$

Using De Moivre's theorem, we get

$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$

$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$

$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta$

$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta$ where $\theta = \tan^{-1}\frac{a}{x}$

Example 8: Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$= \frac{1}{(x-w)(x-w^2)}$ where $w = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{-1-i\sqrt{3}}{2}$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

Substituting $2x+1 = r\,cos\theta,\ \sqrt{3} = r\,sin\theta$such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\ r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\left[cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta\right]$$

$\because \sqrt{3} = r\,sin\theta$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\left(\sqrt{3}\right)^{n+2}}\ 2i\,sin(n+1)\theta\ sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\ sin(n+1)\theta\ sin^{n+1}\theta$$ where $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

Example 9: If $y = x + \tan x$, show that $\cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x \Rightarrow \frac{dy}{dx} = 1 + sec^2 x$

$$\frac{d^2y}{dx^2} = 2\sec x\,(\sec x \tan x) = 2\ sec^2 x \tan x$$

$$\therefore\ \cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\,sec^2 x \tan x - 2(x + \tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10: If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2})\frac{dy}{dx} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2})\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

12.6 LEIBNITZ THEOREM AND RELATED EXAMPLES

If u and v are functions of x such that their n^{th}derivatives exist, then the n^{th}derivative of their product is given by

$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$ where u_r and v_r represent r^{th}derivatives of u and v respectively.

Example 11: Find the n^{th} derivative of $x \log x$

Solution: Let$u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n}x + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}[-(n-1)+n]$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

Example 12: Find the n^{th} derivative of $x^2 e^{3x}\sin 4x$

Solution: Let$u = e^{3x}\sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\,25^{\frac{n}{2}}\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$

$= e^{3x}\,5^n\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$

By Leibnitz's theorem, we have

$$u\,v\;_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x}\sin 4x\right)_n = x^2 e^{3x}\,5^n\sin\left(4x + n\tan^{-1}\frac{4}{3}\right) +$$

$$2nxe^{3x}\,5^{n-1}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) +$$

$$n(n-1)e^{3x}\,5^{n-2}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right) + 0$$

$$= e^{3x}5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13: If $y = a\cos(\log x) + b\, sin(\log x)$, show that

$x^2y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\, cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{acos(logx) + b\ sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + \left(y_{n+1}x + n_{C_1}y_n.1\right) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14: If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem take n^{th}derivative, we get

$$[y_{n+2}(1+x^2) + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2] + \left(y_{n+1}x + n_{C_1}y_n.1\right) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

Example 15: If $y = \ sin(m\, sin^{-1}x)$, then show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$.

Solution: Here $y = \sin(m\, sin^{-1}x)$ 1

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\, sin^{-1}x)$ 2

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2\cos^2(m\, sin^{-1}x)$

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2[1-\sin^2(m\, sin^{-1}x)]$

$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)$......3

$\Rightarrow$ (1- $x^2){y_1}^2 + m^2y^2 = m^2$

Differentiating w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^2 2yy_1 = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$

Using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n 1\right) + m^2y_n = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$......4

Putting $x = 0$ in 1, 2 and 3

$y(0) = 0, y_1(0) = m$ and $y_2(0) = 0$

Putting $x = 0$ in 4

$y_{n+2}(0) = (n^2-m^2)y_n(0)$

Putting $n = 1,2,3 \ldots \ldots$ in the above equation, we get

$y_3(0) = (1^2-m^2)y_1(0)$

$=(1^2-m^2)m \because y_1(0) = m$

$y_4(0) = (2^2-m^2)y_2(0)$

$= 0 \qquad \because y_2(0) = 0$

$y_5(0) = \left(3^2-m^2\right)y_3(0)$

$= m(1^2-m^2)(3^2-m^2)$

$\vdots$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ even \\ m(1^2-m^2)(3^2-m^2)\ldots[(n-2)^2)-m^2], & if\ n\ is\ odd \end{cases}$$

Example 16 If $y = e^{msin^{-1}x}$, show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{msin^{-1}x}$ ...1

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\ \mathrm{e}^{m\, sin^{-1}x}$

$= \frac{my}{\sqrt{1-x^2}}$......②

$\Rightarrow (1-x^2){y_1}^2 = m^2y^2$

Differentiating above equation w.r.t. x , we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$

$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0$......③

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n 1) - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2y_n = 0$$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ② and ③

$y(0) = 1,\ y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in ,we get

$y_{n+2}(0) = (n^2+m^2)y_n(0)$

Putting $n = 1,2,3$ ... in the above equation, we get

$y_3(0) = (1^2+m^2)y_1(0)$

$= (1^2+m^2)m \because y_1(0) = m$

$y_4(0) = (2^2+m^2)y_2(0)$

$= m^2(2^2+m^2) \because y_2(0) = m^2$

$y_5(0) = (3^2+m^2)y_3(0)$

$= m(1^2+m^2)(3^2+m^2)$

$\vdots$

$$\Rightarrow\ y_n(0) = \begin{cases} m^2(2^2+m^2)\ldots[(n-2)^2+m^2], & if\ n\ is\ even \\ m(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2], & if\ n\ is\ odd \end{cases}$$

Example 17: If $y = tan^{-1}x$, show that

$(1-x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x$.....①

$\Rightarrow y_1 = \frac{1}{1+x^2}$ ......②

$y_2 = \frac{-2x}{1+x^2}$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$......③

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$[y_{n+2}(1+x^2) + n_{C_1}y_{n+1}(2x) + n_{C_2}y_n(2)] + 2(y_{n+1}x + n_{C_1}y_n 1) = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$......④

To find $y_n(0)$: Putting $x=0$ in ①, ② and ③, we get

$y(0) = 0, y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$y_{n+2}(0) = -n(n+1)y_n(0)$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = -1(2)y_1(0)$

$= -2 \because y_1(0) = 1$

$y_4(0) = -2(3)y_2(0)$

$= 0 \quad \because y_2(0) = 0$

$y_5(0) = -3(4)y_3(0)$

$= -3(4)(-2) = 4!$

$y_6(0) = -4(5)y_4(0) = 0$

$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$

$\vdots$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $y_{2n}(0) = 0$

Example 18: If $y = (sin^{-1}x)^2$, show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$......②

Squaring both the sides, we get

$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n1) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$......④

To find $y_n(0)$: Putting $x=0$ in ①, ② and ③, we get

$y(0) = 0, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in 4, we get

$y_{n+2}(0) = n^2 y_n(0)$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = 1^2 y_1(0)$

$= 0 \because y_1(0) = 0$

$y_4(0) = 2^2 y_2(0)$

$= 2^2 2 \because y_2(0) = 2$

$y_5(0) = 3^2 y_3(0) = 0$

$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$

$\vdots$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.2^2.4^2 \ldots\ldots\ldots (n-2)^2, & if\ n\ is\ even \end{cases}$$

12.7 LET US SUM UP

In this unit, we have learned about

- the concept of Successive Differentiation
- the nth derivative of functions
- the concept of Successive Differentiation to various problems
- the use of Leibnitz theorem

12.8END UNIT EXERCISE

4.8.1 Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$.

4.8.2 If $x = \sin t,\ y = \sin at$, then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2 y = 0$.

4.8.3 Find the n^{th} derivative of $y = x^3 \cos x$.

4.8.4 Find the n^{th} derivative of $y = x^2 e^x \cos x$.

4.8.5 If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4.8.6 If $y = (sinh^{-1}x)^2$, show that $(1 + x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$

12.9. REFERENCES AND SUGGESTED READINGS

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12.10. GLOSSARY

f^n, y_n, y^n - n^{th} derivative

$y_{n+2} - (n+2)^{th}$ derivative

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