1 VALUES, WISDOM & KNOWLEDGE– The Ultimate Aim of Education

INTRODUCTION

What is the ultimate aim of 'Education' or studying any subject? It is generally thought that education means acquiring a Certificate or Degree after passing an examination. You think that even if you forget everything after the exam, but you have got the certificate or degree, you are educated. This is a wrong notion. After earning a certificate or a degree, you are only a qualified person, not an educated person.

Some others think that education means acquiring knowledge, which is generally equated to getting information about a subject, which is needed to pass an exam with good marks. This is also a wrong notion. 'Knowledge' is not just 'Information'. It is much more than that. Information may be limited to our immediate purpose, which is in our context passing an exam and so limited to the syllabus or even limited to the important or expected questions in the exam.

Knowledge, in fact, is not, and cannot be the end of all education. Knowledge without understanding, without application in real life situations, is useless. Ultimately, the aim of education is to acquire wisdom. Wisdom is also closely related to what is called 'World View', which means a holistic understanding of our world, our universe, and our environment. World View comes by correlating the knowledge from all subjects.

Do you know that there is a close relationship between values, wisdom, and knowledge? Wisdom includes values and knowledge. Wisdom cannot be visualized without values. One cannot be called wise if he/she does not adhere to human values. And of course, knowledge that does not lead to wisdom is valueless.

Values can be defined as a set of norms of behavior that guide the people to do things in such a way that there will be joy, satisfaction, peace of mind, and harmony among individuals and in society.

Derived from a Latin word 'valere', values can be conceptualized as preferences, enduring beliefs, or standards of behavior which guide behaviour and attitude of an individual towards what is right or acceptable.

1.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Elucidate the ultimate aim of education
- Appreciate and explain the relationship between Values, Wisdom and Knowledge
- List and appreciate Universal and Eternal Values
- Explain the Hindu concept of '*Vasudhaiva Kutumbakam'* as a basic Hindu value
- Explain 'True Secularism'
- Discuss in detail *Satyam*, *Shivam* and *Sundaram*
- List Universal 'Vices' that need to be avoided as sins
- List Universal Virtues enumerated in the '*Gita*' as '*Daivi Sampath*' or Divine Virtues
- Discuss Linguistic Connotations and sources of Knowledge and wisdom in Ancient Bharat
- Explain the terms: *Gyan, Pragya, Darshan, Vidya* and *Kala*
- Discuss the Meaning of Knowledge and Disciplines of Knowledge

1.2 UNIVERSAL AND ETERNAL VALUES

There are three eternal universal human values: ***Satyam, Shivam Sundaram* (or Truth, Beauty and Goodness/Godliness)**. All human beings are in search of these three values, though what they mean can vary from place to place as well as time to time. The main branches of Philosophy are based on these three values. Satyam, Shivam, Sundaram has its roots in Indian philosophy and also Western philosophy where it is known as Truth, Beauty and Goodness/Godliness. The main branches of Philosophy are:

(i) Metaphysics/Ontology (Study of Reality)
(ii) Epistemology (Theory of Knowledge/Methods of Knowledge)
(iii) Logic (The most important method of Knowledge)
(iv) Ethics/Axiology (Study of Good and Bad, Right and Wrong)
(v) Aesthetics (Study of Beauty)

Evidently, the first three, i.e., Metaphysics/Ontology, Epistemology and Logic search for Satyam or Truth; the fourth, i.e., Ethics explores Shivam or Goodness; and the fifth, i.e. Aesthetics studies the concept of Sundaram or Beauty. These three values – Satyam, Shivam and Sundaram are very closely linked to our lives.

All truth—material, philosophic, or spiritual—is both beautiful and good. All real beauty—material, art or spiritual symmetry—is both true and good. All genuine goodness—personal morality, social equity & justice, or divine spirituality—is equally true and beautiful. Health, sanity, and happiness are integrations of truth, beauty, and goodness as they are blended in human experience. Such levels of efficient living come about through the unification of energy systems, idea systems, spiritual systems and value systems.

Satyam, Shivam, Sundaram also teach us to strengthen loving attitude towards everyone.

1.2.1 *Vasudhaiva Kutumbakam*: Key to Secularism

Another related wonderful concept given by Indian philosophy is **'*Vasudhaiva Kutumbakam*'**, meaning that the whole world, all human beings are one family. 'Vasudhaiva' means the Mother Earth or the World, implying all human beings, possibly all living beings on Earth. '*Kutumbakam*' means family. Since all human beings are members of one family, we should have a loving attitude towards everyone, irrespective of religious, geographical, social or economic background. **Under this philosophy, all religions are considered to be different paths leading to a common end or goal.** There is one Supreme Power that is called '*Brahm*'. The Supreme Power is also called as God, *Ishwar, Allah*, etc. in various religions. Thus, Hindu religion, which is also called *Sanatan Dharma* (and includes Jainism, Buddhism and Sikhism as a part of the same ethos) is secular by virtue of its philosophy and its very nature. Even if there is a Hindu Country (or *Rashtra*), it will be secular naturally and all citizens will be able to live in harmony and peace. The Hindus never harm anyone of any other religion, unless someone tries to harm them. True Secularism means that:

- There is no discrimination in the name of religion;
- All religions are equal and so there is a common law or civil code for everyone (there is no scope for law based on religion);
- There is religious freedom and everyone is allowed to practice his/her religion;

- No one should interfere with the religion of others and so no one should try to convert others forcibly or by false propaganda
- There is no scope for partisan fundamentalist education, propagating fundamentalism or hatred towards other religions

Now, let us discuss the three Universal Eternal Values in detail, one by one.

1.2.2 *Satyam*/Truth

All of us are in search of truth – at every level. We are all seeking knowledge, which is closely linked to truth. Truth works at least at three levels – facts, reality and ultimate truth.

Facts can be comprehended through our five senses – seeing, hearing, smelling, touching and tasting. All sciences (physical & natural and social) try to establish facts. For instance, Physics studies the physical world; Chemistry studies facts of chemical reactions; and Biology studies the world of flora and fauna, i.e. living world.

Metaphysics goes beyond the physical world and attempts to study (and find) reality, which is beyond the five senses. Epistemology gives us the methods of finding knowledge, i.e. reality. Logic is the most important of these methods. In Indian Thought, there is the concept of '*Para Vidya*' and '*Apara Vidya*'. '*Para*' literally means beyond, and so *Para Vidya* refers to knowledge of the beyond or Metaphysics, whereas *Apara Vidya* is knowledge of the Empirical World, which can be acquired through our 5 senses or physical perception.

Even in everyday life, we like truth to prevail. Those who tell lies are not appreciated or liked in society. Can you identify any religion or faith which says that speaking truth is wrong?

We are also looking for truth about questions like:

- Who am I?
- What is the purpose of life?
- What is death? Is there life after death?
- What is God? Does He/She/It really exist?
- Is there a soul? Is it eternal?

All religions try to answer these questions. These are elements of what we may call the ultimate truth.

CHECK YOUR PROGRESS 1.1

I. Answer the following:

(i) What is the ultimate aim of 'Education'?

(ii) What is the meaning of the word 'Values'?

(iii) What is the relationship between Values, Wisdom and Knowledge.

(iv) Which are the 3 Universal and Eternal Values?

(v) Name the 5 Branches of Philosophy? Also, mention the Universal and Eternal Value that it corresponds to.

(vi) What do you understand by the term ***'VasudhaivaKutumbakam'?***

(vii) What are the different names given to the Supreme Power in different religions?

(viii) Explain how Hinduism, Hindus and Hindu Rashtra are naturally secular.

(ix) What is true secularism?

(x) Distinguish between '*Para Vidya*' and '*Apara Vidya*'.

II. Fill in the blanks:

(i) *Satyam, Shivam, Sundaram* has its roots in …………………..

(ii) ………………..is the branch of philosophy that deals with methods of knowledge.

(iii) Axiology is the branch of philosophy that deals with ……………..

1.2.3 *Shivam*/Goodness/Godliness

Just as we are in search of truth, we are also in search of Goodness/Godliness. Right from childhood, one wants to be a good child; we want to be called good persons all our lives. Even a criminal would try to justify his/her actions and try to prove that he/she is good, not bad.

Ethics provides the moral code, or philosophy, that guides a person's choices and behaviors throughout their life. The idea of a moral philosophy extends beyond the individual to include what is right (and what is wrong) for the community and society at large. Ethics is concerned with rights, responsibilities, use of ethically appropriate language, what it means to live an ethical life and how people make moral decisions.

Morality can be subjective; people may have strong and stubborn beliefs about what's right and wrong that can be in direct contrast to the moral beliefs of others. Yet even though morals can vary from person to person, religion to religion, and culture to culture, there are some universal moral values that stem from basic human emotions. For instance, **non-violence and not harming human beings, irrespective of their religious background, is and should be considered as a universal virtue. If someone kills or harms other human beings, even those who follow another religion, he/she cannot be called virtuous. Any claims that religion teaches killing other human beings are false claims. True religion cannot teach killing other humans, even those humans who follow other religions.**

Those who are considered morally good are said to be virtuous, holding themselves to high ethical standards, while those viewed as morally bad are thought to be wicked, sinful, orcriminal.

1.2.3.1 Vices

There are some vices considered to be deadly vices by most religions. These are:

- *Kaam* (Lust))
- *Krodh* (Anger/Wrath)
- *Lobh* (Greed)
- *Ahankar* (Ego/Vanity)
- *Irshya* (Jealousy)

Hinduism also includes *Moh* (Attachment to worldly objects and pleasures); and Christianity also includes Gluttony (habitual greed or excess in eating) and Sloth (reluctance to work or laziness).

According to Gandhiji, the following are signs of vice or immorality:

- Wealth without Work
- Pleasure without Conscience
- Science without Humanity
- Knowledge without Character
- Politics without Principle
- Commerce without Morality
- Worship without (Self) Sacrifice

1.2.3.2 Virtues

The ***Gita*** enumerates many virtues as *Daivi-Sampat* or divine qualities:

- Fearlessness
- Purity of heart
- Steadfastness and wisdom
- *Vivek* (ability to discriminate between right and wrong)
- *Nishkam Karma* (desireless action or doing one's duty without caring for the result)
- Charity
- Self-restraint
- Sacrifice
- Study of the scriptures
- Austerity or simplicity
- Straightforwardness
- Harmlessness (tendency of not harming anyone)
- Truth
- Absence of anger
- Renunciation
- Peacefulness
- Absence of crookedness
- Compassion to all living beings
- Non-covetousness (not having, or showing, a desire to possess something belonging to someone else)
- Mildness (soft-heartedness and sensitivity towards others)
- Modesty or humility
- Absence of fickleness (changeability, especially with regard to one's loyalties or affections)
- Vigour
- Forgiveness
- Purity
- Absence of jealousy, vanity and arrogance.

These virtues are manifestations of the four fundamental virtues: Non-violence, Truth, Purity, and Self-control. In fact, all the religions in the world speak of virtues. Right actions bring us happiness, peace and well-being. So, it is not surprising that all the major religions talk of virtues. For instance, in Christianity, there are seven virtues: Faith, Hope, Charity, Fortitude, Justice, Prudence, and Temperance. Every religion speaks of virtues, because every religion is a living entity and believes in the happiness and well being of all.

1.2.4 Sundaram/Beauty

Sundaram/Beauty is the third universal and eternal value. Similar to truth and goodness, we are in search of beauty, too, in our lives. What is beauty? The concept of beauty may change from person to person. In that sense, this is the most subjective among these three values, because where one looks for beauty may also change. Thus, some people look for beauty in nature (flora, fauna); some look for it in Arts (Painting, Sculpture, etc.); some search for beauty in literature (poetry, prose, drama); some look for beauty in people; some may find beauty in philosophical or spiritual or religious pursuits; and so on and so forth. Some look for beauty outside and others within. Some like people who are physically beautiful; others may like people who have inner beauty, those who are virtuous or accomplished people. In fact, you will be surprised to know how many different things people can find beauty in – things, people, places, ideas.

The famous Romanticist poet John Keats wrote, "*Beauty is truth, truth beauty. That is all ye know on earth, and all ye need to know*". In another poem, he wrote, "*A thing of beauty is a joy forever*".

Beauty is an emotional element, a pleasure of ours, which nevertheless we regard as a quality of things. The idea of beauty is found in almost every culture and at almost every time in human history, with many similarities. Beauty was, and still is, a term of great esteem linking human beings and nature with artistic practices and works since the early civilizations. From the early cultures, beauty, goodness and truth are customarily related. Beauty carries a double meaning. It is inclusive and exclusive. In the inclusive sense, beauty pertains to anything worthy of approbation, to human virtues and characters, to nobility and goodness, to hidden things and truth, to the natural and divine worlds. In the exclusive, restricted sense, it pertains to how things appear, their manifestations, and to the joys human beings experience when presented with beautiful things, human bodies, artefacts, natural creatures and things. The nature of beauty and its role in philosophy and aesthetics was explained right from the early periods.

CHECK YOUR PROGRESS 1.2

I. Answer the following:

(i) What do you understand by the term '*Shivam*'?

(ii) List the Universal Vices.

(iii) List the signs of vice or immorality as given by *Gandhiji*.

(iv) List the Universal Virtues enumerated in the '*Gita*' as '*Daivi-Sampat*' or Divine Qualities of a person.

(v) Name the 4 fundamental virtues.

(vi) What do you understand by the term '*Sundaram*'?

1.3 LINGUISTIC CONNOTATIONS AND SOURCES OF KNOWLEDGE AND WISDOM IN ANCIENT BHARAT

From ancient times, India has given a great importance to acquisition of knowledge and it has a vast fund of this knowledge in the form of intellectual texts - the world's largest collection of manuscripts, scriptures and thinkers and schools in so many domains of knowledge. *Shree Krishna* in *Bhagwad Gita* tells Arjun that knowledge is the greatest purifier and liberator of the self.

Various terms are closely related to pursuit of knowledge like *Gyan, Pragya, Shiksha, Vidya*, *Kala* and *Darshan*. The root of the great reverence for all knowledge that is *Gyan*, wisdom that is *Pragya*, discipline that is *Vidya*, education that is *Shiksha*, craft that is *Kala* and philosophy that is *Darshan* is attached to the *Guru* in the Indian tradition as he/she is considered as the ultimate authority and source of all knowledge.

- ***Gyan* or *Jnana* (Knowledge):** In Vedic India, education was regarded as the means of self realization and salvation that is *moksha*, which was considered as the highest end of life. Learning was done to acquire *atmagyan* (knowledge about Self) or *brahmagyan* (knowledge about the Supreme).
- ***Pragya* or *Prajna* (Wisdom):** To acquire knowledge merely with the help of sense organs is called *Ajñāna* (ignorance) and to get knowledge by name and form is *Sajna* or *Sjnana*. Acquisition of special knowledge through analysis or contemplation which always remain constant is *Pragya*. Wisdom that is *Pragya* is built on our previous

knowledge to give us new understanding by incorporating various value judgments and experiences and it develops our ability to predict and make inferences.

- ***Darshan*: ** *Darshan* is the system or the point of view which leads to knowledge. Indian philosophers, *rishis* and *saints* use the term *Darshan* for philosophy. They expressed in *Darshan* experiences that they experienced themselves. Perception of truth is known as *Darshan*. In the words of *Dr. Radhakrishnan*, *Darshan* is the logical expression of the nature of reality.
- ***Vidya***: When knowledge is gained in a particular domain and it is organised and systematized for reflection and pedagogy, it is known as *Vidya* which is based on discipline. 18 *Vidyas* have been enumerated, which include:
    - 4 *Vedas*: *Rigveda, Yajurveda, Samaveda, Atharva Veda*
    - 6 *Vedangas*: *Chhandassu, Kalpam, Niruktham, Sikshaa, Vyakarana, Jyotisham*
    - *Ashtadasa* (18) *Puranas*: (History)
    - *Mimamsa* (Study of Actions) and *Vedanta* (Study of Knowledge)
    - 4 Subsidiary *Vedas* - *Ayurveda* (Medicine), *Dhanurved* (Weaponry) *Gandharvaveda* (Music), *ArdhaShastram* (Economy and Polity)
    - *Shilpa* (Architecture)
    - *Nayaya* and *Dharamshastras* (Law and Justice)
    - 6 Auxiliary Sciences - Phonetics, Grammar, Metre, Astronomy, Ritual and Philology.

- ***Kala***: In the Indian context, Knowledge in different domains has been categorised into many disciplines that is *Vidya* and Arts that is *Kala*. Indian tradition talks of 18 major *Vidyas* that is theoretical disciplines (enumerated above) and 64 *Kalas* which are applied or vocational disciplines, that is Art and Craft. Applied Sciences, that is *Kala*, have a direct bearing on day-to-day life of the people and some of them are still a part of contemporary Indian life. The traditional list of *Kala* includes Poetry, Calligraphy, Dancing, Cooking, Carpentry, Agriculture, Animal Husbandry, Fishing, etc. Even for the crafts there are basic texts, for example the popular text *Pingla*.

CHECK YOUR PROGRESS 1.3

I. Answer the following:

(i) Define *Gyan*.

(ii) Define *Pragya*.

(iii) Define *Darshan*.
(iv) Define *Vidya*.
(v) Define *Kala*.

II. Fill in the blanks:

(i) To acquire knowledge with the help of sense organs is called …………...
(ii) Indian *Rishis* and *Saints* use the term *Darshan* for ………………….
(iii) Indian tradition talks of ……….. major *vidyas* and …….. *Kalas*.
(iv) ………………….. is the popular text of craft.

1.4 MEANING OF KNOWLEDGE AND DISCIPLINES OF KNOWLEDGE

1.4.1 Meaning of Knowledge

Knowledge and its transmission is one of the key elements of education, apart from, and only second to, values and wisdom. What aspects of the vast fund of human knowledge are to be selected for transaction and which methods are to be used for this transmission? These questions are very important. According to *Nyaya*, one of the nine Schools of Thought in Indian Philosophy, "Valid methods of knowledge include perception, inference, comparison and memory." Now, let us explore and discuss the meaning of knowledge in detail.

The word 'Knowledge' is derived from the verb 'to know'. It includes all which a person knows and believes to be true. According to the most widely accepted definition, knowledge is justified true belief. It is a kind of belief that is supported by the facts and truths. For example, the sun rises in the east is the knowledge or true belief which is supported by the fact, which is arrived at through daily observation by millions of people. To have a deeper understanding about knowledge let us analyze the following definitions:

- **Oxford Dictionary:** Knowledge means facts, information and skills acquired through experience or education. It is the theoretical or practical understanding of a subject.

- ***Bertrand Russel:*** "Knowledge is that which enlightens the human mind."

- ***William James:*** "Knowledge is another name for practical achievement and success."

- ***Joad:*** "Knowledge is an addition to our existing information and state of experiences."

- ***Socrates:*** "Knowledge is the highest virtue."

- ***Radhakrishnan:*** "Self-knowledge is inseparable from self-existence and it is the only true direct knowledge. It is obtained through all the three experiences i.e., cognition, conation and emotion."

- ***John Wieley:*** "Knowledge is a body of information, technique and experience that coalesces around a particular subject."

- ***Nancy M. Dixon:*** "Knowledge is defined as the meaningful links people make in their minds between information and its application in action in a specific setting."

In the light of above definitions we can say that, knowledge is the familiarity, awareness or understanding of someone or something such as facts, information, description or skills, which is acquired through experience or education by perceiving, discovering or learning.

Knowledge can be conceived as experience organized through language into patterns of thought or structures of concepts thus creating meaning which in turn helps us to understand the world we live in. It can also be conceived of as patterns of activity, or physical dexterity with thought, contributing to acting in the world, and the creating and making of things. Human beings over time have evolved many bodies of knowledge which include a repertoire of ways of thinking, of feeling and of doing things or constructing more knowledge.

Knowledge includes the beliefs about matters of facts, about things, objects, events or about relationships between facts and about principles, laws, theories related to nature and society. It includes the fact or condition of knowing which is gained through experience or education. Knowledge is the sum of human understanding of the world, be it physical, biological, social, mental or spiritual. In simple, but generalized way, knowledge is sum of human understanding of material and mental reality – given and constructed.

1.4.2 Nature of Knowledge

- **Knowledge is both Process and Product:** As a process, it refers to the method of coming to know the phenomenon. Knowledge as a product, is the result of knowing the process. Knowing happens through perception, reason and emotion.
- **Purpose of Knowledge:** The purpose of knowledge is different in different contexts.
- **Knowledge is Dynamic in Nature:** Knowledge keeps changing with the passage of time.
- **Knowledge is of Different Types:** Sources of knowledge are knower's senses and mind. Different sources of knowing construct different forms of understanding and different types of knowledge.
- Knowledge is subjective as well as objective in nature.
- Knowledge is a means to reach the truth.
- **Social Nature of Knowledge:** Knowledge is socially shared understanding as it is developed through collective efforts of people of society. It is acquired by individuals from their own experiences, as well as they build-up this knowledge by associating with other human beings. Therefore, the knowledge is acquired and built-up only in society and it is deep rooted in the social activities of humans.
- **Knowledge is Cumulative:** Knowledge is cumulative as it is preserved and transmitted from one generation to other. New innovations and facts are added with time. Knowledge grows through a process of not only adding to but also perfecting and rectifying the already existing body of knowledge.

1.4.3 Disciplines of Knowledge

Discipline means deep and detailed content knowledge of a particular academic area. Discipline can be defined as a term of learning that is structured in terms of a single type of knowledge. It has a set of concepts that are unique and distinct. Disciplines can be categorized into:

I. **Basic Disciplines** - Basic Disciplines have their own concepts which are unique and distinctive to that discipline only. Principal Basic Disciplines are Humanities, Social Sciences Mathematics (also called Formal Science), Natural and Physical Sciences, Applied Sciences.

II. **Applied Disciplines** - When knowledge of a basic discipline is used in other disciplines, they are called applied disciplines. For example, knowledge of science and technology is used in Engineering, so it is an applied discipline. Some other examples are: Agriculture; Architecture and Design; Business Studies & Accountancy; Education; etc.

Humanities: History, Languages and Literature, Law, Philosophy, Theology, Visual Arts, Performing Arts

Social Science: Anthropology, Archaeology, Economics, Geography, Political science, Psychology, Sociology, Social work

Natural Science: Biology, Chemistry, Physics, Earth Science, Space Science,

Formal Science: Computer Science, Mathematics

Applied Science: Agriculture; Architecture and Design; Business Studies & Accountancy; Education; Engineering and Technology; Environmental Studies and Forestry, Home Science& Human Ecology; Journalism, Media studies and Communication; Library and Information Science; Medicine and Health; Military Science; Public Administration&Public Policy; Social Work, Tourism; Transportation, etc.

1.4.3.1 Interdisciplinary Approach

Interdisciplinary Approach utilizes one discipline or several disciplines as a centre for organizing curriculum. For example, if we consider Economics, then knowledge of Mathematics and other branches of Social Science help to understand the key concepts of this discipline. This is an Interdisciplinary approach where one discipline is the principal organiser, related disciplines are serving as the support system aiding the principal organiser. Interdisciplinary approach is a mode of acquiring integrated knowledge from two or more disciplines, in order to have a better understanding and to solve a problem which would not have been possible through a single discipline.

1.4.3.2 Multidisciplinary Approach

In Multidisciplinary Approach, concepts are selected from various disciplines to create a new field of study. The new field results from intermingling the abstract concepts and is independent of the separate discipline from which it is formed. For example, in the area of Environmental Education, the knowledge of Biology, Geography, Physics, Chemistry and Education are used. There are other areas that are multidisciplinary like Home Economics, Sociology, Biology.

CHECK YOUR PROGRESS 1.4

I. Answer the following:

(i) Define Knowledge.
(ii) What are 'Basic Disciplines'?
(iii) What are 'Applied Disciplines'?
(iv) What is 'Interdisciplinary Approach?
(v) What is 'Multidisciplinary Approach?

II. Fill in the blanks:

(i) Knowledge is the highest …………..
(ii) When knowledge of basic discipline is used in other disiplines, they are called………………..
(iii) ……………………… is an example of multidisciplinarity.

RECAPITULATION POINTS

- Knowledge, in fact, is not, and cannot be the end all of education. Knowledge without understanding, without application in real life situations, is useless. Ultimately, the aim of education is to acquire wisdom. Wisdom is also closely related to what is called 'World View', which means a holistic understanding of our world, our universe, our environment. World View comes by correlating the knowledge from all subjects.

- There is a close relationship between values, wisdom and knowledge. Wisdom includes values and knowledge. Wisdom cannot be visualised without values. One cannot be called wise if he/she does not adhere to human values. Knowledge that does not lead to wisdom is valueless.

- There are three eternal universal human values: ***Satyam, Shivam Sundaram* (or Truth, Beauty and Goodness/Godliness)**. All human beings are in search of these three values, though what they mean can vary from place to place as well as time to time.

- The main branches of Philosophy are: Metaphysics / Ontology (Study of Reality); Epistemology (Theory of Knowledge/Methods of Knowledge); Logic (The most important method of Knowledge); Ethics (Study of Good and Bad, Right and Wrong); and Aesthetics (Study of Beauty)

- Metaphysics/Ontology, Epistemology and Logic search for *Satyam* or Truth; Ethics explores *Shivam* or Goodness; and Aesthetics studies the concept of *Sundaram* or Beauty.

- These three values – *Satyam, Shivam* and *Sundaram* are very closely linked to our lives.

- Another related wonderful concept given by Indian philosophy is **'*Vasudhaiva Kutumbakam*'**, meaning that the whole world, all human beings are one family. '*Vasudhaiva*' means the Mother Earth or the World, implying all human beings, possibly all living beings on Earth. '*Kutumbakam*' means family. Since all human beings are members of one family, we should have a loving attitude towards everyone, irrespective of religious, geographical, social or economic background.

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- Hinduism also includes *Moh* (Attachment to worldly objects and pleasures); and Christianity also includes Gluttony (habitual greed or excess in eating) and Sloth (reluctance to work or laziness).

- According to Gandhiji, signs of vice or immorality: Wealth without Work; Pleasure without Conscience; Science without Humanity; Knowledge without Character; Politics without Principle; Commerce without Morality; and Worship without Sacrifice.

- The ***Gita*** enumerates many virtues as *Daivi-Sampat* or divine qualities: Fearlessness; Purity of heart; Steadfastness and wisdom; *Vivek* (ability to discriminate between right and wrong); *Nishkam Karma* (desireless action or doing one's duty without caring for the result); Charity; Self-restraint; Sacrifice; Study of the scriptures; Austerity or simplicity; Straightforwardness; Harmlessness (tendency of not harming anyone); Truth; Absence of anger; Renunciation; Peacefulness; Absence of crookedness; Compassion to all living beings; Non-covetousness (not having, or showing, a desire to possess something belonging to someone else); Mildness (soft heartedness and sensitivity towards others); Modesty or humility; Absence of fickleness (changeability, especially with regard to one's loyalties or affections); Vigour; Forgiveness; Purity; Absence of jealousy, vanity and arrogance.

- Four fundamental virtues are: Non-violence, Truth, Purity, and Self-control.

- Right actions bring us happiness, peace and well-being. So, it is not surprising that all the major religions talk of virtues. For instance, in Christianity, there are seven virtues: Faith, Hope, Charity, Fortitude, Justice, Prudence, and Temperance. Every religion speaks of virtues, because every religion is a living entity and believes in the happiness and well being of all.

- Beauty is an emotional element, a pleasure of ours, which nevertheless we regard as a quality of things. The idea of beauty is found in almost every culture and at almost every time in human history, with many similarities. Beauty was, and still is, a term of great esteem linking human beings and nature with artistic practices and works since the early civilizations.

- From the early cultures, beauty, goodness and truth are customarily related.

- Beauty carries a double meaning. It is inclusive and exclusive. In the inclusive sense, beauty pertains to anything worthy of approbation, to human virtues and characters, to nobility and goodness, to hidden things and truth, to the natural and divine worlds. In the exclusive, restricted sense, it pertains to how things appear, their manifestations, and to the joys human beings experience when presented with beautiful things, human bodies, artefacts, natural creatures and things. The nature of beauty and its role in philosophy and aesthetics was explained right from the early periods.

- From ancient times, India has given a great importance to acquisition of knowledge and it has a vast fund of this knowledge in the form of intellectual texts - the world's largest collection of manuscripts, scriptures and thinkers and schools in so many domains of knowledge.

- *Shree Krishna* in *Bhagwad Gita* tells Arjun that knowledge is the greatest purifier and liberator of the self.

- Various terms are closely related to pursuit of knowledge like *Gyan, Pragya, Shiksha*, *Vidya, Kala* and *Darshan.*

- The root of the great reverence for all knowledge that is *Gyan*, wisdom that is *Pragya*, discipline that is *Vidya*, education that is *Shiksha*, craft that is *Kala* and philosophy that is *Darshan* is attached to the *Guru* in the Indian tradition as he/she is considered as the ultimate authority and source of all knowledge.

- In the Indian context, Knowledge in different domains has been categorised into many disciplines that is *Vidya* and Arts that is *Kala*. Indian tradition talks of 18 major *Vidyas* that is theoretical disciplines and 64 *Kalas* which are applied or vocational disciplines, that is Art and Craft.

- Knowledge and its transmission is one of the key elements of education, apart from and only second to values and wisdom. What aspects of the vast fund of human knowledge are to be selected for transaction and which methods are to be used for this transmission? These questions are very important. According to *Nyaya*, one of the nine Schools of Thought in Indian Philosophy, "Valid methods of knowledge include perception, inference, comparison and memory."

- Knowledge can be conceived as experience organized through language into patterns of thought or structures of concepts thus creating meaning which in turn helps us to understand the world we live in. It can also be conceived of as patterns of activity, or physical dexterity with thought, contributing to acting in the world, and the creating and making of things. Human beings over time have evolved many bodies of knowledge which include a repertoire of ways of thinking, of feeling and of doing things or constructing more knowledge.

- Basic Disciplines have their own concepts which are unique and distinctive to that discipline only. Principal Basic Disciplines are Humanities, Social Sciences Mathematics (also called Formal Science), Natural and Physical Sciences, Applied Sciences.
- When knowledge of a basic discipline is used in other disciplines, they are called applied disciplines. For example, knowledge of science and technology is used in Engineering, so it is an applied discipline. Some other examples are: Agriculture; Architecture and Design; Business Studies & Accountancy; Education; etc.
- Interdisciplinary Approach utilizes one dicipline or several disciplines as a centre for organizing curriculum.
- In Multidisciplinary Approach, concepts are selected from various disciplines to create a new field of study. The new field results from intermingling the abstract concepts and is independent of the separate discipline from which it is formed.

TERMINAL EXERCISE

I. Answer the following questions:

(a) Write a detailed note on ‘Ultimate aim of Education.

(b) Explain how Values are related to Wisdom, Knowledge and Education.

(c) Write an essay on ‘*Satyam*’.

(d) Write an essay on ‘*Shivam*’.

(e) Write an essay on ‘*Sundaram*’.

(f) Write an essay on ‘*VasudhaivaKutumbakam*’.

(g) Write an essay on ‘True Secularism’.

(h) Write an essay on the Indian concept of Knowledge, explaining related terms like *Gyan, Pragya, Shiksha, Vidya, Kala* and *Darshan*.

(i) Write an essay on ‘Meaning of Knowledge’.

(j) Write an essay on ‘Nature of Knowledge’.

(k) Write an essay on ‘Disciplines of Knowledge’.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 1.1

I.

(i) To acquire wisdom

(ii) Page 1, 1: Paragraph 5

(iii) Page 1, 1:Paragraph 4

(iv) *Satyam, Shivam, Sundaram* (Truth, Beauty and Goodness/Godliness)

(v) Page 2 - Metaphysics / Ontology…..Aesthetic (Study of Beauty)

(vi) The whole world, all human beings are one family

(vii) *Brahman, Ishwar*, God, *Bhagwan, Allah, Khuda*

(viii) Page 3, Paragraph 1

(ix) Page 3, Paragraph 1

(x) Page 3, 3.1: Paragraph 4

II.

(i) Indian philosophy and Western Philosophy

(ii) Epistemology

(iii) Study of values

CHECK YOUR PROGRESS 1.2

I.

(i) Goodness/Godliness

(ii) Page 5, 3.2.1: Paragraph 1

(iii) Page 5, 3.2.1: Paragraph 3

(iv) Page 5, 3.2.2: Paragraph 1

(v) Non-Violence, Truth, Purity & Self-Control

(vi) Page 6, 3.3: Paragraph 1

CHECK YOUR PROGRESS 1.3

I.

(i) Page 7, 4: Paragraph 3

(ii) Page 7, 4: Paragraph 4

(iii) Page 7, 4: Paragraph 5

(iv) Page 8, 4: Paragraph 1

(v) Page 8, 4: Paragraph 2

II.

(i) *Pragya*

(ii) Point of view which leads to knowledge

(iii) 18, 64

(iv) *Pingla*

CHECK YOUR PROGRESS 1.4

I.

(i) Page 8, 5.1: Paragraph 1

(ii) virtue

(iii) Page 10, 5.3 I

(iv) Page 10, 5.3 I

(v) Page 11, 5.3.1, Paragraph 1

II.

(i) Page 22, 5.3.2, Paragraph 1

(ii) Applied Discipline

(iii) Environmental Education/Home Economics/Sociology/Biology

SUGGESTIVE READING LIST

- *Bhagavad Gita* as It is - Pocket Size by Bhaktivedanta Swami Prabhupada (Author)
- https://www.youtube.com/playlist?list=PLETbXIpqYH_qEOkBsrztDD4arUW1cGrm6
- http://www.bhagavatgita.ru/files/Bhagavad-gita_As_It_Is.pdf
- https://vedpuran.files.wordpress.com/2012/03/unencrypted-geeta.pdf

2 KNOWLEDGE AND WISDOM IN MATHEMATICS

INTRODUCTION

In the previous lesson, i.e.,Lesson1, we learnt that the ultimate aim of education and studying any subject is to develop wisdom, which includes knowledge as well as values. This, of course holds true for all subjects. In this lesson, we are concerned with the subject of Mathematics; what construes knowledge and wisdom in Mathematics. A true mathematician is not a juggler of numbers,but a juggler of concepts. According to Ian Stewart, "Mathematics is about ideas; about how certain facts follow inevitably from others; about how certain structures automatically implythe occurrence of particular phenomena.It lets us build mental models of the world, andmanipulate those models in ways that would be impossible in a real experiment."

Where did Mathematics originate from? All *Bharatiyas*, that is Indians, should be proud that it was in ancient Bharat that Mathematics originated. Our sages and rishis gave the knowledge and wisdom of Mathematics to the world. Some famous Mathematicians of ancient Bharat were *Baudhayan* (9^{th} century BC), *Pingala* (6^{th} century BC), *Katayana* (4th century BC), *Aryabhatta* (6^{th} century AD),*Varahamihira* (6^{th} century AD), *Brahmagupta* (7^{th} century AD), *Bhaskara I* (7^{th} century AD), etc.We will read in detail about the origin and evolution of Mathematics in the next lesson. Now, let us try to understand what is Mathematics is all about in terms of knowledge and wisdom.

2.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Explain the concept of Mathematics and its method;
- Distinguish between Pure and Applied Mathematics
- List the various branches of Mathematics and their applications;
- Discuss the contribution of Mathematics in other disciplines;
- Discuss the scope of Mathematics; and

- Draw the relationship between Sustainable Development and Mathematics.

2.2 CONCEPT OF MATHEMATICS

What is Mathematics?In simple terms,Mathematics is the study of topics such as quantity (numbers), structure, space and change. Some other definitions of Mathematics are:

- Mathematics is "the abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics)".
- Mathematics is the science that deals with the logic of shape, quantity and arrangement. It's all around us.
- **Mathematics**is the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter.
- Mathematics is at the heart of science and our daily lives.

2.2.1 Mathematics and Logic

As seen from the definitions above, Mathematics is closely connected with logic and reasoning. All disciplines are based on logic and reasoning. What is so special about Mathematics in this regard? Well! Mathematics, in a way, is a logical system on its own, which other disciplines are not, not even Science. Let us go deeper into it.

Logic is of two kinds –Deductive and Inductive. What is the difference between these two kinds of logic? Deductive reasoning uses facts and theories to reach a conclusion. In inductive reasoning, the conclusion is used to make generalizations of facts and theories.

In Deductive logic, we start with a premise or a set of premises, and then arrive at a conclusion (or prove a hypothesis), using logic.If the premises are true, then the conclusion is also true. When we look at Mathematics, we can see that mathematical knowledge is based on a set of premises, also called axioms or postulates, from where we start and arrive at conclusions or prove what we want to prove. Thus, Mathematics is not only based on deductive logic, but is itself a system of deductive logic.

On the other hand, inductive logic arrives at generalisation from repeated examples. A classic example of inductive logic is the generalisation that the sun rises from the east. We have reached this generalisation on the basis of our observation that the sun rises from the east every day, for centuries and millenniums. All scientific knowledge, which we call empirical knowledge, is based on inductive logic, and so observation and arriving at generalisations (or rules or laws) from the observed examples. This also implies that all scientific knowledge is based on probability. In the example of the sun rising from the east, where the observations are hundred per cent favourable to this generalisation. However, when scientists base their generalisations on observations, they will generalise on the basis of what happens more often than not. Thus,scientific knowledge merely states that it is probable that a particular phenomenon will happen because it has happened most of the times up till now.

This also tells us the difference between Mathematics and Science.

2.2.2 Domain of Concern in Mathematics

Every academic discipline, including Mathematics, has

(i) some specific domain of human experience and enquiry, the problems from which form subject matter of its enquiry, and

(ii) a set of specific assets e.g., tools, techniques, approaches and methods etc. for solving problems from its domain of enquiry.

However, the major source of, or inspiration for, the development of any academic discipline **is the set and type of problemsin its domain of concern, that are felt or encountered by human beings.** For example, for the academic discipline of Physics, its domain of concern is the physical universe of our experience, including matter andenergy, motions of these, interactions between these, and forces acting on these. The discipline of Physics is about dealing with various issues and problems of the physical aspects of the universe; and then also about developing tools, techniques etc. for solving these problems.

For Mathematics, the **domain** of concernis that which is based on our numerical and spatial (related to space) aspects of nature, or universe. And hence, its subject matter is about (solving of) **problems concerning these aspects**. Further these problems **inspire** us to develop the intellectual and other assets e. g. tools, approaches, methods etc. used for solving these problems.

2.2.2.1 The Natural Sources of Inspiration for Mathematics

Problems regarding numbers, shapes, arrangements, movements, and chance have been major sources of inspiration for Mathematics. In the very beginning of the human awakening, the **number** concept must have arisen through **counting** of one's herd of animals, of counting of children, counting of number of days, and later counting of other possessions.

Measurement of lengths and weights led to fractions and the 'real' numbers. Sense of **shape or form** of things encountered led to **Geometry**. In the past, though not much now, **music** had been a major source of inspiration for mathematics. How different musical notes can be rearranged to get different type or quality of music led to **theory of arrangements**, orwhat is now called **combinatorics,** or combinatorial mathematics. Motion of heavenly bodies like planets and stars etc. led to the development of the discipline of **Astronomy**, and then of **Calculus**and**Trigonometry**. Later, systematic studyof **chance,** e. g., as a result of tossing a coin a number of times etc., led to the disciplines of **Probability** and**Statistics. Combinatorics has been** playing significant role in these two disciplines.

Next part of an academic pursuit or discipline is to develop assets including intellectual assets, to solve these problems that form a significant part of subject matter of, in our case, Mathematics. In order to solve problems from any domain of experience, human beings have to define problems and also define tools or methods, e. g. method of solving problems. For the definitionsin the discipline, some fundamental concepts form the basis. One of the fundamental intellectual assets of human beings is concept formation.

2.2.2.2 Some Basic Concepts in Mathematics

Thousands of years back, elementary ideas about, what we now call, Mathematics started with

(i) **counting numbers** like 1, 2,3, … and

(ii) **geometrical shapes** like line, triangle, square and circle etc. These concepts form a part of the set of **fundamental concepts** of Mathematics.

The concept of **set**, which is one of the fundamental concepts of Modern Mathematics, is of relatively recent origin. We, first of all, briefly talk here about the concept of **set** because it facilitates the understanding of other concepts including those of counting numbers and geometrical shapes. The concept of **set** will be discussed in more detail in a later Lesson.

Roughly, a set may be considered as a collection of things or entities, like names of animals, names of books, names of numbers, names of geometrical shapes, or any mixture of these. Any entity in a set is called its member, or its element. Further, in a set, no member is repeated more than once. Also, the entities must be distinctly identifiable. For example, as the drops of water in a glass cannot be distinctly identified, therefore drops of water in a glass **do not** form a set.

The particular reason, for introducing first of all the concept of set, is that the **notation** for writing a set is used frequently in explaining other concepts. For illustration, the **notation {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}** is used to denote the set of all decimal digits, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The notation has at the leftmost position a **left brace**, denoted by **{**, followed by one of the elements, say 0, then a comma and a blank-space, next another element say 1, followed by a comma and blank-space, and so on, till the last element is written, which is followed by the **right-brace**, denoted by **}**. Even if the order of 0, 1, 2 etc in the notation is changed, the set remains the same.

However, problem with the notation arises when the number of elements in a given set is either very large or infinite, because then it is quite difficult or even impossible to specify all the elements for using the notation. In such cases, if there is some pattern among the elements, then additionally **triple dots (…)**may be used. For example, the set of numbers, each of which is less than or equal to 10,000 and is also square of some integer, may be denoted by {0, 1, 4, 9, …, 100, 121, …, 400, …, 10000}, or even more briefly by {0, 1, 4, 9, …, 10000} etc.

Another way of denoting a set, say A, is of the form **A= {*x*: *x* has property p}**, where p is the property of all members of A, and of no other elements. The expression **A= {*x*: *x* has property *p*}** is read as **'A is the set of all *x*'s, where *x* satisfies the property *p*.'**

Apart from 'Set', the fundamental concepts of Mathematics include

1. **Number, & Number System** involving relations like '<', '>' and operations like '+' and '*' etc.
2. **Geometric objects** like points, lines, angles, areas etc.,

3. **Matrices**

4. **Set, relations, operations & structures** on sets like Group, ring, field etc.

Some general techniques of solving mathematical problems are:

1. Direct proof & counter examples
2. Proof by Mathematical Induction
3. Proof by cases
4. Existence proof
5. Proof by contradiction and Proof by contrapositive
6. Proof of equivalences

The subject matter of whole of Mathematics is based on these concepts and these techniques.

CHECK YOUR PROGRESS 2.1

I. Answer the following:

(i) Mathematicians are Jugglers of __________. (Numbers or Ideas)

(ii) What is the ultimate aim of education and studying any subject?

(iii) Where did Mathematics originate from?

(iv) Name five famous Mathematicians of ancient Bharat, along with the century they lived in.

(v) Give a simple definition of Mathematics.

(vi) What are the two kinds ofLogic?

(vii) What is 'Empirical Knowledge' is based on?

(viii) Mathematics is based on which kind of logic?

(ix) What is the major source of, or inspiration for, the development of any academic discipline?

(x) What is the domain of concern for the academic discipline of Physics?

(xi) What is thedomainof concern for Mathematics?

(xii) What are the fundamental concepts of Mathematics?

(xiii) List 5 general techniques of solving mathematical problems are.

2.2.3 Pure and Applied Mathematics

Pure Mathematics is the study of mathematical concepts independently of any applications outside mathematics.

Applied mathematics is that part of mathematics, which is developed & used for the sake of applications outside Mathematics, particularly for solving problems arising out in physical space, social space, and now in cyber space.

The following are some of the branches which are clearly from Applied Mathematics:

- Statics
- Dynamics
- Hydrodynamics
- Fluid Dynamics
- Differential and Integral Calculus
- Differential Equations
- Statistics
- Probability Theory
- Numerical Methods
- Operational Research
- Game Theory

However, if we go just by the names of branches, no branch, except possibly mentioned above, of Mathematics is either purely Pure Mathematics, or purely Applied Mathematics. For example, since ancient times, Number theory has been considered as a branch of pure Mathematics. The well-known prime factorization theorem: that every natural number can be expressed uniquely, apart from the order of primes, as a product of powers of primes, has always been considered concern of pure mathematics. However, now since the advent of computers, number theory has many major application areas, particularly, in cryptography and computer security. In the application of number theory for computer security, the following result of pure mathematics is used: It is quite easy to find product of two very large primes. But, if we are given a number say n which is product of two very large primes p_1 and p_2, which are not known. Then finding p_1 and p_2 from the given number n is quite difficult. Thus, number theory becomes applied.

In order to be still more clear about this type of distinction between pure and applied mathematics, let us consider some more examples. The idea of complex numbers was given by Italian mathematician Cardano in 1545 in context of solving polynomial equations of the type $x^2+1=0$, for which no real roots exist. After that, for centuries, even mathematicians were not convinced about accepting a complex number as a mathematical entity. Then theory of complex numbers was developed, which remained purely a concern of pure mathematics. For centuries, it remained a part of pure mathematics only. But since at least the previous century, the theory is very much being applied in a number of fields including electromagnetism, signal processing, quantum mechanics, control theory, and cartography.

Another well-known example is that of Calculus—developed by Newton and Leibnitz, during the second half of 17th century—for applications to physics, i.e., to the study of phenomenon of motion of physical bodies, and in general the phenomena of continuous change. However, due to many problems arising out of fundamental assumptions of calculus, the discipline of Mathematical Analysis developed, and is a branch of pure mathematics.

Finally, even the problem of fifth/parallel postulate discussed next has always been considered as a problem of (purest of) pure mathematics. The Fifth postulate states: In a plane, through a point not on a given straight line, at most one line can be drawn that never meets the given line. In this respect, it may be mentioned that Euclidean Geometry is based on five postulates (Basic/fundamental fact) including the fifth mentioned above. The other four postulates are:

(i) A straight-line segment may be drawn from any given point to any other,

(ii) A straight line may be extended to any finite length,

(iii) A circle may be described with any given point as its centre and any distance as its radius, and

(iv) All right angles are congruent.

The problem regarding Fifth Postulate arose, because since almost from the beginning many thinkers considered that the so-called fifth postulate is not a postulate (i.e., it is not a fundamental fact), but a theorem which can be proved by using the other four postulates.

However, when the problem was resolved around 1830 (by Lobachevsky, a Russian mathematician and Bolyai, a Hungarian mathematician), after its having remained unresolved for thousands of years (for at least two thousand years), the resolution led to the birth of Non-Euclidean Geometries. And one of these non-Euclidean geometries, Riemann Geometry is applied in establishing the Relativity Theory of Einstein.

Similarly, both Matrix Theory and Group Theory are considered as branches of Pure Mathematics, but have significant applications to Physics.

Also, a branch of Mathematics, which is at present purely theoretical, having no applications at all—as was the case for long time for Number theory—may later find applications. Hence, in general, no branch of Mathematics may be called only pure mathematics or purely applied mathematics.

Actual difference between pure and applied mathematics lies in how parts of a branch are treated and developed. Pure Mathematics is that part of Mathematics, which deals with theories like Number theory, Euclidean Geometry etc. A theory is built upon (i) axioms or postulates like five postulates of Euclidean Geometry, or five Peano's axioms for Natural numbers. These axioms relate some basic or undefined terms like point or straight line in Euclidean Geometry, and, in number theory, two basic concepts are (a) first Natural Number & (b) successor of a natural number; and (ii)rules of inference like Deduction Rule, which states that if we assume (a) P is true & (b) P→ Q is true, then Q must be true.

On these foundations, the rest of theory is developed in terms of theorems, etc. For example, from the Number theory based on Peano's axioms, concepts of addition, multiplication, division and prime number etc. are defined. Then using these concepts, is proved the Prime Number Theorem: Every Natural number can be expressed uniquely, except for order of occurrence, as a product of prime powers.

Many branches of Mathematics have this type of theoretical component, the totality of which constitutes Pure Mathematics.

On the other hand, there is a large part of Mathematics which is applied to solve problems from other disciplines, including to Mathematics itself. For example, Calculus was developed, by Newton & Leibnitz, initially for applications to solve problems relating to motion of natural objects. Here, proving new theorems is not the concern. Thus, calculus may be considered as part of applied mathematics.

On the basis of the above explanation, some of the branches may be categorized as pure mathematics in the sense that dominant part of it is theoretical. Similarly, some other branches may be called applied.

Applied mathematics is the mathematics which is applied to solve problems from various domains of human experience. It has already been applied to model phenomena from various domains of human experience, viz. natural, biological and social etc. These models form subject matter of various disciplines like physics, engineering, business, computer science, social sciences. However, day by day Mathematics is finding applications to almost every domain of human experience and of enquiry.

Rest of the mathematics may be treated as both pure and applied. It is the use of the branch which determines its nature as 'Pure' or as 'Applied'.

2.3 VARIOUS BRANCHES OF MATHEMATICS AND THEIR APPLICATION

Earlier in this Lesson, we had learnt that problems regarding **numbers, shapes, arrangements, movements, and chance** have been major sources of inspiration for Mathematics.These five domains continue to motivate mathematicians to this day, of course, along with some other approaches for the development of Mathematics and its new branches, e. g., abstraction, generalization and mathematical requirement of other disciplines made significant contribution to the continued development of Mathematics. Let us look at the branches of Mathematics,based on these domains.

2.3.1 Numbers

The idea of **'Number'** led to development of various number systems; (conventional) algebra including matrices, determinants etc.; the concept of set and then of Modern Algebra, including group theory, ring theory, field theory, vector space.

Since the advent of computer revolution, along with the idea of arrangement,**Discrete Mathematics** hasbecome an active field of research and includes the sub-disciplines: Theoretical Computer Science, Information Theory, Logic, Combinatorics, Graph Theory, Probability, Game Theory, Decision Theory, Utility Theory, Social Choice Theory, Calculus of Finite Differences, Discrete Calculus or Discrete Analysis, and Operations Research.

All of these may be called sub-disciplines of **Discrete Mathematics**, latest sub-discipline of which is the very important one, viz. **Theoretical Computer Science (TCS),** which is the foundational part of the current computer, communication and ICT revolution.It may be pointed out, in passing, that all computer programs and even computer machines are essentially modelled by mathematical objects. The most well-known discipline **Artificial Intelligence (AI)** is also an application domain of Discrete Mathematics.

2.3.2 Shape

The idea of **'shape',** led to the development of Euclidean Geometry, and then from 19th century onwards gave birth to innumerable geometries or geometry-like disciplines including Riemannian geometry—a non-Euclidean geometry—which forms the basis of General Relativity Theory. **Other geometries** include algebraic geometry, differential geometry, computational geometry, discrete geometry, projective geometry, affine geometry, & algebraic topology, etc.

2.3.3 Motion

The idea of**'movements'**or **'motion'**first led to the development of Calculus, and then of various analyses including Real analysis, complex analysis, Vector analysis, Tensor analysis, Metric spaces, Functional analysis, Measure theory, Differential equations, Numerical analysis.

2.3.4 Chance

The idea of**'chance'**led first to thestudy of Probability Theory and Statistics. Later, it led to the development of Mathematics of Imperfect Knowledge, where knowledge is said to be imperfect if it is random, uncertain, incomplete, vague, imprecise, ambiguous, inconsistent and subjective. The mathematical disciplines for handling **imperfection of knowledge** include Numerical analysis, Probability Theory, Fuzzy Set Theory; Rough Set Theory, Chaos Theory etc.

CHECK YOUR PROGRESS 2.2

I. Answer the following:

(i) Define Pure Mathematics.

(ii) Define Applied Mathematics.

(iii) List 5 branches of Mathematics, which are clearly from applied mathematics.

(iv) Which Geometry is applied in establishing the Relativity Theory of Einstein?

(v) Whichfield is the foundational part of the current computer, communication and ICT revolution?

(vi) Which is the latest sub-discipline of Discrete Mathematics?

(vii) Artificial Intelligence (AI) is an application domain of which Branch of Mathematics?

(viii) Calculus was an outcome of which idea?

(ix) What is knowledge in the 'Mathematics of Imperfect Knowledge'?

(x) Name 5 mathematical disciplines for handling imperfection of knowledge.

2.4 CONTRIBUTION OF MATHEMATICS IN OTHER DISCIPLINES

In the previous section 'Branches of Mathematics andTheir Applications', we have already enumerated so many contributions of Mathematics to other academic disciplines.First of all, let us recall some of the characteristics of Mathematics, viz. abstraction, generalization, use of standard notations, use of deductive/ logical reasoning, and brevity of expression. Because of these, it has been found an appropriate discipline for modelling various aspects and phenomena, which are subject matter of other academic disciplines. Even specialized branches in most academic disciplines concerning Mathematics have been developed, e.g., Mathematical Physics, Mathematical Chemistry, Mathematical Biology, Mathematical Sociology, Mathematical Political Science. The natural sciences mostly use analysis, calculus, differential equations, statistics, numerical methods, etc. The Social Sciences use dominantly statistics and numerical methods.

Some fields of knowledge where Mathematics has contributed immensely are:

Mechanics: Mechanics is a branch of Physics consisting of Statics, Dynamics and Hydrodynamics, etc. It is purely mathematical, and before the advent of Computer Science, it was taught in Mathematics Departments of universities.

Engineering: Engineering, by definition, is use of scientific principles to design and build machines and structures of practical use. In modern times, in addition to scientific principles, Mathematics has become an essential basis of all engineering disciplines.

Apart from the conventional applications of Mathematics, computational methods, of course based on mathematics, are increasingly used in almost all other disciplines. Even special branches in various disciplines have developed, e. g. Computational Physics, ComputationalChemistry, Computational Biology, Computational Sociology etc.

Next, we give examples for applications of Mathematics to some specific disciplines.

- **Chemistry:**Maths is used in topics of Physical Chemistry, like Quantum or Statistical Mechanics, which uses Group Theory and Linear Algebra. Statistical Mechanics relies heavily on Probability Theory.
- **Biological Sciences:** In Biological Sciences,Biomathematics is used in Mathematical Genetics;Mathematical Ecology, for development of computer software for special biological and medical problems; and for developing mathematical theory of epidemics.
- **Mathematical Ecology:** In Mathematical Ecology, it is used in developing prey predator models and models where species in geographical space are considered.
- **Social sciences:** In Economics,Econometrics is almost mathematical in nature**.** In Business Studies,Financial Mathematics is an important sub-discipline.
- **Music:**Relation between Mathematics and Music is possibly the oldest one between any two disciplines. Rather, earlier in ancient times, music was considered as a source or basis of new mathematics.It is now used by music scholars to understand musical scales.Also, mathematical concepts/disciplines like the Golden Ratio, Fibonacci Numbers and Fourier Analysis are found quite useful in composing music.

2.5 SCOPE OF MATHEMATICS

By ‘scope’, we understand that it is in the sense of further development and expansion of the discipline of Mathematics and its applications to various walks of life and to other academic disciplines. In this respect, its scope is almost unlimited. Human imagination is the only limitation.

In this respect, let us see what Prof. Ian Stewart, British mathematician and a popularscience and science-fiction writer, has said about the Golden Age of Mathematics. According to him, "during the last 50 years more Mathematics has been produced than the Mathematics produced in all the previous ages combined together. There are about 1500 Mathematical journals publishing more than 25,000 articles annually.There were only 12 categories/branches of Mathematics in 1868, and now there are at least 80 categories/ branches. So, Mathematics itself is expanding and so are its application".

The current reason for expansion of the discipline of Mathematics and its applications is computer or ICT revolution, through which efficient solutions are being sought for problems from almost every domain of human experience. In this respect, we should remember the following facts:

(i) For a computer-based solution of a problem, first a computational model of the problem and its domain has to be developed.

(ii) For developing a computational model, a mathematical model has to be developed first, and

(iii) For developing a Mathematical model, the required Mathematics may not be available in the existing body of Mathematical knowledge, and hence, new Mathematics, required for the Mathematical model, has to be developed.

Hence, scope for development and expansion of Mathematics and its applications is almost unlimited.

CHECK YOUR PROGRESS 2.3

I. Answer the following:

(i) Give 5 characteristics of Mathematics.

(ii) Name 2 fields of knowledge where Mathematics has contributed immensely.

(iii) Give 2 examples of application of Mathematics in Social Sciences.

(iv) Give 3 examples of application of Mathematics in Biological Sciences.

(v) Give 3 mathematical concepts/disciplines that are found useful in composing music.

(vi) Who is Prof. Ian Stewart?

(vii) What has Prof. Ian Stewartsaid about the Golden Age of Mathematics?

(viii) What is the current reason for expansion of the discipline of Mathematics and its applications and why?

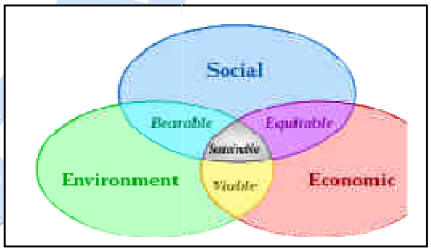
2.6 SUSTAINABLE DEVELOPMENT AND MATHEMATICS

Sustainable Development is highly crucial for the survival and continuance of life on Earth. The topic of sustainability is being discussed at many levels including in UNO, other international and national political, intellectual, and social platforms, and in various academic bodies and forums, etc. Since the submission of the report of UN constituted *World Commission on Environment and Development: Our Common Future (also known as Brundtland Report) in 1987, sustainability has acquired global attention, and even has become serious concern at global level.*

*The topic is quite vast;*we will restrict ourselves to some essential definitions, relevant issues, and to discussing briefly the role that Mathematics can play in achieving the goals of sustainability.

2.6.1 Concept ofSustainableDevelopment

According to *Brundtland Report, 's*ustainable development is the development which meets the needs of the present generation without compromising the ability of future generations to meet their own needs.'



One of the basic assumptions of the report is that life-sustaining natural resources are limited.However, these can be replenished, if used properly.

Sustainability does not have only 'environment' dimension, but has social and economic dimensions too, as shown in the figure above.

2.6.2 Why Sustainability Issue

All resources, especially life-sustaining resources, are limited. However, due to careless overuse, and misuse, of these resources, the air in the environment and waterin oceans, rivers and at ground-level, contain dangerously increasing levels of pollutants threatening life on Earth. Also, increasing pressure on land for habitation and use of fertilizer and disinfectants for increasing agricultural products, land for forests and open spaces etc. is decreasing, and fauna and flora are gradually vanishing. All these factors have become serious issues for continued life on earth. These issues have become serious concerns at various levels, and appropriate measures are being considered. For example, to achieve sustainability,17 goals were determined by Department of Economic and Social Affairs of United Nations.

2.6.3 How Sustainability Can Be Achieved

According to twi-global.com, in 2018, the EU (European Union) Commission released six key transformations to be made. If properly implemented, these steps will allow better sustainability to be achieved by 2050.

a) Sustainable development is a societal challenge, not simply an environmental one - improvements of education and healthcare are,therefore, required to achieve higher income and better environmental decisions

b) Responsible consumption and production, and the importance of doing more with fewer resources, are important to adopt a circular economy and reduce demand

c) Decarbonisation of the energy industry, through clean energy resources and renewable processes, will be necessary to provide clean and affordable energy for all

d) There should be food and clean water for all while protecting the biosphere and the oceans, which will require efficient and sustainable food systems, achievable through the increasing of agricultural productivity and reduction of meat consumption

e) Smart cities: Settlement patterns should be transformed for the good of the population and the environment, which may be done through 'smart' infrastructure and internet connectivity

f) A digital revolution in science, technology, and innovation would be required to support sustainable development, as it is hoped that the world will use the development of Information Technology to facilitate sustainability

2.6.4 Role of Mathematics in Sustainable Development

Mathematics is essentially required in various ways. Here, we discuss only two of these ways.

(i) Mathematics is involved in modelling complex phenomena like global weather and natural disasters occurring throughout the world. Through mathematical modelling, now highly reliable weather forecasting, disaster forecasting and management systems have been developed. In modelling of complex phenomena, quite advanced Mathematics is used. And many a time, the required Mathematics for the purpose does not exist, then the required Mathematics is developed. Thus, it also leads to development of new Mathematics.

(ii) Mathematics, especially Mathematics of Imperfect Knowledge (discussed earlier in this Lesson), is required for measuring progress of various countries in the direction of achieving the various goals, and in respect of following various steps both mentioned earlier.

CHECK YOUR PROGRESS 2.4

I. Answer the following:

(i) Which Report made sustainability acquire global attention, and even has become serious concern at global level?

(ii) How does the *Brundtland Report define 'Sustainable Development?*

(iii) What are the dimensions of Sustainability?

(iv) What is the basic assumption of the *Brundtland*Report?

(v) How many Goals have been determined by the Department of Economic and Social Affairs of United Nationsto achieve sustainability?

(vi) How many key transformations/stepswere released by EU Commission in 2018, which according to them, if properly implemented, will allow better sustainability to be achieved by 2050?

RECAPITULATION POINTS

- The ultimate aim of education and studying any subject is to develop wisdom, which includes knowledge as well as values. This, of course holds true for all subjects.

- Where did Mathematics originate from? All *Bharatiyas,* that is Indians, should be proud that it was in ancient Bharat that Mathematics originated. Our sages and rishis gave the knowledge and wisdom of Mathematics to the world. Some famous Mathematicians of ancient Bharat were *Baudhayan* (9^{th} century BC), *Pingala* (6^{th} century BC), *Katayana*(4th century BC), *Aryabhatta*(6^{th} century AD), *Varahamihira*(6^{th} century AD), *Brahmagupta* (7^{th} century AD), *Bhaskara I* (7^{th} century AD), etc. We will read in detail about the origin and evolution of Mathematics in the next Lesson. Now, let us try to understand what is Mathematics is all about in terms of knowledge and wisdom.

- A true mathematician is not a juggler of numbers, but a juggler of concepts. According to Ian Stewart, "Mathematics is about ideas; about how certain facts follow inevitably from others; about how certain structures automatically imply the occurrence of particular phenomena. It lets us build mental models of the world, and manipulate those models in ways that would be impossible in a real experiment."
- In simple terms,Mathematics is the study of topics such as quantity (numbers), structure, space and change.
- Mathematics is at the heart of science and our daily lives.
- Mathematics is closely connected with logic and reasoning. All disciplines are based on logic and reasoning. But, Mathematics, in a way, is a logical system on its own, which other disciplines are not, not even Science.
- Logic is of two kinds –Deductive and Inductive.
- In Deductive logic, we start with a premise or a set of premises, and then arrive at a conclusion (or prove a hypothesis), using logic. If the premises are true, then the conclusion is also true. When we look at Mathematics, we can see that mathematical knowledge is based on a set of premises, also called axioms or postulates, from where we start and arrive at conclusions or prove what we want to prove. Thus, Mathematics is not only based on deductive logic, but is itself a system of deductive logic.
- All scientific knowledge, which we call empirical knowledge, is based on inductive logic, and so observation and arriving at generalisations (or rules or laws) from the observed examples, through our 5 senses – seeing, hearing, smelling, tasting and sensing through touch. This also implies that all scientific knowledge is based on probability.
- For Mathematics, the domain of concern is that which is based on our numerical and spatial (related to space) aspects of nature, or universe. And hence, its subject matter is about (solving of) problems concerning these aspects. Further these problems inspire us to develop the intellectual and other assets e. g. tools, approaches, methods etc. used for solving these problems.
- The fundamental concepts of Mathematics includeNumber, & Number System involving relations like '<', '>' and operations like '+' and '*' etc.;

Geometric objects like points, lines, angles, areas etc.; Matrices; and Set, relations, operations & structures on sets like Group, ring, field etc.

- Some general techniques of solving mathematical problems are:Direct proof & counter examples; Proof by Mathematical Induction; Proof by cases; Existence proof; Proof by contradiction and Proof by contrapositive; Proof of equivalences.
- The subject matter of whole of Mathematics is based on these concepts and these techniques.
- **Pure Mathematics** is the study of mathematical concepts independently of any applications outside mathematics.
- **Applied mathematics** is that part of mathematics, which is developed & used for the sake of applications outside mathematics, particularly for solving problems arising out in physical space, social space, and now in cyber space.
- Sustainable Development is crucial for the survival and continuance of life on Earth. The topic of sustainability is being discussed at many levels including in UNO, other international and national political, intellectual, and social platforms, and in various academic bodies and forums, etc. Since the submission of the report of UN constituted*World Commission on Environment and Development: Our Common Future (also known as Brundtland Report) in 1987, sustainability has acquired global attention, and even has become serious concern at global level.*
- According to *Brundtland Report,* 'sustainable development is the development which meets the needs of the present generation without compromising the ability of future generations to meet their own needs.'
- Sustainability does not have only 'environment' dimension, but has social and economic dimensions too.
- Mathematics is involved in modelling complex phenomena like global weather and natural disasters occurring throughout the world. Through mathematical modelling, now highly reliable weather forecasting, disaster forecasting and management systems have been developed. In modelling of complex phenomena, quite advanced Mathematics is used. And many a time, the required Mathematics for the purpose does not exist, then the required Mathematics is developed. Thus, it also leads to development of new Mathematics.

- Mathematics, especially Mathematics of Imperfect Knowledge is required for measuring progress of various countries in the direction of achieving the various goals, and in respect of following various steps.

TERMINAL EXERCISE

I. Answer the following:

(i) Write a note on the 'Concept of Logic'.

(ii) Distinguish between 'Deductive' and 'Inductive' Logic.

(iii) Write a note on 'Mathematics and Logic'.

(iv) What is the difference between Mathematics and Science in terms of Logic?

(v) Write a note on 'Domain of Concern in Mathematics'.

(vi) Write a note on 'The Natural Sources of Inspiration for Mathematics'.

(vii) Write a note on 'Concepts and Techniques of Mathematics'.

(viii) Write an essay on 'Pure and Applied Mathematics'.

(ix) Write an essay on 'Various branches of Mathematics and Their Application'.

(x) Write an essay on 'Contribution of Mathematics in other Disciplines'.

(xi) Write an essay on 'Scope of Mathematics'.

(xii) Explain the concept of 'Sustainable Development'.

(xiii) Why do we need 'Sustainability'?

(xiv) How can sustainability be achieved according to EU Commission?

(xv) How does Mathematics help in sustainable development?

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 2.1

I.

(i) Ideas

(ii) The ultimate aim of education and studying any subject is to develop wisdom, which includes knowledge as well as values.

(iii) Ancient Bharat, that is India.

(iv) Any 5 out of:*Baudhayan* (9^{th} century BC), *Pingala* (6^{th} century BC), *Katayana* (4th century BC), *Aryabhatta* (6^{th} century AD), *Varahamihira* (6^{th} century AD), *Brahmagupta* (7^{th} century AD), *Bhaskara I* (7^{th} century AD)

(v) Mathematics is the study of topics such as quantity (numbers), structure, space and change.

(vi) Deductive and Inductive

(vii) Empirical knowledge is based on observation and arriving at generalisations (or rules or laws) from the observed examples, through our 5 senses – seeing, hearing, smelling, tasting and sensing through touch.

(viii) Deductive

(ix) The major source of, or inspiration for, the development of any academic discipline is the set and type of problems in its domain of concern, that are felt or encountered by human beings.

(x) The domain of concern for the academic discipline of Physics is the physical universe of our experience, including matter and energy, motions of these, interactions between these, and forces acting on these. The discipline of Physics is about dealing with various issues and problems of the physical aspects of the universe; and then also about developing tools, techniques etc. for solving these problems.

(xi) Thedomainof concern for Mathematics is that which is based on our numerical and spatial (related to space) aspects of nature, or universe. And hence, its subject matter is about (solving of) problems concerning these aspects. Further, these problems inspireus to develop the intellectual and other assets e. g. tools, approaches, methods etc. used for solving these problems.

(xii) a) Number, & Number System involving relations like '<', '>' and operations like '+' and '*' etc.; b) Geometric objects like points, lines, angles, areas etc.;c) Matrices; and d)Set, relations, operations & structures on sets like Group, ring, field etc.

(xiii) Any 5 out of: a) Direct proof & counter examples; b) Proof by Mathematical Induction; c) Proof by cases; d) Existence proof; e) Proof by contradiction and Proof by contrapositive; and f) Proof of equivalences

CHECK YOUR PROGRESS 2.2

II.

(i) Pure Mathematics is the study of mathematical concepts independently of any applications outside mathematics.

(ii) Applied mathematics is that part of mathematics, which is developed & used for the sake of applications outside Mathematics, particularly for solving problems arising out in physical space, social space, and now in cyber space.

(iii) Any 5 out of: a) Statics;b) Dynamics; c) Hydrodynamics; d) Fluid Dynamics; e) Differential and Integral Calculus;f) Differential equations;g) Statistics; h) Probability Theory;i) Numerical Methods;j) Operational Research; k) Game Theory.

(iv) Riemann Geometry is applied in establishing the Relativity Theory of Einstein.

(v) Theoretical Computer Science (TCS)

(vi) Theoretical Computer Science (TCS)

(vii) Discrete Mathematics

(viii) The idea of'movements' or 'motion'.

(ix) In 'Mathematics of Imperfect Knowledge', knowledge is said to be imperfect if it is random, uncertain, incomplete, vague, imprecise, ambiguous, inconsistent and subjective.

(x) The mathematical disciplines for handling imperfection of knowledge include a) Numerical analysis; b) Probability Theory; c) Fuzzy Set Theory; d) Rough Set Theory; and e) Chaos Theory

CHECK YOUR PROGRESS 2.3

I.

(i) a) Abstraction; b)Generalization; c)Use of standard notations; d)Use of deductive/ logical reasoning; and e) Brevity of expression.

(ii) Mechanics and Engineering

(iii) a) In Economics,Econometrics is almost mathematical in nature; and b)In Business Studies,Financial Mathematics is an important sub-discipline.

(iv) a) Biomathematics is used in Mathematical Genetics;b) Mathematical Ecology, for development of computer software for special biological; and c) medical problems; and for developing mathematical theory of epidemics.

(v) a) Golden Ratio; b) Fibonacci Numbers; and Fourier Analysis

(vi) A British mathematician and a popularscience and science-fiction writer

(vii) "During the last 50 years more Mathematics has been produced than the Mathematics produced in all the previous ages combined together. There are about 1500 Mathematical journals publishing more than 25,000 articles annually.There were only 12 categories/branches of Mathematics in 1868, and now there are at least 80 categories/branches. So, Mathematics itself is expanding and so are its application".

(viii) The current reason for expansion of the discipline of Mathematics and its applications is computer or ICT revolution, through which efficient solutions are being sought for problems from almost every domain of human experience.

CHECK YOUR PROGRESS 2.4

(i) Report of UN constituted *World Commission on Environment and Development: Our Common Future (also known as Brundtland Report) in 1987*

(ii) 'Sustainable development is the development which meets the needs of the present generation without compromising the ability of future generations to meet their own needs.'

(iii) Environment,Society and Economy

(iv) Life-sustaining natural resources are limited. However, these can be replenished, if used properly.

(v) 17 Goals

(vi) Six

5 INTRODUCTION TO *VEDIC* MATHEMATICS

INTRODUCTION

We have had a very rich tradition of Mathematics in India since the *Vedic* Period. Many Indian mathematicians have provided their valuable contributions to the development of Mathematics. *Vedic* Mathematics helps students to develop logical reasoning using *Sutras* and sub–*Sutras*. This subject is enjoyable and interesting for the students. Topics related to arithmetic, algebra, and geometry, trigonometry, differential and integral calculus have been made easy and fast through the *Sutras* and sub–*Sutras*. Therefore, to sort out the student's problems in Mathematics in the modern curriculum, there is a need of *Vedic* Mathematics *Sutras*. It contains also verification system using *Sutras* of *Vedic* Mathematics. Hence the creative sense of the learners increases very highly and so the interest of the learners also increase. It provides a different approach rather than the traditional methods. So, people like to study these methods. The dignitaries in India and abroad appreciate the *sutras* of *Vedic* Mathematics and hence it is propagated in many countries.

5.1 LEARNING OBJECTIVES

After completing the lesson, you will be able to:

- know the history of *Vedic* Mathematics,
- identify sutras and sub–sutras of *Vedic* Mathematics,
- add numbers fast,
- add and subtract the numbers from the left–hand side,
- obtain products fast using many methods,
- calculate square and square root,
- calculate the cube and cube root.

5.2HISTORY OF *VEDIC* MATHEMATICS

Shri Bharati Krishan Tirthji Maharaj interpret these sutras to the whole world in this regard. He was *Shankracharya* of *Goverdhan Math*, *Puri*. His famous book '*Vedic Mathematics*' is available with the new approach. He framed 16 *Sutras* and 13 *sub–sutras* in the Sanskrit language. Many scholars asked *Swami Ji* whether Mathematics exists in *Vedas*. Swami Ji concentrated on these questions from 1911 to 1918. Then he answered that which content you study in Mathematics that exists in *Vedas*. After that he wrote a book on *Vedic*Mathematics. This book contains 40 chapters using these *sutras* and *sub–sutras*. He solved different problems of Mathematics using sutras and sub–sutras. The approach to solve problems is quite different from the traditional methods. The methods of *Vedic* Mathematics are very easy and interesting. Hence people are interested and attracted to use it. These sutras give one–line answers directly. In *Vedic*Maths solutions to the problems are obtained easily and fast. Hence learning mathematics becomes interesting and inspiring. Its natural methods develop the person's mind automatically. The applications of *sutras* and *sub–sutras* develop interest, inspiration and self–confidence to learners. *Vedic* Mathematics sutras are also applied in cross–checking methods and this process enhance the confidence of the learners.

5.2.1*Sutras*of *Vedic*Mathematics

Sr.	**Name of *Sutras***	**Meaning of *Sutras***
1	*EkadhikenaPurvena*	By one more than the previous one.
2	*NikhilamNavatascaramamDasatah*	All from 9 and the last from 10.
3	*Urdhva–tiryagbhyam*	Vertically and Crosswise.
4	*ParavartyaYojayet*	Transpose and adjust.
5	*SunyamSamyasamuccaye*	When the sum is the same, that sum is zero.
6	*(Anursubye) Sunyamanyat*	If one is in ratio, the other is zero.
7	*Sankalana–vyavakalanabhyam*	By addition and by subtraction.
8	*Puranapuranabyham*	By the completion or non–completion.
9	*Calana–Kalanabhyam*	Differential Calculus
10	*Yavadunam*	Whatever the extent of its deficiency.

11	*Vyastisamasti*	Part and whole.
12	*SesanyankenaCaramena*	The remainders by the last digit.
13	*Sopantyadvayamantyam*	The ultimate and twice the penultimate.
14	*EkanyunenaPurvena*	By one less than the previous one
15	*Gunitasamuccayah*	The Product of the Sum
16	*Gunakasamuccayah*	All the Multipliers

5.2.2 *Subsutras* of *Vedic* Mathematics

Sr. No.	Name of *Subsutras*	Meaning of *Subsutras*
1	*Anursubyena*	Proportionately
2	*SisyateSesasamjnah*	The Remainder Remains Constant
3	*Adyamadyenantyamantyena*	First by First and Last by Last
4	*KevalaihSaptakamGunyat*	Only Multiples of Seven
5	*Vestanam*	Osculation
6	*YavadunamTavadunam*	Whatever the deficiency, lessen it further
7	*YavadunamTavadunikritya VargancaYojayet*	Lessen it further to that extent and set sub the square of deficiency.
8	*Antyayordasake'pi*	When the sum of last digits is ten.
9	*Antyayoreva*	Only the last term

10	*Samuccayagunitah*	Sum of the coefficients in the product
11	*Lopanasthapanabhyam*	By Elimination and Retention
12	*Vilokanam*	By observation
13	*Gunitasamuccayah* *Samuccayagunitah*	The product of the sum of coefficients inthe factor is equal to the sum ofCoefficients in the product.

5.2.3 Aims and Objectives of *Vedic* Mathematics

1. To increase interest in mathematics.
2. To increase the level of mathematical thinking.
3. To save the time of learners in solving mathematical problems.
4. To develop the logical thinking of learners.
5. To develop the self–confidence of a person to face the problems.
6. To increase the person interest towards the study of mathematical development.
7. To speed up the calculating speed and ability of the learners.
8. To enable the learner for saving time by increasing the speed of calculation.

5.3 *SHUNYANT* METHOD

5.3.1 Addition Using Shunyant Method

The process of addition using *ShunyantVedic* Mathematics *Sutras* is very interesting and easy. Using traditional methods, the solution in the questions of addition is obtained from the right–hand side. Using *Vedic* Mathematics *Sutras* and sub–*Sutras*, these problems can be solved from the left–hand side. This concept can be understood in the following illustrations:

Example 1: Solve 36 + 67 + 39 + 21 using *Sutras* of *Vedic* Mathematics

Solutions:

Step 1: 3 + 6 + 3 + 2 = 14

Step 2: Zero is written to the right side of the result in step 1 and add one's place digits of the given numbers140+6+7+9+1 = 163

Example 2: Solve 814 + 645+907+ 379 using *Vedic* Mathematics

Solution:

Step 1: 8 + 6 + 9 + 3 = 26

Step 2: Write zero to the right side of the result in step 1 and add ten's place digits of the given numbers260 +1+4+0+7 = 272

Step 3: Write zero to the right side of the result in step–2 and add one's place digits of the given numbers 2720 + 4 +5 + 7 + 9 = 2745

CHECK YOUR PROGRESS 5.1

I. Solve the following questions

(i) 56 +63+89+47

(ii) 96+36+73+81

(iii) 90+65+33+15

(iv) 856+956+375+908

(v) 569+ 425 + 873

(vi) 945+337+65

(vii) 8646+1279+6534+ 6123

(viii) 6687+6542+6853

5.3.2 Subtraction by *Shunyant*Method

The process of subtraction using concept of *Shunyant* in *Vedic* Mathematics *Sutras* is very interesting and easy. Using traditional methods of subtraction, the result is obtained from the right–hand side. But using *Vedic* Mathematics, it can be solved by left hand side. It can be understood in the following illustrations:

Example 3: Solve 856 – 289

Solution:

Step 1: 8 – 2 = 6

Step 2: Zero is written to the right side of the result in step 1 and ten's place digits of the given numbersare added and subtracted according the instructions in the question in the following way:60 + 5 – 8 = 57

Step 3: Zero is written to the right side of the result in step 2 and one's place digits of the given numbersare added and subtracted according the instructions in the question in the following way:570 + 6 – 9 = 567

CHECK YOUR PROGRESS 5.2

I. Solve the following questions

(i) 87 – 38

(ii) 63 – 26

(iii) 586 – 398

(iv) 645 – 186

(v) 758 – 292

(vi) 8172 – 3564

(vii) 8314 – 3478

(viii) 2873 – 1995

5.4MULTIPLICATIONS USING *SUTRA*

5.4.1 Multiplications Using *Sutra*–'One Less Than the Previous One'

Vedic Mathematics *Sutra*–'*EkanyunenaPurvena*'is very simple easy and interesting. The meaning of *EkanyunenaPurvena* is one less than the previous one. Using this sutra, the multiplier consists of entirely of nines digits number and the multiplicand may be any number. Let us take an example to obtain the products using this sutra:

Example 4: Solve 6405 × 9999

Solution: Using this sutra, following will be steps for solving this problem

Step 1: For obtaining left part of products, one is subtracted from other number which does not consist all 9's digits 6405 – 1= 6404

Step 2: Subtract result obtained in step1 from 9999

9999 – 6404 = 3595

Step 3: 64043595 Ans

CHECK YOUR PROGRESS 5.3

I. Solve the following questions

(i) 478 × 999

(ii) 485 × 999

(iii) 4787×9999

(iv) 8624 × 9999

(v) 785 × 999

(vi) 4781 × 9999

(vii) 4567×9999

(viii)8692 × 9999

5.4.2 Multiplication Using Sutra 'Vertically & Crosswise'

Vedic mathematics using sutra "*UrdhvaTiryagbhyam*" is general sutra. It can be applied to obtain the product of any numbers. The Sutra '*UrdhvaTiryagbhyam*' means vertically and cross–wise. Using this sutra, multiplication can be done from right hand side or left–hand side. Let us understand it in the following way:

5.4.3 Two–Digit Numbers Multiplication

The multiplication of 2 digit numbers is obtained in 3 steps as in the following steps using *Vedic* Mathematics *Sutra–Urdhva–tiryagbhyam*:

Example 5: *Solve:* 61 ×54

Solution:

	6	1		6	1		6	1
×	5	4	×	5	4	×	5	4
	6 × 5 = 30			6 × 4 + 1 × 5 = 29			1 × 4 = 4	

Step 1: Both the digits of 1st column from right hand side are multiplied i.e., 1×4=4.

Step 2: The addition of the cross multiplication is written and carry from 1st step is added to this number if any i.e., 6× 4 + 1 × 5 = 29.

Step 3: Third step is the vertical multiplication of ten's place digits i.e., 6 × 5= 30.

Step 4: In each step except extreme left, the result will be in single digits, if the results in other steps more than one digits, the remaining digits will be carried to the left result

Therefore, result is 30/ $_2$9/ 4 or 3294

5.4.4 Multiplication of Three Digits Numbers

The multiplication of three digits numbers is done in five steps as we have shown in the following diagram:

1 2 3 2 1

Example 6: Solve: 605 × 561

Solution:

Step 1:

6	0	5
5	6	1

5 × 1 = 5

Step 2:

6	0	5
5	6	1

6 × 5 + 1 × 0 = 30

Step 3:

6	0	5
5	6	1

6 × 1 + 5 × 5 + 0× 6 = 31

Step 4:

6	0	5
5	6	1

6 × 6 + 5 × 0 = 36

Step 5: 6 0 5

5 6 1

$6 \times 5 = 30$

Therefore, solution is 6 0 5

× 5 6 1

$30/\,_{3}6/\,_{3}1\,/\,_{3}0/5$ or 339405

Example7: Solve:3471 ×5122

Solution: 3 4 7 1

× 5 1 2 2

$15/\,_{2}3/\,_{4}5/\,_{2}6/\,_{2}3/\,_{1}6/\,2$

Or

17778462

Step 1: $1 \times 2 = 2$

Step 2: $7 \times 2 + 2 \times 1 = 16$

Step 3: $4 \times 2 + 1 \times 1 + 7 \times 2 = 23$

Step 4: $3 \times 2 + 5 \times 1 + 4 \times 2 + 1 \times 7 = 26$

Step 5: $3 \times 2 + 7 \times 5 + 4 \times 1 = 45$

Step 6: $3 \times 1 + 5 \times 4 = 23$

Step 7: $3 \times 5 = 15$

CHECK YOUR PROGRESS 5.4

I. Solve the following questions

(i)	43 × 27	(v)	403 × 732
(ii)	48 × 12	(vi)	663 × 481
(iii)	47× 64	(vii)	3127× 1275
(iv)	861 × 142	(viii)	1083 × 2578

5.4.5 Multiplication by *Sutra–Nikhilam*

There are two steps in this type of multiplication the complete name of the sutra is '*NikhilamNavatascaramamDasatah*which means "all from 9 and the last from 10" In this case, both the numbers multiplicand and multiplier are nearer to the base. The bases of numbers are taken as a certain power of ten.Using this sutra, numbers (multiplicand and multiplier) are taken above or below the base. The difference between number and the base is called deviation. The deviation may be positive or negative depending subon the greater or smaller numbers from the base respectively.

Example 8: Solve: 108 × 112

Solution:	**Number**	**Deviation**	
	108	+8	
	112	+12	Base = 100

(108 + 12)   (8 × 12)   or 12096

Step 1: Deviation of 108 is 8 and deviation of 112 is 12.

Step 2: Deviations are multiplied in the right part of the answer i.e., 8 × 12 = 96.

Step 3: Deviation is cross–added for the left part of the answer

108 + 12 = 120 or 112 + 8 = 120.

Note: The base is 100. So, the right part of the answer should be in 2 digit. The number of digits depends subon the number of zero in its base.

Example 9: Solve: 988 × 980

Solution: Numbers	**Deviations**	
9 8 8	– 1 2	
× 9 8 0	– 2 0	Base =1000

988 – 20 / (–12) × (–20)

= 9 6 8 2 4 0

CHECK YOUR PROGRESS 5.5

I. Solve the following using *Vedic* Mathematics

(i) 97 × 95

(ii) 112 × 106

(iii) 94 ×96

(iv) 107 × 103

(v) 992 × 975

(vi) 991 × 985

(vii) 1008×1030

(viii)995 × 981

5.4.6 Multiplication Using Sutra–1 'One More Than The Previous One'

Multiplication using *Sutra–EkadhikenaPurvena* is specialized multiplication. In this type of multiplication, we take such numbers in which whose one's place digits sum is 10, 100, 1000—etc. and digits in tens, hundreds and thousands places in the both numbers (multiplicand and multiplier) are same.

Example 10: Solve 74× 76

Solution:

Step1: 4 × 6 = 24,

Step2: 8 (*Ekadhikena* of 7)× 7 = 56

Step3: 5624

Example 11: Solve 207 × 203

Solution: **Step1:** $7 \times 3 = 21$, **Step2:** $21 \times 20 = 420$

Step 3: 42021 Ans.

CHECK YOUR PROGRESS 5.6

I. Solve the following questions using one less method

(i) 98×92

(ii) 63×67

(iii) 94×96

(iv) 107×103

(v) 992×998

(vi) 1991×1999

(vii) 5008×5002

(viii) 2995×2995

5.5SQUARE USING *SUTRA*

5.5.1 Square Using *Sutra*–'*Yavadunam*'

The term 'square' means the number is multiplied by itself. Square using sutra–*Yavadunam* is very easy and interesting. The students feel very confident to obtain the results of squaring through the *Vedic* Mathematics Sutra. Using this sutra of squaring, the problem is solved using base. The bases of the numbers may be 10,100, 1000, 10000, ———————. The working of this sutra is explained in the following examples;

Example 12: Find the value of 93^2

Solution:$93^2 = (93 - 7) \mid (7)^2$

$= 8649$

Step 1: Deficit of 93 from in 100 is $93 - 100 = -7$

Step 2: Decrease 93 by deficit 7, we get $93 - 7 = 86$

Step 3: Square of deficit we get is 49. Adjusting it on right side of 86 we get8649.

Precautions

- The number of digits in the right side in squaring should be according to the number of zeros in the base
- If the number of digits in the square of deviation is less than that of zeros of the base of the number, then zero is added for the compensation. If the number of digits in the square of deviation is more than the base, then extra digits are carried over to the left part of answer.

Example 13: Find the square of 985

Solution:$985^2 = (970+15) \mid 15^2$

= 970 | 225

= 970225

Example 14: Find the square of 1021

Solution: $1021^2 = (1021 + 21) \mid 21^2 = 1042441$

CHECK YOUR PROGRESS 5.7

I. Find the Square the following numbers using Sutra–Vinculum Method

(i) 98

(ii) 91

(iii) 994

(iv) 988

(v) 986

(vi) 991

(vii) 989

(viii) 980

(ix) 1015

(x) 10

5.5.2 Squaring Using Duplex Method

Duplex of a number is the sum of the double product of the equidistant digits from its both sides. If a digit is not paired from both sides and remains alone, its square is added to the sum of double product of the remaining. This concept is explained in the following way:

- Duplex of one digit D (5) = 5^2 = 25
- Duplex of two digits D (61) = 2 (6 × 1) = 2 (6) = 12
- Duplex of three digits D(431) = 2(1 × 4) + 3^2 = 8 + 9 = 17
- Duplex of four digits D(1086) = 2 (1 × 6) + 2 (0 × 8) = 12

5.5.3 Square of Numbers Using Duplex

While squaring of two digits numbers, first Duplex of one digit is being computed, then duplex of two digits are calculated and then one digit of number is computed as shown in the following example:

Example 15: Solve $(61)^2$

Solution $(61)^2$ =D (6) | D (61) | D (1)

= $6^2 \mid 2\,(6 \times 1) \mid (1)^2$

= 36 | $_{1}2$ | 1

= 3721

Example 16: Solve $(423)^2$

Solution: $(423)^2$ = D (4) | D (42) | D (423) | D (23) | D (3)

$= 4^2 \mid 2 (4 \times 2) \mid 2 (3 \times 4) + 2^2 \mid 2 (2 \times 3) \mid (3)^2$

$= 16 \mid {}_{1}6 \mid {}_{2}8 \mid {}_{1}2 \mid 9$

= 178929

CHECK YOUR PROGRESS 5.8

I. Find the Square the following questions using DuplexMethod

(i)	42	(v)	706
(ii)	91	(vi)	816
(iii)	82	(vii)	523
(iv)	54	(viii)	431

5.6 CUBE USING *SUTRAS*

5.6.1 Cube Using *Sutras–Nikhilam*Method

Cubing using *Nikhilam* method is very easy and interesting. In this method, we take the numbers near the bases. The bases are 10, 100, 1000, ——etc. This concept can be understood in the following examples:

Example17: Find the value of 105^3

Solution: Deviation =105–100= 5

Step 1: = Number+2(deviation) =105+2(5) = 115

Step 2 = $3(\text{deviation})^2$ = $3 (5)^2 = 75$

Step 3 = $(\text{deviation})^3$= $(5)^3 = 125$

$(105)^3 = 115 \mid 75 \mid {}_{1}25 = 1157625$

Note: Since base is100 in the above case. Hence the first and second part from the right side must be in two digits. If there are more digits in these steps, the remaining digits must be carried over to the left side.

Example18: Find the value of 1008^3

Solution:1008^3

$= 1008+2(8) \mid 3(8)^2 \mid (8)^3$

= 1024 | 192 | 512

= 1024192512

Example19: Find the value of 994^3

Solution: 994^3

= 994+2(–6) | 3(–6)² | (–6)³

= 982 | 108 | (–216) = 982108000 – 216 = 982107784

CHECK YOUR PROGRESS 5.9

I. Find the cube of the following

(i) 96
(ii) 102
(iii) 109
(iv) 996
(v) 112
(vi) 997
(vii) 114
(viii) 985

5.6.2 Cubing Using Sub–Base Method

If the given number is not near the base, then select the sub–base (multiple of base near to the numbers) and apply it according to the following steps:

Example20: Evaluate of 41^3

Solution: Sub–base 40= 4×10 (4×base), ratio = 4

Step1: Cube of deviation i.e. $(1)^3 = 1$

Step2: The product of the ratio and three times and the square of deviation is $4\times3\times(1)^2 = 12$

Step3: Square of ratio × (Number + twice the deviation) = $4^2\,(41+2(1)) = 688$

$41^3 = 4^2\,(41+2(1))\ |\ (4)\,(3)\,(1)^2\ |\ (1)^3$

= 688 | 12 | 1

= 68921

Example 21: Find the value of $(297)^3$

Solution: Here, Sub–base = 3×100=300, ratio =3, deficit =297–300= –3

∴ $(297)^3$

$= (3)^2\,(297+2(-3))\ |\ 3\,[3(-3)^2]\ |\ (-3)^3$

= 9 (291) | 81 | (–27)

= 2619 | 81 | (–27)

= 26198100–27 = 26198073

CHECK YOUR PROGRESS 5.10

I. Find the Cube of the following questions

(i) 26
(ii) 202

(iii) 304
(iv) 705
(v) 412
(vi) 497
(vii) 501
(viii) 585

5.7 SQUARE ROOT USING *SUTRA VILOKANAM*

Square root can be calculated using *Sutra* by *Vilokanam*. *Vilokanam* means by observation. The process is explained in the following way:

5.7.1Table of Computing One's Place Digit in Square Root of Perfect Squared Numbers

One's place digit in squared Number	1	4	6	9
One's place digit in its square root	1 or 9	2 or 8	4 or 6	3 or 7

Example 22: Find the square root of 9216

Solution:

Step 1: Unit digit of 9216 is 6. Hence the unit digit of its square root will be4 or 6.

Step 2: We have 92 in the second pair. The less number is chosen from the expression $9^2 < 92 < 10^2$ hence 9 is the greatest number whoseSquare is less than 92.

Step 3: Hence adjust a 4 and 6 on right side of 9, we get two number 94 and 96 among which one will be required square root.

Step4: Now unique number 95 ending with 5 as it must digit lying between two is 95 and its square is 95^2 = 9025. Hence $\sqrt{9216}$= 96

Example 23: Find the value of $\sqrt{25281}$

Solution:

Step1: Now as its unit digit is 1, the unit digit of its square root can be 1 or 9.

Step2: Now $15^2 > 252 < 16^2$ but the smaller number will be selected between15 and 16, so, we get 151 and 159 and one will be the square root of required number.

Step3: The unique number with unit digit 5 between these two numbers is 155 its square = (15×16)| 25 = 24025 is less than 225281.

Hence greater number 159 will be the square root of 25281.

CHECK YOUR PROGRESS 5.11

I. Calculate the square root of the following numbers:

(i) 2601 (v) 8836

(ii) 4489 (vi) 1156

(iii) 8281 (vii) 5776

(iv) 1089 (viii)7744

5.8CUBE ROOT OF A CUBE BY *VILOKANAM*

To obtain the cube root of a perfect cube number, the concept of prime factors is evaluated using traditional concept. The different approach is adopted to find the cube root of a perfect cube numbers by *Vedic* Mathematics *Sutra–Vilokanam*. Let us understand it using the following concept:

5.8.1 Table Determining One's Place Digit in Cube Root

One's place digit in cube number	1	2	3	4	5	6	7	8	9
One's place digit in cube root	1	8	7	4	5	6	3	2	9

It can be remembered easily as follows:

(1) If one's place digit of the given cube is 1,4,5,6 or 9, the unit digit in its cube root will be same.

(2) If one's place digit of the given cube is 2,3,7, or 8 then the unit digit in the cube root can be obtained by subtraction from 10

Example 24: Find cube root of 12167

Solution: Set this number in two groups of three digits starting from the righthand side i.e. 12 167

Step1: Its unit digit is 7, so the unit digit of given cube number is 10 –7=3.

Step2: For determining the ten's place, we get 12 and $8 < 12 < 27$ mean

$2^3 < 12 < 3^3$. Hence the smaller number will be selected between2 and 3.

Step3: Adjusting above unit digit 3in its right side, we get 23 whichis required cube root.

Example 25: Find cube root of 205379

Solution:

Step1: Unit digit of the cube is 9. So, the unit digit of its cube root willbe 9

Step2: Remaining part is 205 and$125 < 205 < 216$ means $5^3 < 205 < 6^3$ and thesmaller number will be selected between 5 and 6.

Step3: Adjusting above, unit digit on its right part, we get 59 which isthe required cube root.

CHECK YOUR PROGRESS 5.12

I. Calculate the cube root of the following numbers by using *Sutra–Vilokanam*:

(i) 226981 (v) 474552

(ii) 79507 (vi) 140608

(iii) 830584 (vii) 438976

(iv) 300763 (viii)54872

5.9 SQUARE & SQUARE ROOT OF ALGEBRAIC EXPRESSION

We have been introduced to calculate square in the arithmetic. The square of algebraic expressions is calculated using identities. Generally, use the identities is applied in two terms and three terms to obtain the square. But the question is that how to calculate the square of algebraic expressions if there are more than three terms. Using *Vedic* Mathematics, there is a sutra of squaring of such expression. Using *Vedic* Mathematics Sutra–Duplex, this concept can be understood in the following way:

Example 26: Solve the following using Duplex method: $(5x + 7)^2$

Solution: There are three steps of doing square of algebraic expression

Dsub (5x) + Dsub (5x&7) + Dsub (7) = $25x^2 + 70x + 49$

Example27: Solve the following using Duplex method: $(5x + 2y + 4z)^2$

Solution: We calculate the square of the above expression in the following way:

Dsub(5x)+Dsub(5x&2y)+Dsub(5x,2y&4z)+Dsub(2y&4z)+Dsub(4z)

$25x^2 + 20xy + 40xz + 4z^2 + 16yz + 16z^2$

Or

$25x^2 + 4y^2 + 16z^{2+}$ $20xy + 40xz + 16yz$

5.9.1 Square Root of Algebraic Expressions

In this, we will compute the square root of perfect squared algebraic expressions. We use Duplex also in finding the square root of perfect squared algebraic expressions. If there are two or three terms in the algebraic expression, there will two terms in the square root. If there are five or six terms in the perfect squared algebraic expressions, there will three terms in the square root of the algebraic expressions.

Example28: Find the square root of $4x^2+12xy+9y^2$

Solution: In this problem, we need not apply the identity to find the square root of the algebraic expression. There are two steps of finding the square root of the algebraic expression.

Step1: Square root of $4x^2 = 2x$

Step2: Twice the first term of the square root of the first term which is called divisor and divide the second term of the question by this squared term. $12xy \div 4x = 3y$. Hence square root of the algebraic expression is $2x+3y$.

Note: If we want to check whether it is perfect squared number or not, we do Duplex of the second term of the square root and subtract it from the third term. If it is zero, then it is perfect squared number. $9y^2–9y^2 = 0$.

4x	$4x^2+12xy+9y^2$
	$2x + 3y$

Example 29: Find the square root of $4x^4–12x^3+33x^2–36x+36$

Solution: In this problem there are five terms. There will three terms in square root of thisproblem.

Step1: The square root of $4x^4 = 2x^2$

Step2:$–12x^3 \div$ (twice of $2x^2 = 4x^2$ iscalled divisor) $= –3x$

Step3: Duplex of the second quotient is subtracted from the third term of thequestion and divided by the divisor$4x^2$ as $(33x^2 – 9x^2) \div 4x^2 = 6$

$$4x^4 –12x^3 + 33x^2 – 36x + 36$$

$$2x^2 – 3x + 6$$

Hence the square root of the required algebraic expression is $2x^2 – 3x + 6$

CHECK YOUR PROGRESS 5.13

1. Compute value of the following expression using Duplex method

(i) $(x+2y + 6z)^2$ (ii) $(4x –5y + z)^2$

(iii) $(5x + 2y - 4z)^2$ (v) $(3a + 6b - 2c)^2$

(iv) $(x + 6y - z)^2$ (vi) $(x^2 + 5x + 2)^2$

2. **Compute the square root of the following algebraic expressions**

(i) $16a^2+24ab+9c^2$ (iv) $144x^2+312xy+169y^2$

(ii) $25a^2+90ab+81c^2$ (v) $a^4 - 4a^3 + 14a^2 - 20a + 25$

(iii) $64x^2+176xy+121y^2$ (vi) $9x^4 - 12x^2y + 6x^2z + 4y^2 - 4yz + z^2$

TERMINAL EXERCISE

1. **Fill in the blanks of the following**

(i) The product of 51 × 63 is_______

(ii) The product of 75 × 34 is________

(iii) The square of 986 is __________

(iv) The square of 73 is __________

(v) The square root of 12544_______

(vi) The cube root of 12167_______

2.Tick True or False

(i) The product of 103 × 109 is11227 [True / False]

(ii) The square root of 992016 is 994 [True / False]

(iii) The cube root of 12167 is 33 [True / False]

(iv) The square of 986 is 982016 [True / False]

(v) The product of 43 × 92 is 3946 [True / False]

(vi) The square of 52 is 2704[True / False]

3. **Solve the following questions using *Sutra–EkadhikenaPurvena***

(i) 52 × 58 (v) 204× 206

(ii) 73 × 77 (vi) 994 × 996

(iii) 62× 68 (vii) 904× 906

(iv) 112 × 118 (viii)302 × 308

4. **Solve the following questions using *Urdhvatiryagbhyam***

(i) 26 × 74 (v) 2041× 4175

(ii) 62 × 34 (vi) 2274 × 8024

(iii) 712× 553 (vii) 5523× 4431

(iv) 903 × 127 (viii)5127 × 6007

5. **Solve the following questions using *Sutra Nikhilam***

(i) 105×108

(ii) 112×104

(iii) 94×98

(iv) 126×105

(v) 1013×1021

(vi) 1031×1020

(vii) 991×980

(viii) 970×995

ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 5.1

(i) 255

(ii) 286

(iii) 203

(iv) 3095

(v) 1867

(vi) 1347

(vii) 22582

(viii) 20082

CHECK YOUR PROGRESS 5.2

(i) 49

(ii) 37

(iii) 188

(iv) 459

(v) 4666

(vi) 4608

(vii) 4836

(viii) 878

CHECK YOUR PROGRESS 5.3

(i) 477522

(ii) 484515

(iii) 47865213

(iv) 86231376

(v) 784215

(vi) 47805219

(vii) 45665433

(viii) 86911308

CHECK YOUR PROGRESS 5.4

(i) 1161

(ii) 576

(iii) 3008

(iv) 122262

(v) 294996

(vi) 318903

(vii) 3986925

(viii) 2791974

CHECK YOUR PROGRESS 5.5

(i) 9215

(ii) 11872

(iii) 9024

(iv) 11021

(v) 967200
(vi) 976135
(vii) 1038
(viii) 86911308

CHECK YOUR PROGRESS 5.6

(i) 9016
(ii) 4221
(iii) 9024
(iv) 11021
(v) 10016
(vi) 3980009
(vii) 25050016
(viii) 8970025

CHECK YOUR PROGRESS 5.7

(i) 9604
(ii) 8281
(iii) 988036
(iv) 976144
(v) 972196
(vi) 982081
(vii) 978121
(viii) 960440
(ix) 1030225
(x) 103632

CHECK YOUR PROGRESS 5.8

(i) 1764
(ii) 8281
(iii) 6724
(iv) 2916
(v) 498436
(vi) 665856
(vii) 273529
(viii) 185761

CHECK YOUR PROGRESS 5.9

(i) 884736
(ii) 1061208
(iii) 1295029
(iv) 988047936
(v) 1404928
(vi) 991026973
(vii) 1481544
(viii) 955671625

CHECK YOUR PROGRESS 5.10

(i) 17576
(ii) 8242408
(iii) 28094464
(iv) 350402625
(v) 69934528
(vi) 122763473
(vii) 125751501
(viii) 200201625

CHECK YOUR PROGRESS 5.11

(i) 51
(ii) 67
(iii) 91
(iv) 33
(v) 94
(vi) 34
(vii) 76
(viii) 88

CHECK YOUR PROGRESS 5.12

(i) 61
(ii) 43
(iii) 94
(iv) 67
(v) 78
(vi) 52
(vii) 76
(viii) 38

CHECK YOUR PROGRESS 5.13

1.

(i) $x^2 + 4y^2 + 36z^2 + 4xy + 24yz + 12xz$

(ii) $16x^2 + 4y^2 + 16z^2 – 40xy – 20yz + 8xz$

(iii) $25x^2 + 49y^2 + z^2\ 48xy –16yz – 40xz$

(iv) $x^2 +36y^2 + z^2 + 12xy – 12yz – 2xz$

(v) $9a^2 + 36b^2 + 4c^2 +36ab –24bc –12ca$

(vi) $x^4 + 25x^2 + 4 + 10x^3 +20x + 4x^2$

2.

(i) $4a +3b$

(ii) $5ya + 9zb$

(iii) $8a + 11b$

(iv) $12x + 13y$

(v) $a^2 – 2a + 5$

(vi) $3x^2–2y + z$

6 NUMBER SYSTEM

INTRODUCTION

The creation of number system is the greatest inventions in the history of civilization. It is very difficult to count if a person does not know about the number system. Hence there is dire need to know about the number system. In this chapter we shall introduce the number system. From the excavation of Harappa and *Mohen-jodaro*, the evidence has uncovered the evidence of the use of number system in practical mathematics. The people of Indus valley civilization used the bricks of 4:2:1. They also know the values in fraction also. They also use regular and irregular geometrical shapes.

6.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Recognize the natural numbers, whole numbers, integers, rational numbers, irrational numbers and real numbers.
- Express the expansion of real numbers
- Know about the exponents
- Know the laws of exponents
- Convert the decimal expansion of rational number of $\frac{p}{q}$ form
- Develop the number system using *Vedic* Mathematics system

6.2 NATURAL NUMBERS

The counting N = {1,2,3,4, ------) is called the natural numbers. These numbers are very important in our routine life. The natural numbers are unlimited. If you think any number, the next number can be obtained by adding one.

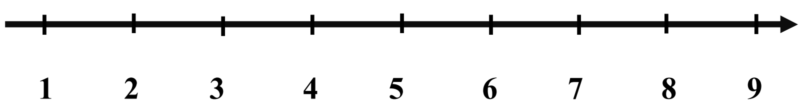
6.2.1 Approach of *Vedic* Mathematics in Developing Natural Numbers

The natural number can be obtained using *Vedic* Mathematics *Sutra*-1 named '*EkadhikenaPurvena*'. The meaning of this *Sutra* is one more than the previous one. Using this *Sutra* one can be added to get the next number and 1 is called successor also. The following features can be described to know the natural numbers.

- Addition of natural numbers such as $8 + 3 = 11$ is also a natural number.
- Subtraction of natural numbers such as $11 – 4 = 7$ is also natural number.
- Product of natural numbers such as $21 \times 2 = 42$ is also a natural number.
- Division of natural number such as $40 \div 5 = 8$ is also natural number.
- Division of natural number such as $21 \div 4 = 5.25$ is not a natural number.
- Addition and subtraction of a natural number can be represented on a number line.
- Addition and multiplication of natural numbers yield also natural numbers
- Subtraction and division of natural numbers may or may not yield natural numbers.
- The natural number can be represented on number line also.

6.2.2 Properties of Natural numbers

- If a and b are natural numbers, then addition of natural numbers is commutative and this can be represented as $a + b = b + a$
- Addition of natural numbers a, b and $c$is associative and it can be represented as $(a + b)+ c= a + (b + c)$
- The product of natural numbers a and b is commutative and can be represented as $a \times b = b \times a$
- The product of natural numbers is associative and it can be represented as $(a \times b) \times c = a \times (b\times c)$
- If a is any natural number and $1\times a = a\times 1$and 1 is multiplicative identity for natural numbers
- Multiplication of natural numbers is distributive over addition for a, b and c, then as $a \times (b + c) = a \times b + a \times c$



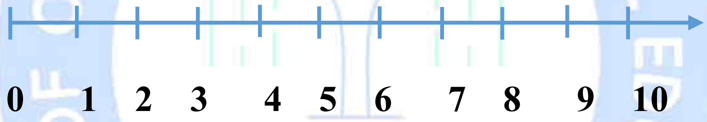
6.3 WHOLE NUMBERS

When digit 0 is added to the list of natural numbers, then these numbers are called whole number.

The numbers W = {0, 1, 2, 3, 4, 5--------------------} are called whole numbers. The whole are unlimited. Theseis no end points of the whole numbers such as natural numbers.

6.3.1 Approach of *Vedic* Mathematics in Developing Natural Numbers

The natural number can be obtained using *Vedic* Mathematics *Sutra*-1 named *'EkadhikenaPurvena'*. The meaning of this *Sutra* is one more than the previous one. Using this *Sutra* one can be added to get the next number and 1 is called successor also. The meaning of the *Sutra* also includes that one digit is one more than the previous one. If we have the previous one and the previous of one is zero. Hence the whole numbers can be developed using the *Sutra"EkadhikenaPurvena"*



6.3.2 Properties of Whole numbers

- If a and b are natural, then addition of whole numbers is commutative and this can be represented as $a + b = b + a$
- Addition of whole numbers a, b and c is associative and it can be represented as $(a + b) + c = a + (b + c)$
- The product of whole numbers a and b is commutative and can be represented as $a \times b = b \times a$
- The product of whole numbers is associative and it can be represented as $(a \times b) \times c = a \times (b \times c)$
- If a is any whole number and $1 \times a = a \times 1 = a$ and 1 is multiplicative identity for whole number
- The sum of whole number and 0 is the number itself and 0 is additive identity.
- Multiplication of whole numbers is distributive over addition for$a$, b and c, then as $a \times (b + c) = a \times b + a \times c$

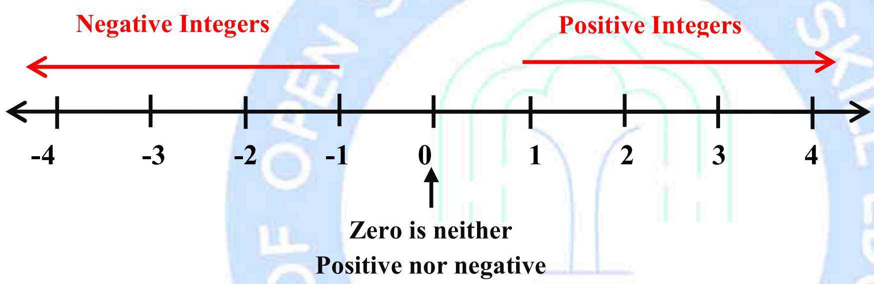
6.4 INTEGERS

The positive and negative natural numbers are called the integers. The positive numbers are represented on the right side of zero and negative numbers are represented on the left side of the zero. The integers can be represented as *Z* or *I.*

6.4.1 Approach of *Vedic* Mathematics in Developing Integers

The source *Sutras* of *Vedic* Mathematics "*EkadhikenaPurvena*" and "*EkanyunenaPurvena*" develop the concept of positive and negative integers. The previous of zero digit is -1 and the previous of -1 and -2. The *Sutra* "*EkadhikenaPurvena*" develops positive digits.

Hence integers *Z* or *I* = {------------------ -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5--------------------}



6.4.2 Properties of Integers

- If *a* and *b* are integers, then addition of integers is commutative and this can be represented as $a + b = b + a$
- Addition of integers a, *b* and *c* is associative and it can be represented as $(a + b) + c = a + (b + c)$
- The product of integers a and *b* is commutative and can be represented as $a \times b = b \times a$
- The product of integers is associative and it can be represented as $(a \times b) \times c = a \times (b \times c)$
- If *a* is any integers and $1 \times a = a \times 1$ and 1 is multiplicative identity for integers
- The sum of integers and 0 is the integers itself and 0 is additive identity.
- Multiplication of integers is distributive over addition for *a, b* and *c*, then as $a \times (b + c) = a \times b + a \times c$

6.5 RATIONAL NUMBERS

A number in $\frac{p}{q}$ form is called rational number where p and q are integers and q cannot be zero. The collection of all rational numbers is denoted by Q.

6.5.1 Approach of *Vedic* Mathematics in Rational Number

The source *Sutras* of *Vedic* Mathematics "*EkadhikenaPurvena*" and "*EkanyunenaPurvena*" develop the concept of rational numbers. These *Sutras* are applied in numerator and denominator by addition and subtraction to numerator and denominator. One more *upSutra* named "*Anurupyena* is applied also. The meaning of this *Sutra* is by proportionately. It means that number series can be extended according to the ratio also.

$$Q = \{\frac{1}{5}, \frac{-3}{2}, \frac{8}{4}, \frac{18}{11}, \text{------------}\}$$

6.5.2 Positive and Negative Rational Number

(i) A rational number $\frac{p}{q}$ is said to be positive if p and q are positive or negative. Thus $\frac{8}{7}$, $\frac{-3}{-5}$, $\frac{7}{5}$, $\frac{-25}{-13}$ are all positive integers.

(ii) A rational number $\frac{p}{q}$ is said to be negative if p and q have opposite sign. Thus, $\frac{4}{-3}$, $\frac{9}{-2}$, $\frac{14}{-17}$, $\frac{-61}{23}$ are all negative rational numbers.

6.5.3 Standard Form of Rational Number

A rational number $\frac{p}{q}$ is said to be in standard or simplest form if p and q have no common factor other than 1. It means p and q are co-primes. $\frac{5}{-4}, \frac{17}{3}, \frac{15}{8}, \frac{-14}{5}$ are in standard form. The lowest form can be calculated by cancelling the common factor of numerator and denominator. This is described in the following example

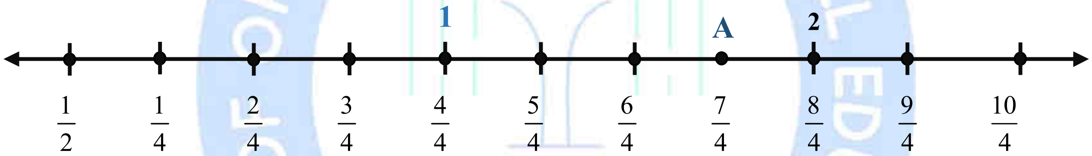
Example1: Write down the lowest form or standard form of $\frac{85}{34}$.

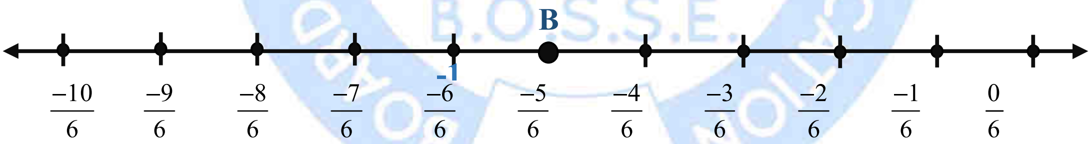
Solution: The numerator and denominator $\frac{85}{34}$ is factorized as $\frac{5\times17}{2\times17}$, 17 is common in numerator and denominator and it is cancelled and remaining part of $\frac{5\times17}{2\times17}$ is $\frac{5}{2}$ in standard form.

6.5.4 Rational Number on Number line

Rational Numbers can be represented on number line. The positive rational numbers can be represented on the right side of zero and negative rational numbers on the left side of zero. The representation of rational numbers on the number line is demonstrated in the following example:

Example2: Represent the rational number $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$ and $\frac{8}{4}$ on the number line.



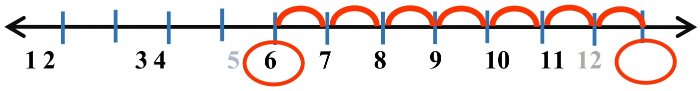


6.5.5 Addition on the number line

The addition of the natural number, whole numbers, integers and rational numbers can be added on the number line. The addition of natural numbers on the number lines is added in the following way:

Example3: Add 5 and 7 on the number line

Solution: Draw number line and mark natural number 1 to 12. The first natural number 5 is circled first and 7 is added as shown on the number line. This result of addition is12 according to the final circled number.



5+7=12

6.5.6 Equivalent Forms of a Rational Numbers

Equivalent forms of a rational number can be generated by multiplying or dividing the numerator and denominator of the given rational number by the same number. It can be understood as in the following examples:

Example4: Write three equivalent fractions of $\frac{5}{-4}$

Solution: $\frac{5}{-4} = \frac{5\times2}{-4\times2} = \frac{10}{-8}, \frac{5\times4}{-4\times4} = \frac{20}{-16}, \frac{5\times5}{-4\times5} = \frac{25}{-20}$

$\frac{10}{-8} = \frac{20}{-16} = \frac{25}{-20}$, are all equivalent fractions.

6.5.7 Comparison of Rational Numbers

(i) Compare the numerator of rational numbers if the denominators are same. The greater numerator of rational number is greater than the other rational number.

Example5: Which is greater $\frac{15}{46}$ or $\frac{11}{46}$

Solution: $\frac{15}{46} > \frac{11}{46}$

(ii) If the denominators of rational numbers are not equal, make their denominators equal by multiplying or dividing. Then rational number having greater numerator is the greater.This concept is described in the following illustration

Example6: Which is greater between the following rational numbers: $\frac{21}{17}$ and $\frac{32}{34}$

Solution: For making the denominator same, the rational number $\frac{21}{17}$ will be multiplied by 2 to its numerator and denominator. Hence $\frac{21\times2}{17\times2} = \frac{42}{34}$, Now the denominator of $\frac{42}{34}$ and $\frac{32}{34}$ are same. Now by observation, it is concluded that $\frac{42}{34}$ is greater. Consequently, $\frac{21}{17} > \frac{32}{34}$

CHECK YOUR PROGRESS 6.1

(i) From the following numbers, circle the rational numbers:

(a) 5 (b) $\frac{5}{4}$ (c) $\frac{4}{6}$ (d) $\frac{9}{0}$

(ii) Add the following on the number lines:

(a) 2 + 8 (b) -3 + 7 (c) $\frac{3}{7}+\frac{6}{7}$

(iii) Convert the following rational numbers into the lowest form:

(a) $\frac{104}{84}$ (b) $\frac{200}{175}$ (c) $\frac{-1024}{800}$ (d) $\frac{2024}{-608}$

(iv) Which are the integers among the following numbers:

(a) $\frac{16}{8}$ (b) $\frac{32}{15}$ (c) $\frac{2}{4}$ (d) $\frac{-21}{3}$

(v) Write three equivalent rational numbers of the following:

(a) $\frac{3}{4}$ (b) $\frac{6}{11}$ (c) $\frac{-6}{11}$

(vi) Compare the following rational number:

(a) $\frac{2}{3}$ and $\frac{4}{5}$ (b) $\frac{8}{34}$ and $\frac{12}{68}$ (c) $\frac{-12}{25}$ and $\frac{-14}{30}$

6.5.8 Addition of Rational Numbers

Firstly, the numerator of rational numbers with same denominators is added only and the common denominator is written as given in the rational numbers. It is described in the following examples.

Example7: Add $\frac{75}{13}$ and $\frac{27}{13}$

Solution: Since the denominator is common in both the rational numbers. Hence the numerator is added in the following way and $\frac{75+27}{13}=\frac{102}{13}$

If the denominator is not common of rational numbers are not same, equal denominators are obtained firstly and after that the rational numbers are added. Such rational is added as inthe following example.

Example8: Add $\frac{12}{5} + \frac{4}{7}$

Solution: To equate the denominator, the following steps are explained

$\frac{12\times 7}{5\times 7} = \frac{84}{35}$ and $\frac{4\times 5}{7\times 5} = \frac{20}{35}$

Now $\frac{84}{35} + \frac{20}{35} = \frac{104}{35}$

6.5.9 Subtraction of Rational Numbers

Firstly, the numerator of rational numbers with same denominators is subtracted only and the common denominator is written as given in the rational numbers. It is described in the following examples.

Example9: Subtract $\frac{13}{19}$ from $\frac{24}{19}$

Solution: $\frac{24-13}{19} = \frac{11}{19}$

If the denominator is not same, firstly equal denominators are obtained and after that the rational numbers are subtracted. Such rational numbers are subtracted in the following example:

Example10: Subtract $\frac{1}{11}$ from $\frac{24}{33}$

Solution: For making the denominator equal,

$$\frac{1\times 3}{11\times 3} = \frac{3}{33}$$

Then $\frac{24}{33} - \frac{3}{33} = \frac{24-3}{33} = \frac{21}{33}$

6.5.10 Multiplication and Division of Rational Numbers

The multiplication of two or more than two rational numbers can be multiplied. If the rational numbers are $\frac{p}{q}, \frac{r}{s}, \frac{t}{u}$, then these rational numbers can be multiplied in the following way: $\frac{p\times r\times t}{q\times s\times u}$

Example11: Multiply $\frac{4}{25}$ from $\frac{25}{8}$

Solution: $\frac{4}{5}\times\frac{25}{8}=\frac{4\times25}{5\times8}=\frac{4\times5\times5}{5\times4\times2}=\frac{5}{2}$ (The common in numerator and denominator are cancelled)

The division of two rational numbers can be obtained in the following way $\frac{p}{q},\frac{r}{s}$

$\frac{p}{q},\div\frac{r}{s}$, The sign of division is replaced with sign of multiplication and after the sign of multiplication, the rational number is reciprocalled. $\frac{p}{q}\times\frac{s}{r}$,

Example12: Solve $\frac{21}{15}\div\frac{35}{12}$

Solution: According the concept, the problem $\frac{21}{15}\div\frac{35}{12}$ can be written as $\frac{21}{15}\times\frac{12}{35}=\frac{21\times12}{15\times35}$

$=\frac{3\times7\times3\times4}{5\times3\times7\times5}=\frac{12}{25}$

CHECK YOUR PROGRESS 6.2

(i) Add the following rational numbers:

(a) $\frac{7}{12}$ and $\frac{11}{12}$ (b) $\frac{14}{36}$ and $\frac{13}{18}$ (c) $\frac{-7}{9}$ and $\frac{41}{-15}$ (d) $\frac{31}{5}$ and $\frac{16}{25}$

(ii) Subtract the following rational numbers:

(a) $\frac{85}{42}$ from $\frac{110}{21}$ (b) $\frac{40}{50}$ from $\frac{305}{8}$ (c) $\frac{16}{12}$ from $\frac{32}{15}$

(iii) Simplify the following rational numbers:

(a) $\left(\frac{3}{7}+\frac{-8}{14}\right)+\frac{5}{70}$ (b) $\left(\frac{-5}{16}+\frac{7}{24}\right)+\frac{10}{48}$

(iv) Multiply the following rational numbers:

(a) $\frac{14}{40}\times\frac{10}{56}$ (b) $\frac{15}{45}\times\frac{54}{28}$ (c) $\frac{70}{35}\times\frac{105}{225}$

(v) Divide the following rational numbers:

(a) $\frac{50}{42} \div \frac{20}{42}$ (b) $\frac{45}{56} \div \frac{80}{82}$ (c) $1\frac{05}{6} \div \frac{12}{33}$

(vi) Find the area of rectangle whose dimensions are $\frac{36}{45}$ cm and $\frac{50}{42}$ cm

6.5.11 Terminating Decimals

The decimal presentation of rational can be obtained by dividing numerator from denominator. This process can be the process of long division. Using this process, when the remainder is zero and the resulting decimal has finite number of decimal places, such rational numbers are known as terminating decimals. i.e., $\frac{782}{100} = 7.82$

6.5.12 Non-Terminating Decimals

The decimal presentation of rational can be obtained by dividing numerator from denominator. This process can be the process of long division. Using this process, when the remainder is not zero and the resulting decimal has infinite number of decimal places, such rational numbers are known as terminating decimals. i.e., $\frac{7}{3} = 2.333333333$---------

Example13: Represent $\frac{16}{25}$ in decimal form and state the type of rational number.

Solution:

```
        0.64
     ________
25  |1 6.0 0
    |
    | -150
    |________
    |
```

1 0 0

- 1 0 0

00

$\frac{16}{25}$ = 0.64, Terminating

Example14: Represent $\frac{16}{25}$ in decimal form and state the type of rational number.

Solution:

$$\begin{array}{r|l} & 0.64 \\ \hline 25 & 16.00 \\ & -150 \\ \hline & 100 \\ & -100 \\ \hline & 00 \end{array}$$

$\frac{16}{25}$= 0.64, Terminating

Example15: Represent $\frac{1}{3}$ in decimal number and state the type of rational number.

Solution:

$$\begin{array}{l} 0.33\text{------} \\ \hline 1.00 \\ 9 \\ \hline 10 \\ -\ 9 \\ \hline \end{array}$$

1(This remainder repeats)

$\frac{1}{3} =$ 0.333-------, non-terminating

Example16: Express 0.6666--------- in $\frac{p}{q}$ form

Solution: Let x = 0.6666666---------- (1)

Multiplying by 10 to equation (1)

$10x$ = 6.6666666------------------- (2)

Subtraction equation (1) from (2)

$9x = 6$

$x = \frac{6}{9} = \frac{2}{3}$

CHECK YOUR PROGRESS 6.3

(i) Represent the following rational numbers into decimal form:

(a) $\frac{12}{3}$ (b) $\frac{25}{4}$ (c) $\frac{14}{25}$ (d) $\frac{7}{9}$ (e) $\frac{35}{11}$ (f) $\frac{72}{15}$

(ii) Represent the following in $\frac{p}{q}$ form

(a) 0.356 (b) 0.33333----------. (c) $4.\overrightarrow{33}$ (d) 0. 315315-----

6.5.13Rational Numbers Between Two Rational Numbers

To obtain one rational number between two rational numbers, addition of both the rational numbers is halved. If a and b are two rational numbers, then the one rational number between them is $\frac{1}{2}(a+b)$

Example17: Find a rational number between $\frac{4}{7}$ and $\frac{5}{6}$

Solution: One rational between $\frac{4}{7}$ and $\frac{5}{6}$ is $\frac{1}{2}\left(\frac{4}{7}+\frac{5}{6}\right) = \frac{1}{2} \times \frac{59}{42} = \frac{59}{84}$

The way to calculate the rational numbers between two rational numbers is obtained by equating the denominator. The equivalent rational number may be found out if it needs. This method is described in the following way:

Example18: Find the three rational between $\frac{2}{3}$ and $\frac{2}{15}$

Solution: To make the denominator equal, the fraction $\frac{2}{3}$ is multiplied by 5 as $\frac{2\times5}{3\times5} = \frac{10}{15}$

Then three rational numbers between $\frac{10}{15}$ and $\frac{2}{15}$ are $\frac{3}{15}, \frac{4}{15}$ *and* $\frac{5}{15}$

CHECK YOUR PROGRESS 6.4

(i) Find a rational number between the following rational numbers:

(a) $\frac{2}{3}$ and $\frac{2}{15}$ (b) $\frac{4}{5}$ and $\frac{2}{7}$ (c) $\frac{8}{9}$ and $\frac{9}{10}$

(ii) Find three national numbers between the following rational numbers:

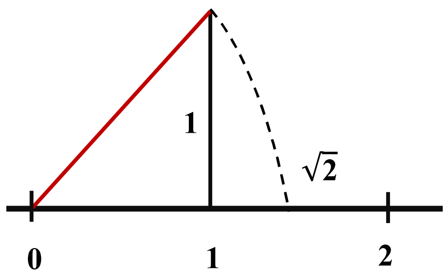
(a) $\frac{1}{4}$ and $\frac{3}{5}$ (b) $\frac{5}{3}$ and $\frac{4}{9}$ (c) $\frac{4}{10}$ and $\frac{8}{5}$

6.6 IRRATIONAL NUMBERS

An irrational is a number which is neither terminating nor repeating decimal presentation i.e. 0.202002000, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ ------

6.6.1 Irrational Number on Number Line

Irrational numbers can also be represented on the number lines. The approach of constructing the unit of irrational number is Pythagoras Theorem or *Baudhayan* Theorem.To draw $\sqrt{2}$ on number line, the Pythagoras Theorem is applied. To represent $\sqrt{2}$ on number line, the number 2 is splitted according to Pythagoras Theorem and it can be represented as $\sqrt{1^2 + 1^2}$.



CHECK YOUR PROGRESS 6.5

(i) Represent the following irrational numbers on the number line.

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $1+\sqrt{3}$

6.6.2Irrational Numbers between Rational Numbers

An irrational value between two any numbers can be calculated by the same method as we have calculated in the previous way. Let us understand it in the following example.

Example19: Find an irrational number between 3 and 5

Solution: The irrational number can be calculated by doing the square root of the product of the numbers i.e. $\sqrt{3 \times 5} = \sqrt{15} = 3.87$----------

Example20: Find an irrational number between 3 and $\sqrt{6}$

Solution: An irrational number can be calculated as $\frac{1}{2}(3+\sqrt{6}) =$

$\frac{1}{2}(3 + 2.449 - - - - -) = \frac{1}{2}(5.449 - - - - -) = 2.724$----------------

CHECK YOUR PROGRESS 6.6

1. Find an irrational number between the following numbers:

(a) 4 and 3

(b) 5 and 7

(c) $\sqrt{5}$ and 6

(d) $\sqrt{15}$ and 8

6.7 EXPONENTS

When a number is multiplied many times, this number can be written in exponential form. The exponent of a number describes how many times to use that number in a multiplication. It is written above the number in small size than the number. It is described in the following example:

$3\times3\times3\times3\times3\times3\times3\times3 = 3^8$

$5\times5\times5\times5\times5\times5\times5=5^7$

In the above examples, 3 is repeated 8 times.3^8 will be read 3 raised to 8 and 5 is repeated 7 times. Hence it will be read as 5 raised to 7.

Example21: Find the value of $\left(\frac{4}{7}\right)^5$

Solution: $\left(\frac{4}{7}\right)^5 = \frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7} = \frac{1024}{16807}$

Example22: Express 625 in the exponential notation

Solution: $5\times5\times5\times5\times5 = (5)^4$

Example23: Express in the exponential form of $\frac{64}{324}$

Solution: $\frac{2\times2\times2\times2\times2\times2}{3\times3\times3\times3\times3} = \left(\frac{2}{3}\right)^5$

CHECK YOUR PROGRESS 6.7

I. Find the value of the following:

(i) $\left(\frac{2}{5}\right)^4$ (ii) $\left(\frac{-9}{4}\right)^5$ (iii) $\left(\frac{-3}{5}\right)^5$ (iv) $\left(\frac{-2}{7}\right)^3$

II. Express the following rational numbers in the exponential form:

(i) $\frac{27}{8}$ (ii) $\frac{-125}{729}$ (iii) $\frac{343}{125}$ (iv) $\frac{1331}{-2197}$ (v) $\frac{1000}{729}$

6.7 LAWS OF EXPONENTS

6.7.1 Law of Exponents-1

Law of exponents can be understood in the following illustrations:

(i) $5^4 \times 5^2 = (5 \times 5 \times 5 \times 5) \times (5 \times 5) =$

$$= 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$$

(ii) $(-3)^3 \times (-3)^4 = (-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3) \times (-3) =$

$$= (-3)^7$$

Using this expression, it can be generalized as

$a^m \times a^n = a^{(m+n)} (Law l)$

In this generalization, if $a$is non-zero rational number and m and are two positive integers, then exponents are added.

6.7.2 Law of Exponents-2

Now observe the following

$$(4)^6 \div (4)^3 = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4}$$

$$= (4 \times 4 \times 4) = (4)^3$$

$$(5)^4 \div (5)^2 = \frac{5 \times 5 \times 5 \times 5}{5 \times 5}$$

$$= (5)^2$$

In the above illustrations, the base of each part is same and during the division, the power has been reduced as in the following illustration.

Law-2 $(a)^m \div (a)^n = (a)^{m-n}$ where a is any non-zero rational number and m and n are positive integers.

Example24: Solve

Solution: Here $a = \frac{6}{7}$, $m = 2$, $n = 3$

According to the law of exponents

$$\left(\frac{6}{7}\right)^{2+3} = \left(\frac{6}{7}\right)^5 = \left(\frac{6}{7}\right)\times\left(\frac{6}{7}\right)\times\left(\frac{6}{7}\right)\times\left(\frac{6}{7}\right)\times\left(\frac{6}{7}\right) = \frac{7776}{16807}$$

Example25: Evaluate $\left(\frac{-4}{5}\right)^3 \times \left(\frac{-4}{5}\right)^2$

Solution: $\left(\frac{-4}{5}\times\frac{-4}{5}\times\frac{-4}{5}\right)\times\left(\frac{-4}{5}\times\frac{-4}{5}\right) = \left(\frac{-4}{5}\right)^{3+2}$

$$= \left(\frac{-4}{5}\right)^5$$

$$= \frac{-1024}{3125}$$

Example26: Evaluate $(6)^5 \div (6)^2$

Solution: According to the laws of exponents $(6)^{5-2} = 6^3 = 6\times6\times6 = 216$

Example27: Evaluate $\left(\frac{15}{16}\right)^{14} \div \left(\frac{15}{16}\right)^{12}$

Solution: Here a = $\left(\frac{15}{16}\right)$, $m = 14,\ n = 12$

According to law of exponent

$(a)^m \div (a)^n = (a)^{m-n}$, Hence $\left(\frac{15}{16}\right)^{14-12}$

$\left(\frac{15}{16}\right)^2 = \left(\frac{15}{16}\right)\times\left(\frac{15}{16}\right) = \frac{225}{256}$

Example28: Evaluate $\left(\frac{7}{11}\right)^6 \div \left(\frac{7}{11}\right)^9$

Solution: Here $a = \left(\frac{7}{11}\right), m = 6, n = 9$

Hence $\left(\frac{7}{11}\right)^{6-9} = \left(\frac{7}{11}\right)^{-3} = \left(\frac{11}{7}\right)^3$

$= \left(\frac{11}{7}\right)\times\left(\frac{11}{7}\right)\times\left(\frac{11}{7}\right) = \frac{1331}{343}$

6.7.3 Law of Exponents-3

Let us observe the following illustrations

$$\left(4^2\right)^3 = 4^2 \times 4^2 \times 4^2 = 4^{2+2+2} = 4^6 = 4^{3\times 2}$$

$$\left(5^3\right)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{3+3+3+3} = 5^{12} = 5^{3\times 4}$$

The generalization form of above concept, if a is non-zero rational number and m and n are two positive integers, thus $(a^m)^n = a^{mn}$

Example29: Solve $\left[\left(\frac{9}{5}\right)^2\right]^3$

Solution: $\left(\frac{9}{5}\right)^{2\times 3} = \left[\left(\frac{9}{5}\right)^6 = \left(\frac{9}{5}\right)\times\left(\frac{9}{5}\right)\times\left(\frac{9}{5}\right)\times\left(\frac{9}{5}\right)\times\left(\frac{9}{5}\right)\times\left(\frac{9}{5}\right)\right]$

$$= \frac{531441}{15625}$$

6.7.4 Zero Exponent

If a is non-zero rational number and m and n are positive integer and $m = n$.

This can be expressed in the division

$$(a)^m \div (a)^n = a^m \div a^m = \frac{a^m}{a^m} = 1 = a^{m-m} = a^0$$

It is observed that for any rational number other zero, $a^0 = 1$

Example30: find the value of $\left(\frac{5}{6}\right)^0$

Solution: $\left(\frac{5}{6}\right)^0 = 1 \; (\because a^0 = 1)$

Example31: Evaluate $\left(\frac{496}{765}\right)^{47-47}$

Solution: $\left(\frac{496}{765}\right)^{47-47} = \left(\frac{496}{765}\right)^0 = 1 \; (\because a^0 = 1)$

CHECK YOUR PROGRESS 6.8

(i) Simplify and express the result in the exponential form

(a) $(5)^3 \times (5)^4$

(b) $\left(-\frac{5}{6}\right)^1 \times \left(-\frac{5}{6}\right)^3$

(c) $\left(\frac{4}{3}\right)^3 \times \left(\frac{4}{3}\right)^5$

(ii) Simplify and express the result in the exponential form.

(a) $(-4)^5 \div (-4)^3$

(b) $\left(-\frac{6}{11}\right)^{24} \div \left(-\frac{6}{11}\right)^{21}$

(c) $\left(-\frac{402}{19}\right)^{102} \div \left(-\frac{402}{19}\right)^{101}$

(iii) Simplify and express the result in the exponential form.

(a) $\left(6^2\right)^3$

(b) $\left[\left(\frac{4}{5}\right)^6\right]^3$

(c) $\left[\left(-\frac{8}{13}\right)^2\right]^6$

(iv) Simplify the following

(a) $\left(\frac{11}{4}\right)^0$

(b) $\left(\frac{6}{17}\right)^0 \times \left(\frac{6}{17}\right)^0$

(c) $(2)^0 \times (3)^0 \times (4)^0 \times (5)^0$

6.7.5 Applications of law of Exponents

The applications using the laws of exponents are very useful to solve the complex problems. Let us solve such problems in the following way:

Example32: Evaluate the following problems

(i) $(343)^{\frac{1}{3}}$ (ii) $\left(\frac{4}{25}\right)^{\frac{3}{2}}$ (iii) $\left[\left(-\frac{125}{1331}\right)^6\right]^{\frac{1}{9}}$

Solution:

(i) $(343)^{\frac{1}{3}} = (7 \times 7 \times 7)^{1/3} = (7^3)^{1/3} = 7^{3\times\frac{1}{3}} = 7$

(ii) $\left(\frac{4}{25}\right)^{\frac{3}{2}} = \left(\frac{2\times2}{5\times5}\right)^{\frac{3}{2}} = \left[\left(\frac{2}{5}\right)^2\right]^{\frac{3}{2}} = \left(\frac{2}{5}\right)^{2\times\frac{3}{2}} = \left(\frac{2}{5}\right)^3 = \frac{8}{125}$

(iii) $\left\{\left[\left(\frac{-5}{11}\right)^3\right]^6\right\}^{\frac{1}{9}} = \left(\frac{-5}{11}\right)^{3\times 6\times\frac{1}{9}} = \left(\frac{-5}{11}\right)^2 = \frac{25}{121}$

CHECK YOUR PROGRESS 6.9

1. Simplify the following problems:

(i) $(1331)^{\frac{4}{3}}$

(ii) $\left(-\frac{216}{729}\right)^{\frac{2}{3}}$

(iii) $\left(\frac{6561}{625}\right)^{\frac{1}{4}}$

(iv) $\left(\frac{6}{5}\right)^{\frac{1}{3}} \times \left(\frac{6}{5}\right)^{\frac{1}{3}} \times \left(\frac{6}{5}\right)^{\frac{1}{3}}$

(v) $\left(\frac{7}{6}\right)^{\frac{3}{4}} \times \left(\frac{7}{6}\right)^{\frac{2}{4}} \times \left(\frac{7}{6}\right)^{\frac{3}{4}}$

TERMINAL EXERCISE

I. Fill in the blanks in the following questions

(i) Compare the rational numbers $\frac{2}{7}$ ------- $\frac{4}{5}$

(ii) The lowest form of $\frac{105}{80}$ is _____

(iii) The smallest rational number of $\frac{-21}{3}, \frac{54}{4}, \frac{25}{10}$ ------

(iv) The result of $\frac{7}{6} + \frac{8}{12}$ ----

(v) The result of $\frac{7}{30} - \frac{2}{15}$ ------

(vi) The result of $\frac{15}{40} \times \frac{5}{6}$ ---------

(vii) The result of $\frac{15}{6} \div \frac{60}{42}$ --------

II. Express the following in exponential form:

(i) $3 \times 5 \times 7 \times 7 \times 5 \times 3 \times 5$

(ii) $\frac{1}{2} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{5}{6}$

(iii) $-3 \times -6 \times -5 \times -3 \times -5 \times -6$

III. Simplify the following:

(i) $5^0 + 2^0 + 14^0 - 7^0$

(ii) $(5^0 + 2^0 + 14^0)(5^0 + 2^0 + 14^0 - 7^0)$

(iii) $\frac{7^0 + 3 + 9^0}{1 + 2^0}$

(iv) $(1^0 + 10^0 - 21^0 + 7^0) + (100^0 - 41^0 + 85^0 + 3^0)$

IV. Simplify the following:

(i) $\left(\frac{4}{5}\right)^2 \times \left(\frac{7}{8}\right)^3 \times \left(\frac{10}{14}\right)^2$

(ii) $\left(\frac{10}{11}\right)^2 \times 33 \times \left(\frac{11}{20}\right)^2$

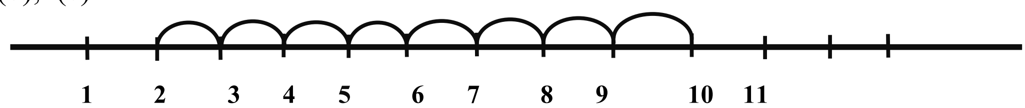
(iii) $\left(\frac{1}{2}\right)^2 \times \frac{32}{3} \times \frac{12}{16}$

ANSWER TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 6.1

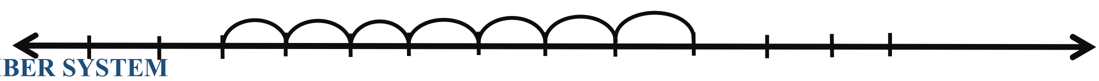
(i) (a), (b), (c)

(ii) (a)



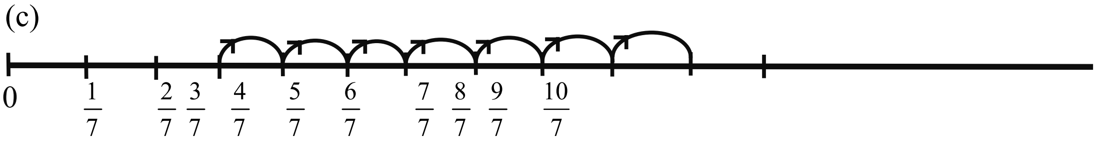
OA = 10

(b)



-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7

(c)



(iii) (a) $\frac{26}{21}$ (b) $\frac{8}{21}$ (c) $\frac{-32}{25}$ (d) $\frac{256}{76}$

4. (d) $\frac{-21}{3}$

5. (a) $\frac{9}{12}, \frac{12}{16}, \frac{15}{20}$ (b) $\frac{12}{22}, \frac{18}{33}, \frac{24}{44}$ (c) $-\frac{18}{14}, -\frac{27}{21}, -\frac{36}{28}$

6. (a) $\frac{2}{3} < \frac{4}{5}$ (b) $\frac{8}{34} > \frac{12}{68}$ (c) $-\frac{12}{25} < \frac{-14}{30}$

CHECK YOUR PROGRESS 6.2

(i) (a) $\frac{88}{45}$ (b) $\frac{-158}{36}$ (c) $\frac{88}{45}$ (d) $\frac{88}{45}$

(ii) (a) $\frac{135}{42}$ (b) $\frac{7465}{200}$ (c) $\frac{48}{60}$

(iii) (a) $\frac{5}{70}$ (b) $\frac{9}{48}$

(iv) (a) $\frac{1}{16}$ (b) $\frac{9}{14}$ (c) $\frac{14}{15}$

(v) (a) $\frac{5}{2}$ (b) $\frac{369}{448}$ (c) $\frac{121}{24}$

(vi) $\frac{20}{21}\, cm^2$

CHECK YOUR PROGRESS 6.3

(i) (a) 4 (b) 6.25 (c) 0.56 (d) 0.777------- (e) 3.1818----- (f) 4.8

(ii) (a) $\frac{89}{250}$ (b) $\frac{1}{3}$ (c) $\frac{429}{99}$ (d) $\frac{35}{111}$

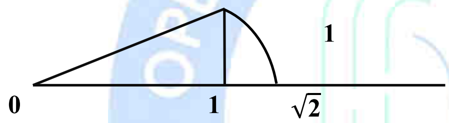
CHECK YOUR PROGRESS 6.4

(i) (a) $\frac{2}{5}$ (b) $\frac{19}{35}$ (c) $\frac{161}{180}$

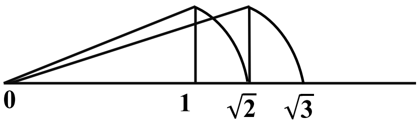
(ii) (a) $\frac{6}{20}, \frac{7}{20}, \frac{8}{20}$ (b) $\frac{5}{9}, \frac{6}{9}, \frac{7}{9}$ (c) $\frac{5}{10}, \frac{6}{10}, \frac{7}{10}$

CHECK YOUR PROGRESS 6.5

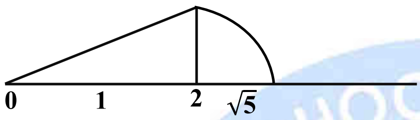
1. (a) $\sqrt{1^1 + 1^1}$ =



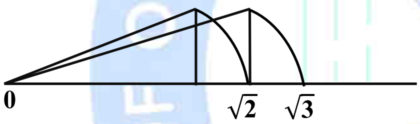
(b) $\sqrt{\sqrt{2}^2 + 1^2}$



(c)$\sqrt{2^2 + 1^2}$



(d)$1+ \sqrt{3} =$ OA



CHECK YOUR PROGRESS 6.6

(a) 3.8729----- (b) 5.9160----- (c) 4.1180-------- (d) 5.9364-----

CHECK YOUR PROGRESS 6.7

I.

(a) $\frac{16}{625}$ (b) $\frac{-59049}{1024}$ (c) $\frac{-243}{3125}$ (d) $\frac{-8}{343}$

II. (a) $\left(\frac{3}{2}\right)^3$ (b) $\left(\frac{-5}{9}\right)^3$ (c) $\left(\frac{7}{5}\right)^3$ (d) $\left(\frac{11}{-13}\right)^3$ (e) $\left(\frac{10}{9}\right)^3$

CHECK YOUR PROGRESS 6.8

(i) (a) 5^7 (b) $\left(\frac{-5}{6}\right)^4$ (c) $\left(\frac{4}{3}\right)^8$

(ii) (a) $(-4)^2$ (b) $\left(\frac{-6}{11}\right)^3$ (c) $\frac{-402}{19}$

(iii) (a) $(6)^6$ (b) $\left(\frac{4}{5}\right)^{18}$ (c) $\left(\frac{-8}{12}\right)^{12}$

(iv) (a) 1 (b) 1 (c) 1

CHECK YOUR PROGRESS 6.9

(i) 14641 (ii) $\frac{36}{81}$ (iii) $\frac{9}{5}$ (iv) $\frac{6}{5}$ (v) $\frac{49}{36}$

7 RATIONAL EXPONENTS

INTRODUCTION

Achildwasaskedtowritethefollowing:

(i) 7tobemultiplied4timesbyitself

i.e. $7\times7\times7\times7=2401$

(ii) 3tobemultipliedbyseventimes

$3\times3\times3\times3\times3\times3\times3=2187$

(iii) 11tobemultipliedby6times

$11\times11\times11\times11\times11\times11=161051$

Anotherchildwasaskedtomultiply15byitself18times.

Third child was asked to multiply 17 by 21 times. such writing becomes quite boring andexhaustiveforachild.

Theaboveexamplecanbewrittenindifferentform:

(i) $7\times7\times7\times7=7^4$

(ii) $3\times3\times3\times3\times3\times3=3^6$

(iii) $11\times11\times11\times11\times11\times11=11^6$

(iv) $15\times15\times15\times....18\text{times}=15^{18}$

(v) $17\times17\times17\times17\times...21\text{times}=17^{21}$

Writing the multiplication of number by itself is called exponential notation. In the lesson we shall study the meaning of notation, its application. We shall prove laws of exponents and learn to apply these laws. We shall alsolearn radicals, its laws andfind simplest form of radical. We shall learn fundamental application of a radical. We shall also learnrationalisationofradicals.

7.1 LEARNING OBJECTIVES

Aftercompletingthislesson,wewillbeableto:

- Writeprimefactorizationofanumber.
- Understandaboutradicals
- Learnthemeaningofnegativeintegersasexponent.
- Learnlawsofexponentsandtheirapplication.
- Learnthemeaningof$p/q$where$q \neq 0$.
- Knowthelawsofexponentforintegralexponents
- Learnaboutpureandmixedradicals.
- Knowtheorderofaradical.
- Simplifyradicalexpression
- Learnfundamentaloperationsofaradical.
- Learnrationalizationofradicals

ESSENTIALBACKGROUNDKNOWLEDGE

(a) Primenumbers

(b) Fourfundamentaloperationsonnumbers

(c) Rationalnumbers

(d) Knowledgeoforderrelation

7.2EXPONENTIALNOTATION

Asexplainedearlier

(i) $7\times7\times7\times7=7^4$ (ii) $3\times3\times3\times3\times3\times3\times3=3^7$

Thus 7^4isreadas7raisedtopower4orfourthpowerof7.Here7isbaseand4is calledexponent(orindex).

Similarly$3^7$is read as 3 raised to power 7 or Seventh power of 3. Here 3 is called base and7iscalledexponent.

Similarly9^{25} isreadas9raisedtopower25or25thpowerof9.Here9isbaseand25 isexponent(or index).

Thus the product of a 9 number by itself several times istaken asExponential Formor Exponentialnotation.

Again–8×–8×–8×..15times=$(-8)^{15}$ herebaseis–8andexponent(orbase)is15. Thismethodofwritinginexponentialformcanbeusedinrationalnumbers.

Thus $\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\ldots 23 \text{ times} = \left(\frac{5}{13}\right)^{23}$

And $\left(\frac{-11}{19}\right)\left(\frac{-11}{19}\right)\left(\frac{-11}{19}\right)$.... 15 times = $\left(\frac{-11}{19}\right)^{15}$

Ingeneralifmisarationalnumbermultipliedbyitselfntimes,itiswrittenasm^n.

Hereagainmiscalledbaseandniscalledexponent.Wetakeafewexamplesto illustrateitmoreclearly.

Example 1: Evaluate: (i) $(5)^4$ (ii) $\left(\frac{3}{5}\right)^5$ (iii) $\left(\frac{-7}{9}\right)^4$

Solution: (i) $5^4 = 5 \times 5 \times 5 \times 5 = 625$

(ii) $\left(\frac{3}{5}\right)^5 = \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) = \frac{3\times3\times3\times3\times3}{5\times5\times5\times5\times5} = \frac{243}{3125}$

(iii) $\left(\frac{-7}{9}\right)^4 = \left(\frac{-7}{9}\right)\left(\frac{-7}{9}\right)\left(\frac{-7}{9}\right)\left(\frac{-7}{9}\right) = \frac{-7\times-7\times-7\times-7}{9\times9\times9\times9} = \frac{2401}{6561}$

Example 2: Write the following in exponential form:

(i) $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ (ii) $\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}$

Solution: (i) $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^{10}$

(ii) $\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}\times\frac{5}{13}=\left(\frac{5}{13}\right)^6$

Example3: Express each of the following in exponential form and write its base and exponent in each case. (i) 625(ii) $\frac{4096}{729}$ (ii) –16807

Solution: (i) 625 = 5 × 5 × 5 × 5 = 5^4, base 5, exponent = 4

(ii) $\frac{4096}{729}=\frac{4\times4\times4\times4\times4\times4}{3\times3\times3\times3\times3\times3}=\frac{4^6}{3^6}=\left(\frac{4}{3}\right)^6$ base = $\frac{4}{3}$, exponent = 6

(iii) –16807 = –7 × –7 × –7 × –7 × –7 = $(-7)^5$. base = –7, exponent = 5

Example 4: Write the reciprocal of each of the following and express them in exponential form: (i) 2^6 (ii) $\left(\frac{5}{7}\right)^2$ (iii) $\left(\frac{-2}{5}\right)^7$

Solution:(i) $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

Reciprocal of $2^6 = 64$ is $\frac{1}{64}=\frac{1}{2^6}=\left(\frac{1}{2}\right)^6$

(ii) $\left(\frac{5}{7}\right)^2=\frac{5}{7}\times\frac{5}{7}=\frac{25}{49}$ Reciprocal of $\left(\frac{5}{7}\right)^2=\frac{25}{49}$ is $\frac{49}{25}=\left(\frac{7}{5}\right)^2$

(iii) $\left(\frac{-2}{5}\right)^7=\frac{-2}{5}\times\frac{-2}{5}\times\frac{-2}{5}\times\frac{-2}{5}\times\frac{-2}{5}\times\frac{-2}{5}\times\frac{-2}{5}=\frac{-128}{78125}$

Reciprocal of $\left(\frac{-2}{5}\right)^7=\frac{-128}{78125}$ is $\frac{78125}{-128}=\left(\frac{5}{-2}\right)^7$

We conclude that if $\left(\frac{a}{b}\right)$ is a non zero rational number and n is a positive integer, then the reciprocal of $\left(\frac{a}{b}\right)^n=\left(\frac{b}{a}\right)^n$

CHECK YOUR PROGRESS7.1

1. Fill in the blanks of the following in exponential form:

(i) $\frac{2}{7}\times\frac{2}{7}\times\frac{2}{7}\times$.....10 times = $\left(\frac{2}{7}\right)^{\cdots\cdots}$

(ii) $\frac{-7}{9}\times\frac{-7}{9}\times\frac{-7}{9}\times$.....25 times = _______

(iii) 13 × 13 × 13 × 13.....p times = _______

2. Write the base and exponential of the following :

(i) $(-8)^4$ (ii) $(6)^{10}$ (iii) $\left(\frac{-c}{d}\right)^m$

3. Write the value of the following:

(i) $\left(\frac{3}{4}\right)^5$ (ii) $\left(\frac{-3}{8}\right)^3$ (iii) $\left(\frac{-5}{2}\right)^4$

4. Write the reciprocal of the following :

(i) 4^3 (ii) $(-4)^5$ (iii) $\left(\frac{-2}{7}\right)^4$

5. Simplify the following:

(i) $\left(\frac{2}{5}\right)^4\times\left(\frac{5}{2}\right)^3$ (ii) $\left(\frac{-3}{7}\right)^4\div\left(\frac{15}{14}\right)^2$

7.3 PRIME FACTORIZATION

(i) 72 = 9 × 8

(ii) 72 = 2 × 2× 2 × 9

(iii) 72 = 8× 3× 3

(iv) 72 = 2 × 2 × 2 × 3 × 3 = $2^3 \times 3^2$

The above four are factors of 72 but prime factors are those factors which cannot be factorised further.

Incase (i) $72 = 9 \times 8$ We can further write factors of 9 and 8.

In (ii) $72 = 2 \times 2 \times 2 \times 9$ = We can factorise 9

In (iii) $72 = 8 \times 3 \times 3$ Here 8 can be further factorised.

In (iv) $72 = 2 \times 2 \times 2 \times 3 \times 3$ All the factors are prime number. It cannot be further factorised.

Similarly 180 can be written as.

2	180
2	90
3	45
3	15
	5

$\therefore 180 = 2 \times 2 \times 3 \times 3 \times 5 = 2^2 \times 3^2 \times 5$

Again

3	3465
3	1155
5	385
7	77
11	11
	1

$3465 = 3 \times 3 \times 5 \times 7 \times 11 = 3^2 \times 5 \times 7 \times 11$

We see that all natural members except 1 can be written as product of powers of prime numbers in a unique manner except the order of occurrence of factors.

Example 5: Express 18900 in exponential form of prime factors.

2	18900
2	9450
3	4725
3	1575

3	525
5	175
5	35
	7

Solution:

$18900 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 7 = 2^2 \times 3^3 \times 5^2 \times 7^1$

CHECK YOUR PROGRESS7.2

1. Express each of the following in exponential form:

   (i) 256 (ii) 1296 (iii) $\frac{-1331}{125}$ (iv) $\frac{243}{16807}$

2. Express each of the following as prime factors: (i) 1430 (ii) 3800 (iii) 22869

7.4NEGATIVE INTEGERS AS EXPONENT

The reciprocal of 23 = 8 is $\frac{1}{8} = \frac{1}{2^3}$

The reciprocal of $\frac{1}{243} = \frac{1}{3^5}$ is 243 = 35

$\frac{1}{2^3}$ can be expressed as 2^{-3}

Similarly 35 can be expressed as $\frac{1}{3^{-5}}$. So the power of base is changed i.e. positive to negative or negative to positive if we change its position from numerator to denominator

$\frac{1}{3^5} = 3^{-5}; \frac{1}{7^5} = 7^{-5}; \frac{1}{x^a} = x^{-a}$ $\frac{1}{5^{-3}} = 5^3; \frac{1}{3^{-6}} = 3^6; \frac{1}{x^{-c}} = x^c$

and

CHECK YOUR PROGRESS7.3

1. Rewrite the following in negative exponent:

(i) $\frac{1}{11^5}$ (ii) $\frac{1}{17^9}$ (iii) $\frac{1}{9^5}$

2. Rewrite the following with positive exponent:

(i) $\frac{1}{3^{-7}}$ (ii) $\frac{1}{7^{-5}}$ (iii) $\frac{1}{23^{-5}}$

3. If $\frac{a}{b}$ is any non zero rational number and p is a positive integer than

$$\left(\frac{a}{b}\right)^{-p} = \frac{a^{-p}}{b^{-p}} = \frac{b^p}{a^p} = \left(\frac{b}{a}\right)^p$$

7.5 LAWS OF EXPONENT

Consider $5^5 \times 5^3 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^8$ $= 5^{5+3}$

Again $(-4)^6 \times (-4)^2 = (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4)$

$= (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) \times (-4) = (-4)^8 = (-4)^{6+2}$

Now $\left(\frac{5}{8}\right)^2 \times \left(\frac{5}{8}\right)^3 = \left(\frac{5}{8}\right) \times \left(\frac{5}{8}\right) \times \left(\frac{5}{8}\right) \times \left(\frac{5}{8}\right) \times \left(\frac{5}{8}\right) = \left(\frac{5}{8}\right)^5 = \left(\frac{5}{8}\right)^{2+3}$

$(p)^4 \times (p)^3 = (p) \times (p) \times (p) \times (p) \times (p) \times (p) \times (p) = (p)^7 = (p)^{4+3}$

Law I: If p is any non-zero rational number and m and n are positive integers then $p^m \times p^n = p^{m+n}$

i.e. in multiplication if bases are same, the powers are added.

Example 6: Evaluate $\left(\frac{-7}{3}\right)^2 \times \left(\frac{-7}{3}\right)^3$

Here $p = \frac{-7}{3}$, $m = 2$, and $n = 3$

$$\therefore \left(\frac{-7}{3}\right)^2 \times \left(\frac{-7}{3}\right)^3 = \left(\frac{-7}{3}\right)^{2+3} = \left(\frac{-7}{3}\right)^5 = \frac{-16807}{243}$$

Example 7: Find the value of $\left(\frac{3}{5}\right)^4 \times \left(\frac{3}{5}\right)^3$

Here $p = \frac{3}{5}$, $m = 4$, $n = 3$

$$\therefore \left(\frac{3}{5}\right)^4 \times \left(\frac{3}{5}\right)^3 = \left(\frac{3}{5}\right)^{4+3} = \left(\frac{3}{5}\right)^7 = \frac{2187}{78125}$$

Now we proceed further

(i) $8^5 \div 8^3 \frac{8^5}{8^3} = \frac{8\times8\times8\times8\times8}{8\times8\times8} = 8\times8 = 8^2 = 8^{5-3}$

(ii) $(-6)^7 \div (-6)^4 = \frac{(-6)^7}{(-6)^4} = \frac{-6\times-6\times-6\times-6\times-6\times-6\times-6}{-6\times-6\times-6\times-6} = (-6)^3 = (-6)^{7-4}$

Law II: When $m>n$, then $pm \div p^n = p^{m-n}$. From the above two examples we get $(p)^m \div (p)^n = p^{m-n}$ for if p is a nonzero rational number and m and n are positive integers $(m>n)$

Law III: When $n>m$

$$p^m \div p^n = \frac{1}{p^{n-m}}$$

Example 8: Find the value of $\left(\frac{9}{22}\right)^{13} \div \left(\frac{9}{22}\right)^{24}$

Solution: Here $p = \frac{9}{22}$, $m = 13$, $n = 24$

$$\therefore \left(\frac{9}{22}\right)^{13} \div \left(\frac{9}{22}\right)^{24} = \left(\frac{9}{22}\right)^{13-24} = \left(\frac{9}{22}\right)^{-11} = \left(\frac{22}{9}\right)^{11}$$

Example 9: Find the value of $\left(\frac{6}{11}\right)^5 \div \left(\frac{6}{11}\right)^{14}$

Solution: Here $p = \frac{6}{11}$, $m = 5$, $n = 14$

$$\therefore \left(\frac{6}{11}\right)^5 \div \left(\frac{6}{11}\right)^{14} = \left(\frac{6}{11}\right)^{5-14} = \left(\frac{6}{11}\right)^{-9} = \frac{1}{\left(\frac{6}{11}\right)^9} = \left(\frac{11}{6}\right)^9$$

Let us consider the following:

(i) $(5^4)^3 = (5^4)\,(5^4)\,(5^4) = 5^{4+4+4} = 5^{12} = 5^{4\times3}$

(ii) $\left(\left(\frac{4}{9}\right)^3\right)^6 = \left(\frac{4}{9}\right)^3\left(\frac{4}{9}\right)^3\left(\frac{4}{9}\right)^3\left(\frac{4}{9}\right)^3\left(\frac{4}{9}\right)^3\left(\frac{4}{9}\right)^3$

$$\left(\frac{4}{9}\right)^{3+3+3+3+3+3} = \left(\frac{4}{9}\right)^{18} = \left(\frac{4}{9}\right)^{3\times6}$$

(iii) $\left(\left(\frac{-2}{5}\right)^2\right)^4 = \left(\frac{-2}{5}\right)^2\left(\frac{-2}{5}\right)^2\left(\frac{-2}{5}\right)^2\left(\frac{-2}{5}\right)^2 = \left(\frac{-2}{5}\right)^{2+2+2+2} = \left(\frac{-2}{5}\right)^8 = \left(\frac{-2}{5}\right)^{2\times4}$

In the above three cases we find:

Law IV: If p is an non zero rational number and m and n are two positive integersthen $(p^m)^n = p^{mn}$

Let us take two examples

$$\left[\left(\frac{4}{7}\right)^3\right]^6 = \left(\frac{4}{7}\right)^{3\times6} = \left(\frac{4}{7}\right)^{18}$$

$$\left[\left(\frac{2}{5}\right)^5\right]^4 = \left(\frac{2}{5}\right)^{5\times4} = \left(\frac{2}{5}\right)^{20}$$

7.6 ZERO EXPONENT

We have read that

(i) $p^m \div p^n = p^{m-n}$ if $m>n$

(ii) $p^m \div p^n = \frac{1}{p^{n-m}}$ if $m<n$

Let us consider the case when $m = n$

$\therefore p^m \div p^n = p^{m-n}$

$\Rightarrow \frac{p^m}{p^m} = p^{m-m} = p^0$

$\Rightarrow 1 = p^0$

$\therefore$ Any base expect 0 raised power 0 is 1

Law V: If p is a rational number other than zero than $p^0 = 1$

$\therefore a^0 = b^0 = q^0 = r^0 = (-10)^0 = 100^0 = (1020)^0 = \left(\frac{-5}{7}\right)^0 = 1$

Example 10: Find the values of: (i) $\left(\frac{3}{5}\right)^0$ (ii) $\left(\frac{-2}{9}\right)^0$

Solution: Using $p^0 = 1$ we get $\left(\frac{3}{5}\right)^0 = 1$ and $\left(\frac{-2}{9}\right)^0 = 1$

CHECK YOUR PROGRESS7.4

1. Fill in the blanks in the following questions:

(i) $\left(\frac{5}{7}\right)^8 \times \left(\frac{5}{7}\right)^3 =$.........

(ii) $(13)^7 \times (13)^6 \times (13)^5 =$ ………..

(iii) $\left(\frac{-3}{8}\right)^2 \times \left(\frac{-3}{8}\right)^1 \times \left(\frac{-3}{8}\right)^6 =$.........

2. Write True (T) or False (F) for the followings:

(i) $7^3 \times 7^2 = 7^6$ (ii) $\left(\frac{5}{13}\right)^6 \times \left(\frac{5}{13}\right)^7 = \left(\frac{5}{13}\right)^{13}$

(iii) $\left(\frac{8}{13}\right)^0 = 0$ (iv) $\left(\frac{-3}{2}\right)^2 = \frac{-9}{4}$

(v) $\left(\left(\frac{8}{17}\right)^5\right)^3 = \left(\frac{8}{17}\right)^8$ (vi) $\left(\left(\frac{3}{11}\right)^7\right)^0 = 0$

(vii) $((4)^3)^6) = (4)^9$ (viii) $\left(\left(\frac{11}{23}\right)^5\right)^3 = \left(\frac{11}{23}\right)^{15}$

(ix) $\left[\left(\frac{15}{23}\right)^3\right]^0 = 1$ (x) $\left(\frac{7}{5}\right)^5 \times \left(\frac{7}{5}\right)^{-5} = 1$

3. Choose the correct option:

(i) $\left(\frac{-5}{7}\right)^3 \times \left(\frac{-5}{7}\right)^2 \times \left(\frac{-5}{7}\right)^5 =$

(a) $\left(\frac{-5}{7}\right)^{30}$ (b) $\left(\frac{-5}{7}\right)^{11}$ (c) $\left(\frac{-5}{7}\right)^{10}$ (d) $\left(\frac{-5}{7}\right)^{13}$

(ii) $\left(\left(\frac{-15}{26}\right)^4\right)^3 =$

(a) $\left(\frac{-15}{26}\right)^7$ (b) $\left(\frac{-15}{26}\right)^{12}$ (c) $\left(\frac{-15}{26}\right)^{64}$ (d) $\left(\frac{-15}{26}\right)^1$

(iii) $\left(\frac{-7}{15}\right)^9 \times \left(\frac{-7}{15}\right)^{-4} \times \left(\frac{-7}{15}\right)^{-5} =$

(a) $\left(\frac{-7}{15}\right)^8$ (b) $\left(\frac{-7}{15}\right)^{18}$ (c) $\left(\frac{-7}{15}\right)^{-18}$ (d) 1

4. Simplify and express the result in exponential for:

(i) $(2^6)^2$ (ii) $\left[\left(\frac{3}{5}\right)^4\right]^5$ (iii) $\left(\frac{13}{7}\right)^7 \times \left(\frac{6}{13}\right)^0$ (iv) $\left(\frac{-5}{9}\right)^0 \times \left(\frac{-5}{9}\right)^2$

7.7 RADICALS

We know that

$5^2 = 25$ $\sqrt{25} = \sqrt{5 \times 5} = 5$ $25^{1/2} = 5$

$3^3 = 27$ $\sqrt[3]{27} = \sqrt[3]{3 \times 3 \times 3} = 3$ $27^{1/3} = 3$

$6^3 = 216$ $\sqrt[3]{216} = \sqrt[3]{6 \times 6 \times 6} = 6$ $(216)^{1/3} = 6$

$4^4 = 256$ $\sqrt[4]{256} = \sqrt[4]{4 \times 4 \times 4 \times 4} = 4$ $(256)^{1/4} = 4$

Can you imagine what is $\sqrt[4]{14641}$?

Think of a number x such that $x^4 = 14641 = (11)^4$

$\therefore x = 11 \Rightarrow \sqrt[4]{14641} = 11$ or $(14641)^{1/4} = 11$

Rule: if a is a positive real number and n be a positive integer then $\sqrt[n]{a} = b$, if $b^n = a$ and $b>0$.

Here the symbol "$\sqrt{}$ $\sqrt{}$ or" is called the radical sign we shall now consider some identities relating to square roots which are useful

If a and b are positive real numbers, then

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

(iv) $(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + (\sqrt{b})^2 + 2\left(\sqrt{a}\right)\left(\sqrt{b}\right) = a + b + 2\sqrt{ab}$

(v) $(\sqrt{a} - \sqrt{b})^2 = (\sqrt{a})^2 - (\sqrt{b})^2 - 2\left(\sqrt{a}\right)\left(\sqrt{b}\right)$

(vi) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{c} + \sqrt{d}\right) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$

Pure and mixed radical

$\sqrt{3}, \sqrt[3]{12}, \sqrt[4]{18}$ are such numbers which can be simplified further, such radicals are called Pure radical.

The radical of the type $\sqrt[3]{75}, \sqrt{20}, \sqrt[4]{162}$ can be expressed as

$$\sqrt[3]{75} = \sqrt{3\times5\times5} = 5\sqrt{3}$$

$$\sqrt{20} = \sqrt{2.2.5} = 2\sqrt{5}$$

$$\sqrt[4]{162} = \sqrt[4]{2.3.3.3.3} = 3\sqrt[4]{2}$$

Such radical which can be written as whole number and radical sign number are called mixed radicals. The radicals which cannot be converted into a mixed radicals are called pure radicals.

The radicals which are same are called like radicals.

7.7.1 Similar or Like radicals

Two radicals are said to similar or like radicals if they can be reduced to the same irrational factor without consideration of co-efficient. For example $3\sqrt{5}$ and $7\sqrt{5}$ are similar radicals. Again $\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$. So $\sqrt{75}$ and $\sqrt{12}$ are similar radicals.

$\sqrt{75} = 5\sqrt{3}$ and $\sqrt{50} = 5\sqrt{2}$ are not similar radicals.

7.7.2 Order of Radicals

For different radicals sign to find the order we convert radical sign into the same denominator. It enables us to find the order of the radicals.

Example 11: Which is greater $\sqrt{\frac{1}{5}}$ or $\sqrt[3]{\frac{1}{7}}$?

Solution: $\sqrt{\frac{1}{5}} = \left(\frac{1}{5}\right)^{1/2}$ and $\sqrt[3]{\frac{1}{7}} = \left(\frac{1}{7}\right)^{1/3}$

We equal the denominator of $\frac{1}{2}$ and $\frac{1}{3}$

L.C.M. of 2 and 3 is 6

$$\therefore \left(\frac{1}{5}\right)^{1/2} = \left(\frac{1}{5}\right)^{3/6} = \left[\left(\frac{1}{5}\right)^{3}\right]^{1/6} = \left(\frac{1}{125}\right)^{1/6}$$

$$\left(\frac{1}{7}\right)^{1/3} = \left(\frac{1}{7}\right)^{2/6} = \left[\left(\frac{1}{7}\right)^{2}\right]^{1/6} = \left(\frac{1}{49}\right)^{1/6}$$

We know $\frac{1}{49} > \frac{1}{125}$ we have $\left(\frac{1}{49}\right)^{1/6} > \left(\frac{1}{125}\right)^{1/6} \Rightarrow \sqrt[3]{\frac{1}{7}} > \sqrt{\frac{1}{5}}$

Example 12: Which is greater $\sqrt[4]{3}$ or $\sqrt[3]{10}$?

Solution: $\sqrt[4]{3} = 3^{1/4}$ and $\sqrt[3]{10} = 10^{1/3}$

L.C.M. of 4 and 3 is 12

$$\sqrt[4]{3} = 3^{1/4} = 3^{3/12} = (3^3)^{1/12} = \sqrt[12]{27}$$

$$\sqrt[3]{10} = 10^{1/3} = 10^{4/12} = (10^4)^{1/12} = (10000)^{1/12} = \sqrt[12]{10000}$$

$$\sqrt[12]{10000} > \sqrt[12]{27} \Rightarrow \sqrt[3]{10} > \sqrt[4]{3}$$

Example 13: Arrange the following in descending order of magnitude:

$$\sqrt[3]{3}, \sqrt[3]{4}, \sqrt[4]{5}$$

$$\sqrt[3]{3} = 3^{1/3}; \sqrt[3]{4} = 4^{1/3} \text{ and } \sqrt[4]{5} = 5^{1/4}$$

L.C.M. of 3, 3, 4 is 12

$$\therefore \sqrt[3]{3} = 3^{1/3} = 3^{4/12} = (3^4)^{1/12} = (81)^{1/12}$$

$$\sqrt[3]{4} = 4^{1/3} = 4^{4/12} = (4^4)^{1/12} = (256)^{1/12}$$

$$\sqrt[4]{5} = 5^{1/4} = 5^{3/12} = (5^3)^{1/12} = (125)^{1/12}$$

We know 256 > 125 > 81

$\therefore (256)^{1/2} > (125)^{1/12} > (81)^{1/12}$

$\sqrt[3]{4}, \sqrt[4]{5}, \sqrt[3]{3}$ are in descending order.

Rules of exponent for real numbers

We studied the following laws of exponent for integral exponents

(i) $a^m.a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$

(iii) $a^m \div a^n = a^{m-n} (m>n)$ (iv) $a^m.b^m = (ab)^m$

(v) $a^0 = 1$ (vi) $\frac{1}{a^n} = a^{-n}$

The above rules are valid when base is a positive real number and exponents are rational numbers.

The following questions give us illustration of above rules.

Example 14: Simplify each of the following:

(i) $(125)^{1/3}$ (ii) $2^{1/3} \times 2^{3/5}$ (iii) $(2^{1/3})^4$ (iv) $\frac{7^{3/4}}{49^{1/4}}$

(v) $3^{1/4}.25^{1/8}$ (vi) $8^{1/3} \div 2$ (vii) $\sqrt[4]{\sqrt[3]{5}}$ (viii) $\frac{\sqrt[4]{16}}{\sqrt[4]{81}}$

Solution: (i) $(125)^{1/3} = (5^3)^{1/3} = 5^{3 \times 1/3} = 5^1 = 5$

(ii) $2^{1/3} \times 2^{3/5} = 2^{\frac{1}{3}+\frac{3}{5}} = 2^{\frac{5+9}{15}} = 2^{\frac{14}{15}} = \sqrt[15]{2^{14}}$

(iii) $(2^{1/3})^4 = 2^{\frac{1}{3} \times 4} = 2^{\frac{4}{3}} = \sqrt[3]{2^4} = 2.\sqrt[3]{2}$

(iv) $\frac{7^{3/4}}{49^{1/4}} = \frac{7^{3/4}}{\left(7^2\right)^{1/4}} = \frac{7^{3/4}}{7^{1/2}} = 7^{\frac{3}{4}-\frac{1}{2}} = 7^{\frac{1}{4}}$

(v) $3^{1/4}.(25)^{1/8} = (3)^{1/4}.(5^2)^{1/8} = 3^{1/4}.5^{1/4} = (3 \times 5)^{1/4} = \sqrt[4]{15}$

(vi) $8^{1/3} \div 2 = (2^3)^{1/3} \div 2 = 2^{3 \cdot \frac{1}{3}} \div 2 = 2 \div 2 = 1$

(vii) $\sqrt[4]{\sqrt[3]{5}} = \left(\sqrt[3]{5}\right)^{\frac{1}{4}} = (5^{1/3})^{1/4} = 5^{1/12} = \sqrt[12]{5}$

(viii) $\dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \left(\dfrac{16}{81}\right)^{1/4} = \left(\dfrac{2^4}{3^4}\right)^{1/4} = \left[\left(\dfrac{2}{3}\right)^4\right]^{1/4} = \left(\dfrac{2}{3}\right)^{4\times\frac{1}{4}} = \left(\dfrac{2}{3}\right)^1 = \left(\dfrac{2}{3}\right)$

7.7.3 Four fundamental operation on radicals

1. Addition and subtraction of radicals

The similar radicals are added and subtracted.

Example 15: (i) add $\sqrt{50}$ and $\sqrt{288}$

(ii) Subtract $\sqrt{18}$ from $\sqrt{98}$

Solution: (ii) $\sqrt{50} = \sqrt{5\times5\times2} = 5\sqrt{2}$

$\sqrt{288} = \sqrt{2\times2\times2\times2\times2\times3\times3} = 12\sqrt{2}$

$\sqrt{50} + \sqrt{288} = 5\sqrt{2} + 12\sqrt{2} = \sqrt{2}(5+12) = 17\sqrt{2}$

(ii) $\sqrt{98} = \sqrt{7\times7\times2} = 7\sqrt{2}$

$\sqrt{18} = \sqrt{3\times3\times2} = 3\sqrt{2}$

$\sqrt{98} - \sqrt{18} = 7\sqrt{2} - 3\sqrt{2} = \sqrt{2}(7-3) = 4\sqrt{2}$

Example 16: Simplify each of the following:

(i) Add $\left(2\sqrt{3}+3\sqrt{2}\right)$ and $\left(\sqrt{2}-3\sqrt{3}\right)$

(ii) Subtract $\left(3\sqrt{2}+\sqrt{5}\right)$ from $\left(3\sqrt{5}-2\sqrt{2}\right)$

Solution (i) $\left(2\sqrt{3}+3\sqrt{2}\right)+\left(\sqrt{2}-3\sqrt{3}\right) = 2\sqrt{3}+3\sqrt{2}+\sqrt{2}-3\sqrt{3}$

$= 2\sqrt{3}-3\sqrt{3}+3\sqrt{2}+\sqrt{2}$

$= \sqrt{3}(2-3)-\sqrt{2}(3+1)$

$= -\sqrt{3}+4\sqrt{2}$

$= \left(2\sqrt{3}+3\sqrt{2}\right)+\left(\sqrt{2}-3\sqrt{3}\right) = -\sqrt{3}+4\sqrt{2}$

(ii) $\left(3\sqrt{5}-2\sqrt{2}\right)-\left(3\sqrt{2}+\sqrt{5}\right) = 3\sqrt{5}-2\sqrt{2}-3\sqrt{2}-\sqrt{5}$

$= 3\sqrt{5}-\sqrt{5}-2\sqrt{2}-3\sqrt{2}$

$= \sqrt{5}(3-1)-\sqrt{2}(2+3)$

$= 2\sqrt{5}-5\sqrt{2}$

$\left(3\sqrt{5}-2\sqrt{2}\right)-\left(3\sqrt{2}+\sqrt{5}\right) = 2\sqrt{5}-5\sqrt{2}$

7.7.4 Multiplication and division of radicals

The number under the same root can be multiplied or divided.

$\sqrt{3}$ and $\sqrt{5}$ can be multiplied as both are under same root $\sqrt[3]{4}$ and $\sqrt[5]{7}$ cannot be multiplied as the root of 4 is 3^{rd} and 7 is 5^{th} root. These can be multiplied only when radicals are changed to same root.

Similar is the case with division.

Example 17: Multiply: (i) $2\sqrt{3}$ by $3\sqrt{2}$ (ii) $2\sqrt{5}$ by $3\sqrt{5}$

Solution (i) $2\sqrt{3}\times3\sqrt{2} = 2\times3\times\sqrt{3\times2} = 6\sqrt{6}$

(ii) $2\sqrt{5}$ by$3\sqrt{5} = 2\times3\times\sqrt{5\times5} = 2\times3\times5 = 30$

Example 18: Multiply $5\sqrt{3}$ by $3\sqrt[5]{8}$

Solution: $\sqrt{3}$ is square root and $\sqrt[5]{8}$ is fifth root of 8. We cannot multiply directly.

$\sqrt{3} = 3^{1/2}$ and $\sqrt[5]{8} = 8^{1/5}$

L.C.M. of 2 and 5 is 10

$\therefore\ 3^{1/2} = 3^{5/10} = \sqrt[10]{3^5} = \sqrt[10]{243}$

$8^{1/5} = 8^{2/10} = \sqrt[10]{8^2} = \sqrt[10]{64}$

$\therefore\ 5\sqrt{3} \times 3\sqrt[5]{8} = 5\sqrt[10]{243} \times 3\sqrt[10]{64}$

Root is same. So we can multiply $= 5\times 3 \times \sqrt[10]{243\times 64} = 15\sqrt[10]{3^5 \times 2^6}$

Example 19: Divide: (i) $8\sqrt{6}$ by $2\sqrt{3}$ (ii) $12\sqrt{6}$ by $\left(\sqrt{2}\times\sqrt{3}\right)$

Solution (i) $8\sqrt{6} \div 2\sqrt{3} = \dfrac{8\sqrt{6}}{2\sqrt{3}}$

6 and 3 have same root

$$= \frac{8}{2}.\sqrt{\frac{6}{3}} = 4\sqrt{2}$$

(ii) $\left(12\sqrt{6}\right) \div \left(\sqrt{2}\times\sqrt{3}\right) = \dfrac{12\sqrt{6}}{\sqrt{2}\times\sqrt{3}} = \dfrac{12\sqrt{6}}{\sqrt{6}} = 12$

Example 20: Multiply $8\sqrt[3]{25}$ by $11\sqrt[6]{32}$

Solution: $8\sqrt[3]{25} = 8(25)^{1/3} = 8(25)^{2/6} = 8\sqrt[6]{625} = 8\sqrt[6]{25^2}$

$11\sqrt[6]{32} = 11\sqrt[6]{32}$

So $8\sqrt[3]{25} = 8\sqrt[6]{625}$ and $11\sqrt[6]{32}$ have same root

$8\sqrt[3]{25} \times 11\sqrt[6]{32} = 8\sqrt[6]{625} \times 11\sqrt[6]{32} = 88\sqrt[6]{625} \times \sqrt[6]{32}$

$8\sqrt[3]{25} \times 11\sqrt[6]{32} = 8\sqrt[6]{5^5 \times 2^5} = 88\sqrt[6]{10^5}$

Example 21: Divide $15\sqrt[3]{13}$ by $6\sqrt[6]{5}$

Solution: $15\sqrt[3]{13} \div 6\sqrt[6]{5} = \dfrac{15\sqrt[3]{13}}{6\sqrt[6]{5}} = \dfrac{5}{2}\dfrac{13^{1/3}}{5^{1/6}}$

$= \dfrac{5}{2}\dfrac{(13)^{2/6}}{(5)^{1/6}} = \dfrac{5}{2}\dfrac{(13^2)^{1/6}}{(5)^{1/6}} = \dfrac{5}{2}\dfrac{(169)^{1/6}}{5^{1/6}}$

$= \dfrac{5}{2}\left(\dfrac{169}{5}\right)^{1/6} = \dfrac{5}{2}\sqrt[6]{\dfrac{169}{5}}$

Example 22: Simplify: $12\sqrt{18} + 6\sqrt{20} + 6\sqrt{147} + 3\sqrt{50} + 8\sqrt{45}$

Solution: $12\sqrt{18} + 6\sqrt{20} + 6\sqrt{147} + 3\sqrt{50} + 8\sqrt{45}$

$= 12\sqrt{3.3.2} + 6\sqrt{2.2.5} + 6\sqrt{7.7.3} + 3\sqrt{2.5.5} + 8\sqrt{3.3.5}$

$= 36\sqrt{2} + 12\sqrt{5} + 42\sqrt{3} + 15\sqrt{2} + 24\sqrt{5}$

$= 36\sqrt{2} + 15\sqrt{2} + 42\sqrt{3} + 12\sqrt{5} + 24\sqrt{5}$

$= \sqrt{2}(36+15) + 42(\sqrt{3} + \sqrt{5}(12+24)$

$= 51\sqrt{2} + 42\sqrt{3} + 36\sqrt{5}$

CHECK YOUR PROGRESS7.5

1. Use >, < or = sign in the blank space:

   (i) $\sqrt[3]{4}$........$\sqrt[4]{5}$ (ii) $\sqrt[4]{3}$........$\sqrt[3]{10}$ (iii) $\sqrt[3]{6}$........$\sqrt{8}$

   (iv) $\sqrt[3]{3}$........$\sqrt[4]{4}$

2. Arrange the following in ascending order of magnitude: $\sqrt[3]{2}$, $\sqrt[4]{3}$, $\sqrt[3]{4}$

3. Arrange the following in descending order of magnitude: $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{6}$

4. Simplify each of the following:

   (i) $\sqrt{32} + \sqrt{20} - \sqrt{128}$

(ii) $\sqrt[3]{24}+\sqrt[3]{81}-8\sqrt[3]{3}+2\sqrt[3]{375}$

(iii) $6\sqrt[3]{54}-2\sqrt[3]{16}+4\sqrt[3]{128}-5\sqrt[3]{8}$

(iv) $\sqrt[3]{432}-6\sqrt[3]{250}$

(v) $12\sqrt{18}+6\sqrt{20}-6\sqrt{147}+30\sqrt{50}+8\sqrt{45}$

5. Multiply the following :

(i) $5\sqrt[3]{54}$ by $8\sqrt[3]{40}$ (ii) $2\sqrt{50}\times\sqrt{18}\times 2\sqrt{98}$

6. Divide : (i) $\sqrt[3]{128}\div\sqrt[3]{432}$ (ii) $\sqrt[4]{405}\div\sqrt[4]{112}$

7.7.5 Rationalisation of Radicals

We have already read that:

(i) $5^{1/2}\times 5^{1/2} = 5^{1/2+1/2} = 5^1 = 5$ (ii) $9^{5/13}\times 9^{8/13} = 9^{\frac{5+8}{13}} = 9^1 = 9$

(iii) $15^{2/3}\times 15^{1/3} = 15^{\frac{2}{3}+\frac{1}{3}} = 15^{\frac{3}{3}} = 15$

In the above cases we see that on multiplying two radicals, we get the result as rational number.

In such cases each radical is called the rationalizing factor of the other radical.

(i) $\sqrt{7}$ is a rationalizing radical of $\sqrt{7}$

(ii) $\sqrt[9]{13^5}$ is rationalizing radical of $\sqrt[9]{13^4}$ and vice versa.

(iii) $\sqrt[4]{15}$ is a rationalizing factor of $\sqrt[4]{15^3}$ and vice versa.

In short the method of converting radical to radical number is called rationalization and two numbers when multiplied gives the rational number is called the rationalizing factor of other. For example, the rationalizing factor of $\sqrt{y}$ is $\sqrt{y}$ and of $\sqrt{5}-\sqrt{3}$ is $\sqrt{5}+\sqrt{3}$

Remember:

(i) The quantities $a-\sqrt{b}$ and $a+\sqrt{b}$ are called conjugate radicals. Their sum and product is always rational.

i.e. $a-\sqrt{b}+a+\sqrt{b}=a+a=2a$ a rational number

$(a-\sqrt{b})(a+\sqrt{b})=(a)^2-(\sqrt{b})^2=a^2-b$ a rational number

(ii) Rationalization is usually done of the denominator of an expression involving irrational radical. We are able to understand it more clearly by taking a few example.

Example 23: Rationalize the denominator of the following:

(i) $\frac{1}{\sqrt{5}}$ (ii) $\frac{1}{\sqrt{18}}$ (iii) $\frac{1}{\sqrt{3}-\sqrt{5}}$ (iv) $\frac{1}{\sqrt{7}-2}$

Solution:(i) $\frac{1}{\sqrt{5}}.\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$

(ii) $\frac{1}{\sqrt{18}}=-\frac{1}{\sqrt{9\times 2}}=\frac{1}{3\sqrt{2}}$

The rationalizing factor of $3\sqrt{2}$ is $\sqrt{2}$

$\therefore \frac{1}{\sqrt{18}}=\frac{1}{3\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{3\sqrt{2^2}}=\frac{\sqrt{2}}{3\times 2}=\frac{\sqrt{2}}{6}$

(iii) The rationalizing factor of $\sqrt{3}-\sqrt{5}$ is $\sqrt{3}+\sqrt{5}$

$\therefore \frac{1}{\sqrt{3}-\sqrt{5}}=\frac{1}{\sqrt{3}-\sqrt{5}}\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}}=\frac{\sqrt{3}+\sqrt{5}}{(\sqrt{3})^2-(\sqrt{5})^2}=\frac{\sqrt{3}+\sqrt{5}}{3-5}$

$=\frac{\sqrt{3}+\sqrt{5}}{-2}=\frac{-1}{2}=\sqrt{3}+\sqrt{5}$

(iv) Rationalising factor of $\sqrt{7}-2$ is $\sqrt{7}+2$

$$\therefore \frac{1}{\sqrt{7}-2}=\frac{1}{\sqrt{7}-2}\times\frac{\sqrt{7}+2}{\sqrt{7}+2}=\frac{\sqrt{7}+2}{(\sqrt{7})^2+2^2}=\frac{\sqrt{7}+2}{7-4}=\frac{\sqrt{7}+2}{3}$$

$$=\frac{1}{3}(\sqrt{7}+2)$$

Example 24: Reduce $\frac{7+5\sqrt{3}}{7-5\sqrt{3}}$ into a fraction with a rational denominator.

Solution: The rationalizing factor of $7-5\sqrt{3}$ is $7+5\sqrt{3}$

$$\therefore \frac{7+5\sqrt{3}}{7-5\sqrt{3}}\times\frac{7+5\sqrt{3}}{7+5\sqrt{3}}=\frac{(7+5\sqrt{3})^2}{(7)^2-(5\sqrt{3})^2}$$

$$=\frac{49+75+70\sqrt{3}}{49-75}=\frac{124+70\sqrt{3}}{-26}$$

$$\frac{2(62+35\sqrt{3})}{-26}=-\frac{1}{13}(62+35\sqrt{3})$$

Example 25: Rationalise $\frac{1}{1+\sqrt{2}-\sqrt{3}}$ into a fraction with a rational denominator.

Solution: Here $\frac{1}{1+\sqrt{2}-\sqrt{3}}=\frac{1}{(1+\sqrt{2})-\sqrt{3}}$

The rationalizing factor of $(1+\sqrt{2})-\sqrt{3}$ is $(1+\sqrt{2})+\sqrt{3}$

$$=\frac{1}{1+\sqrt{2}-\sqrt{3}}=\frac{1}{(1+\sqrt{2})-\sqrt{3}}\times\frac{(1+\sqrt{2})+\sqrt{3}}{(1+\sqrt{2})+\sqrt{3}}=\frac{(1+\sqrt{2})+\sqrt{3}}{(1+\sqrt{2})^2+\sqrt{3}^2}$$

$$=\frac{1+\sqrt{2}+\sqrt{3}}{(1+2+2\sqrt{2})-3}=\frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}$$

Again rationalizing the denominator

$$=\frac{1+\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}+2+\sqrt{6}}{4}=\frac{2}{4}+\frac{\sqrt{2}+\sqrt{6}}{4}=\frac{1}{2}+\frac{\sqrt{2}+\sqrt{6}}{4}$$

Example 26: If $x = 3+\sqrt{5}$ find the value of $x+\frac{1}{x}$

Solution: $x = 3+\sqrt{5}$

$$\Rightarrow \frac{1}{x} = \frac{1}{3+\sqrt{5}} = \frac{1}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{3-\sqrt{5}}{(3)^2-(\sqrt{5})^2}$$

$$= \frac{3-\sqrt{5}}{9-5} = \frac{3-\sqrt{5}}{4}$$

$$\therefore x+\frac{1}{x} = \frac{3+\sqrt{5}}{1} = \frac{3-\sqrt{5}}{4}$$

$$= \frac{12+4\sqrt{5}+3-\sqrt{5}}{4} = \frac{15+\sqrt{5}(4-1)}{4} = \frac{15+3\sqrt{5}}{4}$$

Example 27: Show that $\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2} = 5$

Solution: $\frac{1}{3-\sqrt{8}} = \frac{1}{3-\sqrt{8}} \times \frac{3+\sqrt{8}}{3-\sqrt{8}} = \frac{3+\sqrt{8}}{(3)^2-(\sqrt{8})^2} = \frac{3+\sqrt{8}}{9-8} = \frac{3+\sqrt{8}}{1}$

Similarly by $\frac{1}{\sqrt{8}-\sqrt{7}} = \frac{1}{\sqrt{8}-\sqrt{7}} \times \frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}-\sqrt{7}}$

$$= \frac{\sqrt{8}+\sqrt{7}}{(\sqrt{8})^2-(\sqrt{7})^2} = \frac{\sqrt{8}+\sqrt{7}}{1}$$

Again $\frac{1}{\sqrt{7}-\sqrt{6}} = \sqrt{7}+\sqrt{6}$

$$\frac{1}{\sqrt{6}-\sqrt{5}} = \sqrt{6}+\sqrt{5}$$

$$\frac{1}{\sqrt{5}-2} = \sqrt{5}+2$$

$\therefore$ LHS of given expression =

$$(3+\sqrt{8}) - (\sqrt{8}+\sqrt{7}) + (\sqrt{7}+\sqrt{6}) - (\sqrt{6}+\sqrt{5}) + (\sqrt{5}+2)$$

$$= 3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2$$

$$=3+2$$

$$=5=\text{RHS}$$

Example 28: If $\frac{4+3\sqrt{2}}{4-\sqrt{2}}=x+y\sqrt{2}$ find the values of x and y.

Solution: $\frac{4+3\sqrt{2}}{4-\sqrt{2}}=\frac{4+3\sqrt{2}}{4-\sqrt{2}}\times\frac{4+\sqrt{2}}{4+\sqrt{2}}$

$$=\frac{16+4\sqrt{2}+12\sqrt{2}+6}{(4)^2-(\sqrt{2})^2}$$

$$=\frac{22+16\sqrt{2}}{16-2}=\frac{2(11+8\sqrt{2})}{2\times 7}$$

$$=\frac{11+8\sqrt{2}}{7}=\frac{11}{7}+\frac{8\sqrt{2}}{7}$$

$$\therefore \frac{11}{7}+\frac{8\sqrt{2}}{7}=x+y\sqrt{2}$$

$$\therefore x=\frac{11}{7} \text{ and } y=\frac{8}{7}$$

Example 29: If $x=5+2\sqrt{6}$ find the value of $x^2+\frac{1}{x^2}$

Solution: $x=5+2\sqrt{6}$

$$\therefore \frac{1}{x}=\frac{1}{5+2\sqrt{6}}$$

Rationalising $\frac{1}{5+2\sqrt{6}}$

$$\frac{1}{x}=\frac{1}{5+2\sqrt{6}}=\frac{1}{5+2\sqrt{6}}\times\frac{5-2\sqrt{6}}{5-2\sqrt{6}}$$

$$= \frac{5-2\sqrt{6}}{(5)^2-(2\sqrt{6})^2}$$

$$= \frac{5-2\sqrt{6}}{25-24} = \frac{5-2\sqrt{6}}{1}$$

$$\therefore x+\frac{1}{x} = (5+2\sqrt{6})+(5-2\sqrt{6})$$

$$\left(x+\frac{1}{x}\right)^2 = (10)^2$$

$$x^2+\frac{1}{x^2}+2 = 100$$

$$x^2+\frac{1}{x^2} = 100-2 = 98$$

Example 30: $\frac{7+\sqrt{5}}{7-\sqrt{5}}-\frac{7-\sqrt{5}}{7+\sqrt{5}} = a+\frac{7}{11}\sqrt{5}b$

Solution: $\frac{7+\sqrt{5}}{7-\sqrt{5}} = \frac{7+\sqrt{5}}{7-\sqrt{5}}\times\frac{7+\sqrt{5}}{7+\sqrt{5}} = \frac{(7+\sqrt{5})^2}{(7)^2-(\sqrt{5})^2}$

$$= \frac{49+5+14\sqrt{5}}{49-5} = \frac{54+14\sqrt{5}}{44}$$

Again $\frac{7-\sqrt{5}}{7+\sqrt{5}} = \frac{7-\sqrt{5}}{7-\sqrt{5}}\times\frac{7-\sqrt{5}}{7+\sqrt{5}} = \frac{(7-\sqrt{5})^2}{(7)^2-(\sqrt{5})^2}$

$$= \frac{49+5-14\sqrt{5}}{49-5} = \frac{54-14\sqrt{5}}{44}$$

$$\therefore\therefore \frac{7+\sqrt{5}}{7-\sqrt{5}}-\frac{7-\sqrt{5}}{7+\sqrt{5}} = \frac{54+14\sqrt{5}}{44}-\frac{54-14\sqrt{5}}{44}$$

$$= \frac{54+14\sqrt{5}-54+14\sqrt{5}}{44} = \frac{28\sqrt{5}}{44} = \frac{7\sqrt{5}}{11}$$

$$\therefore a+\frac{7}{11}\sqrt{5}b=0+\frac{7}{11}\sqrt{5}$$

by comparing we get $a = 0$, $b = 1$

CHECK YOUR PROGRESS7.6

1. Write the rationalizing factor of each of following:

   (i) $\sqrt[3]{25}$ (ii) $\sqrt{3}+2$ (iii) $\sqrt[5]{x^2}+\sqrt[5]{(y^2)}$

2. Simplifying by rationalizing the denominator of the following:

   (i) $\frac{5\sqrt{7}}{\sqrt{15}}$ (ii) $\frac{3}{\sqrt{11}}$ (iii) $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}+\sqrt{3}}$ (iv) $\frac{\sqrt{11}+3}{\sqrt{11}-3}$

3. Simplify : $\frac{3+\sqrt{5}}{3-\sqrt{5}}+\frac{3-\sqrt{5}}{3+\sqrt{5}}$

4. Rationalise the denominator of $\frac{1}{(\sqrt{3}+\sqrt{5})-\sqrt{8})}$

5. If $x = 2 + \sqrt{3}$ find the value of :

   (i) $x+\frac{1}{x}$ (ii) $x^2+\frac{1}{x^2}$

6. If $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}=x+y\sqrt{15}$ find the value of x and y.

7. If $\sqrt{5}$ = 2.236 find the value of $\frac{\sqrt{15}-\sqrt{3}}{2\sqrt{3}}$

7.7 RECAPITULATION POINTS

(a) $b \times b \times b \ldots n$ times $= b^n$ is called exponential form where base is b and n is exponent of power.

(b) Laws of exponents:

(i) $b^m \times b^n = b^{m+n}$

(ii) $\frac{b^m}{b^n} = b^{m-n}$

(iii) $(b^m)^n = b^{mn}$

(iv) $(be)^n = b^n.e^{-n}$

(v) $\left(\frac{b}{c}\right)^n = \frac{b^n}{c^n}$

(vi) $b^0 = 1$

(vii) $b^{-n} = \frac{1}{b^n}$

(viii) $\frac{1}{e^n} = e^{-n}$

(ix) $b^{\frac{p}{q}} = \sqrt[q]{b^p}$

(c) $\sqrt[n]{y}$ is called radical where y is a rational number and nth root of y is not a rational number.

(d) In $\sqrt[n]{y}$, n is called index and y is called radical.

(e) A radical which can be written as a whole number and radical sign number is called a mixed radical.

(f) The order of the radical is the number that indicates the root. i.e. the order of $\sqrt[n]{y}$ is n.

(g) Laws of radicals (b> 0, c> 0)

(i) $(\sqrt[n]{b})^n = b$ (ii) $\sqrt[n]{b} + \sqrt[n]{c} = \sqrt[n]{bc}$ (iii) $\frac{\sqrt[m]{b}}{\sqrt[m]{c}} = \sqrt[m]{\frac{b}{c}}$

(h) Operation on radicals

(i) $\sqrt[n]{x}.\sqrt[n]{y} = x^{1/n}.y^{1/n} = (xy)^{1/n}$ (ii) $(y^{m})^{\frac{1}{n}} = y^{\frac{1}{mn}} = (y^{n})^{\frac{1}{m}}$

(iii) $\left(\dfrac{y^{1/m}}{x^{1/m}}\right) = \left(\dfrac{y}{x}\right)^{1/m}$ (iv) $(y^{m})^{1/n} = y^{m/n}$

(v) $(y^{a})^{1/m} = y^{a/m} = y^{an/mn} = (y^{an})^{1/mn}$

(i) Radicals are similar if they have the same irrational factor.

(j) Radical of the same order are multiplied and divided.

(k) For comparison of radicals we change radicals to radicals of same order. Then the radicals are compared according to their radical and along with co-efficient.

(l) If the product of two radicals is rational, Each is called the rationalizing factor of the other.

(m) $\sqrt{b}+\sqrt{c}$ is called rationalizing factor of $\sqrt{b}-\sqrt{c}$ and vice versa.

(n) To rationalize the denominator $\dfrac{1}{a+b\sqrt{x}}$ we multiply it by $\dfrac{a-b\sqrt{x}}{a-b\sqrt{x}}$ where a and b are integers.

TERMINAL EXERCISE

1. Write the following in exponential form:

(i) 7 × 11 × 11 × 7 ×7 × 8 × 8 × 11 (ii) $\left(\dfrac{-7}{3}\right)\left(\dfrac{-7}{3}\right)\left(\dfrac{-7}{3}\right)\left(\dfrac{-7}{3}\right)\left(\dfrac{-7}{3}\right)$

(iii) $\left(\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{5}{6}\times\dfrac{5}{6}\right)\times\dfrac{25}{36}$

2. Simplify the following

(i) $\left(\dfrac{-5}{9}\right)^{3}\times\left(\dfrac{3}{7}\right)^{2}\times\left(\dfrac{7}{5}\right)^{5}$ (ii) $\left(\dfrac{5}{9}\right)^{2}\times\dfrac{36}{125}\times\left(-\dfrac{3}{8}\right)^{3}$

3. Simplify and express the result in exponential form:

(i) $(48)^3 \times (6)^3 \times (12)^4$ (ii) $\left(\frac{-23}{19}\right)^{15} \div \left(\frac{-23}{19}\right)^{15}$ (iii) $\left(\left(\frac{5}{11}\right)^5\right)^3$

4. Simplify each of the following:

(i) $\left[\left(\frac{2}{5}\right)^5 \div \left(\frac{2}{5}\right)^5\right] + \left[7^4 \div 7^4\right] - \left[\left(\frac{4}{11}\right)^{-4} \div \left(\frac{4}{11}\right)^{-4}\right]$

(ii) $(5^0 + 11^0 + 13^0)(17^0 + 7^0 - 6^0)$

5. Simplify each of the following:

(i) $(81)^{-7} \div (81)^{-11}$ (ii) $(97)^7 \div (97)^{-4}$ (iii) $\left(\frac{-5^5}{11}\right) \times \left(\frac{-5}{11}\right)^{-7}$

6. Find the value of y in the following:

(i) $\left(\frac{3}{5}\right)^{-7} \times \left(\frac{3}{5}\right)^{13} = \left(\frac{3}{5}\right)^{y}$ (ii) $\left(\frac{5}{11}\right)^{-4} \times \left(\frac{5}{11}\right)^{-7} = \left(\frac{5}{11}\right)^{2y+1}$

7. Express the following as a product of primes and write the answer in exponential form:

(i) 1296000 (ii) 227276 (iii) 57624

8. Express as pure radical:

(i) $2\sqrt[5]{2}$ (ii) $7\sqrt[3]{4}$ (iii) $2\sqrt[3]{5}$

9. Write the following as missed radicals:

(i) $\sqrt[4]{2025}$ (ii) $\sqrt[4]{640}$ (iii) $\sqrt[4]{729}$

10. Which of the following radicals pairs are similar?

(i) $\sqrt{245}, \sqrt{605}$ (ii) $\sqrt[3]{2000}, \sqrt[3]{432}$ (iii) $\sqrt[6]{216}, \sqrt{250}$

11. Simplify: (i) $\sqrt{63} + \sqrt{28} - \sqrt{175}$ (ii) $\sqrt{8} + \sqrt{128} - \sqrt{50}$

Tick the Correct Option In Q. 12, 13, 14, 15

12. The value of $\frac{\sqrt{729}+\sqrt{441}}{\sqrt{729}-\sqrt{441}}$ is:

(i) $\sqrt{\frac{1170}{228}}$ (ii) 8 (iii) $\frac{1}{8}$ (iv) 80

13. If $\sqrt{3}$ = 1.732 and $\sqrt{2}$ = 1.414 the value of $\frac{1}{\sqrt{3}-\sqrt{2}}$ is:

(i) 0.318 (ii) 3.146 (iii) $\frac{1}{3.146}$ (iv) $\sqrt{1.732}-\sqrt{1.414}$

14. Which of the following is equal to *x*:

(i) $(\sqrt[3]{x})^{\frac{2}{3}}$ (ii) $x^{5/12}.x^{12/5}$ (iii) $x^{12/5}-x^{7/5}$ (iv) $\sqrt[12]{(x^4)^{\frac{1}{3}}}$

15. After rationalizing $\frac{3}{3\sqrt{3}-2\sqrt{3}}$ we get the denominator as:

(i) 5 (ii) 13 (iii) 19 (iv) 36

Fill in the Blanks in Q. No. 16 to 20.

16. If $\sqrt{2}$ = 1.414 the value of $\frac{\sqrt{10}-\sqrt{5}}{2\sqrt{5}}$ is _______

17. If $\sqrt{5}$ = 2.236 the value of $(\sqrt{7}\times\sqrt{35})$ is _______

18. $\frac{5\sqrt{3}}{\sqrt{5}}$ can be written as _______ with rational denominator.

19. If $x = 2+\sqrt{3}$ the value of $x+\frac{1}{x}$ is _______

20. The value of $\sqrt[4]{5^3}\div\sqrt{5}$ is pure radical form is _______

State True/False in Q. No. 21 to 24.

21. $\sqrt{3}\times(2\sqrt{2}+\sqrt{3})=2\sqrt{6}+3$

22. $3\sqrt{54} \div 2\sqrt{6} = \frac{9}{2}\sqrt{6}$

23. $\sqrt[3]{625} \div \sqrt[3]{5} = \frac{1}{5}$

24. $\frac{1}{\sqrt{3}+\sqrt{12}} = -\frac{\sqrt{3}}{9}$

25. Which is greater:

(i) $\sqrt{3}\ or \sqrt[3]{5}$ (ii) $\sqrt[3]{6}$ or $\sqrt[4]{8}$

26. Write the following in ascending order:

(i) $\sqrt{5}, \sqrt[3]{7}, \sqrt[4]{11}$ (ii) $\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{5}$

27. Write the following in descending order: $\sqrt[3]{16}, \sqrt{12}, \sqrt[6]{320}$

28. Simplify the denominator by rationalizing:

(i) $\frac{3}{7+2\sqrt{5}}$ (ii) $\frac{5}{\sqrt{2}-\sqrt{7}}$ (iii) $\frac{\sqrt{10}-3}{\sqrt{10}+3}$ (iv) $\frac{1}{\left(\sqrt{3}+\sqrt{5}\right)-\sqrt{5}}$

29. Simplify: $\frac{7\sqrt{3}}{\sqrt{10}+3} - \frac{2\sqrt{5}}{\sqrt{6}+\sqrt{5}} - \frac{3\sqrt{2}}{\sqrt{15}+3\sqrt{2}}$

30. If $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = x + y\sqrt{15}$ find the value of x and y.

31. If $x = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}-\sqrt{2}}, y = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ find $x^2 + y^2$.

32. If $x = \frac{1}{7+4\sqrt{3}}, y = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ show that $x^3 - 4x + 1 = 0$ and hence evaluate $x^3 - 9x^2 - 69x + 5$.

7.8 CASE STUDY-I

Ateacher divided X class students into four groups A, B, C & D. Each group consists of bright, average bright, average and weak students in Maths. All the groups appeared almost of same level. The teacher told them that they will be tested in the chapter of rational exponent. Each student is to answer only one question. No consultation in the classroom is allowed. The group getting maximum marks will get group name written in the corner of blackboard for a weak. The group members participating were helping each other to understand the topic. The following questions were asked and the answer of different groups are written below the question indicating group as A, B, C & D. Correct answer carries three marks and 1 mark is deducted for wrong answer.

1. Convert the following pure radicals into mixed radicals: $\sqrt[3]{24000}$

   (a) $10\sqrt[3]{24}$ (b) $20\sqrt[3]{60}$ (c) $20\sqrt[3]{3}$ (d) $20\sqrt{60}$

2. The value of $\dfrac{\sqrt{121}+\sqrt{289}}{\sqrt{289}-\sqrt{121}}$

   (a) $\sqrt{\dfrac{410}{168}}$ (b) $\dfrac{14}{3}$ (c) $\dfrac{3}{14}$ (d) $\sqrt{\dfrac{168}{410}}$

3. If $\sqrt{3}$ = 1.732 and $\sqrt{5}$ = 2.236, the value of $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ is:

   (a) 1.984 (b) 3.968 (c) 0.504 (d) $\dfrac{1}{1.984}$

4. Which is greatest $\sqrt[3]{6}$ or $\sqrt[4]{11}$ or $\sqrt[6]{31}$ or $\sqrt[12]{1071}$?

   (a) $\sqrt[112]{1071}$ (b) $\sqrt[6]{31}$ (c) $\sqrt[3]{6}$ (d) $\sqrt[4]{11}$

5. Simplify the following & write the result in pure radical form

   $\sqrt{27}+\sqrt{48}-\sqrt{108}+\sqrt{75}$

   (a) $\sqrt{75}$ (b) $\sqrt{108}$ (c) $\sqrt{72}$ (d) $\sqrt{48}$

6. Write in descending order $\left(\left(\dfrac{6}{7}\right)^4\right)^3$ or $\left(\dfrac{6}{7}\right)^{4^3}$ or $\left(\left(\dfrac{6}{7}\right)^5\right)^2$ or $\left(\dfrac{6}{7}\right)^{5^2}$

   (a) $\left(\dfrac{6}{7}\right)^{5^2} > \left(\left(\dfrac{6}{7}\right)^4\right)^3 > \left(\dfrac{6}{7}\right)^{4^3} > \left(\left(\dfrac{6}{7}\right)^5\right)^2$

(b) $\left(\frac{6}{7}\right)^{4^3} > \left(\left(\frac{6}{7}\right)^5\right)^2 > \left(\left(\frac{6}{7}\right)^4\right)^3 \times \left(\frac{6}{7}\right)^{5^2}$

(c) $\left(\left(\frac{6}{7}\right)^4\right)^3 \times \left(\left(\frac{6}{7}\right)^5\right)^2 > \left(\frac{6}{7}\right)^{4^3} > \left(\frac{6}{7}\right)^{5^2}$

(d) $\left(\left(\frac{6}{7}\right)^4\right)^3 \times \left(\left(\frac{6}{7}\right)^5\right)^2 \times \left(\frac{6}{7}\right)^{4^3} \times \left(\frac{6}{7}\right)^{5^2}$

Write the name of winner group.

7.9 CASE STUDY-II

Four friends Nitya, Divij, Balwant Kaur and Param decided to celebrate the birthday of each of them in a different way. All of them are studying in class X. They decided to hold a competition among themselves on radicals. They requested their common friend of class XII to frame six questions of multiplication on the topic. Each of them would the answer for the questions. Then answers are to be indicated as per first letter of their name i.e. N for Nitya, D for Divij, B for Balwant Kaur and P for Param. The winner shall contribute Rs. 50 and rest will pay for Rs. 150 each. They shall hold a party together. The correct same carries 3 marks and 1 mark is deducted for wrong answer. The questions are as follows:

1. Simplify the following:

$$\left(\frac{5}{9}\right)^2 \times 2\left(\frac{6}{5}\right)^3 \times \left(\frac{2}{3}\right)^{-2}$$

(N) $\frac{81}{8}$ (D) $\frac{5}{12}$ (B) $\frac{8}{81}$ (P) $\frac{12}{5}$

2. Simplify:

$$\left\{\left(\frac{3}{7}\right)^6 \div \left[\left(\frac{3}{7}\right)^3\right]^2\right\} \div \left[\left(\frac{2}{5}\right)^4 \times \left(\frac{5}{2}\right)^4\right] - \left[\left(\frac{2}{9}\right)^5 \div \left(\frac{2}{9}\right)^{-5}\right]$$

(N) 1 (D) 3 (B) 2 (P) 0

3. Find the value of y in the following $\left(\frac{4}{7}\right)^{-5} \times \left(\frac{4}{7}\right)^{-4} = \left(\frac{4}{7}\right)^{2y+1}$

(N) 4 (D) –4 (B) 5 (P) –5

4. Express the following as a product of primes and write answer in exponential form: 14400

(N) $2^5.3^3.5^2$ (D) $2^4.3^3.5^3$ (B) $2^7.3.5^2$ (P) $2^6.3^2.5^2$

5. Which of the following radical pairs are similar:

(i) $\sqrt[3]{216}, \sqrt[3]{750}$ (ii) $\sqrt{800}, \sqrt{1250}$ (iii) $\sqrt{245}, \sqrt[3]{363}$ (iv) $\sqrt[4]{1250}, \sqrt[5]{320}$

(N) $\sqrt{245}, \sqrt{363}$ (D) $\sqrt[4]{1250}, \sqrt[5]{320}$ (B) $\sqrt{800}, \sqrt{1250}$ (P) $\sqrt[3]{216}, \sqrt[3]{750}$

6. After rationalizing $\frac{4}{5\sqrt{2}+4\sqrt{3}}$ we get the denominator as :

(N) 98 (D) 2 (B) 18 (P) –2

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 7.1

1.

(i) $\left(\frac{2}{7}\right)^{10}$

(ii) $\left(\frac{-7}{9}\right)^{25}$

(iii)$(13)^p$

2. (i) base = –8, exponent = 4; (ii) base = 6,exponent = 10

(iii) base = $\frac{-c}{d}$, exponent = m

3. (i) $\frac{343}{1024}$ (ii) $\frac{-27}{512}$ (iii) $\frac{625}{16}$

4. (i) $\left(\frac{1}{4}\right)^3$ (ii) $\frac{-27}{272}$ (iii) $\left(\frac{-7}{2}\right)^4$

5. (i) $\frac{2}{5}$ (ii) $\frac{36}{1225}$

CHECK YOUR PROGRESS 7.2

1. (i) 2^8 (ii) $2^4 \times 3^4$ (iii) $\left(\frac{-11}{5}\right)^3$ (iv) $\left(\frac{3}{7}\right)^5$

2. (i) $2 \times 5 \times 11 \times 13$ (ii) $2^3 \times 5^2 \times 19$ (iii) $3^3 \times 7 \times 11^2$

CHECK YOUR PROGRESS 7.3

1. (i) 11^{-5} (ii) 17^{-9} (iii) 9^{-5}

2. (i) 3^7 (ii) 7^5 (iii) $(23)^5$

CHECK YOUR PROGRESS 7.4

1. (i) $\left(\frac{5}{7}\right)^{11}$ (ii) $(13)^{18}$ (iii) $\left(\frac{-3}{8}\right)^9$

2. (i) F (ii) T (iii) F

(iv) F (v) F (vi) F

(vii) F (viii) T (ix) T

(x) T

3. (i) $c\left(\frac{-5}{7}\right)^{10}$ (ii) $b\left(\frac{-15}{26}\right)^{12}$ (iii) $d\,(1)$

4. (i) 2^{12} (ii) $\left(\frac{3}{5}\right)^{20}$ (iii) $\left(\frac{13}{7}\right)^{7}$ (iv) $\left(\frac{-5}{9}\right)^{2}$

CHECK YOUR PROGRESS 7.5

1. (i) $>$ (ii) $<$ (iii) $<$ (iv) $>$

2. (i) $\sqrt[3]{2}$, $\sqrt[4]{3}$, $\sqrt[3]{4}$ 3. $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{6}$

4. (i) $22\sqrt{2}$ (ii) $7\sqrt[3]{3}$ (iii) $-24\sqrt{2}$

(iv) $-24\sqrt[3]{2}$ (v) $186\sqrt{2} - 42\sqrt{3} + 36\sqrt{5}$

5. (i) $240\sqrt{5}$ (ii) $420\sqrt{2}$

6. (i) $\frac{2}{3}$ (ii) $\frac{3}{2}\sqrt[4]{\frac{5}{7}}$

CHECK YOUR PROGRESS 7.6

1. (i) $\sqrt[3]{25^2}$ (ii) $\sqrt{3} - 2$ (iii) $\sqrt[5]{x^2} - \sqrt[5]{y^2}$

2. (i) $\frac{\sqrt{105}}{3}$ (ii) $\frac{3\sqrt{11}}{11}$ (iii) $3 - 2\sqrt{2}$ (iv) $10 + 3\sqrt{11}$

3. 7 4. $\frac{\sqrt{3} + \sqrt{5} + \sqrt{8}}{2\sqrt{15}}$

5. (i) 4 (ii) 14

6. $x = 4, y = 1$ 7. 0.618

ANSWERS TO QUESTIONS FOR CASE STUDY

CASE STUDY I

1. (C) $20\sqrt[3]{3}$ 2. (B) $\frac{14}{3}$ 3. (A) 1.984 4. (D) $\sqrt[4]{11}$

5. (B) $\sqrt{108}$ 6. (B) $\left(\frac{6}{7}\right)^{4^3} > \left(\left(\frac{6}{7}\right)^5\right)^2 > \left(\left(\frac{6}{7}\right)^4\right)^3 \times \left(\frac{6}{7}\right)^{5^2}$

CASE STUDY II

1. (P) $\frac{12}{5}$ 2. (N) 1 3. (P) –5

4. (P) $2^6.3^2 5^2$ 5. (B) $\sqrt{100}, \sqrt{1250}$ 6. (D) 2

Param pays Rs. 50

8 ALGEBRAIC EXPRESSION AND POLYNOMIALS

INTRODUCTION

The numbers such as 0, 1, 2, $\sqrt{3}$ _ _ _ _ have been introduced to the previous classes. The operations such as addition, subtractions, multiplication and division in arithmetic have also been introduced to students in the previous classes. In Algebra, literal numbers are used to represent a number. In algebra, *a*, *b*, *c*, _ _ _ _, *x, y, z* are supposed to represent a number. Let the perimeter of a square $p = 4\times s$. The algebraic format can be represented as the generalized form of arithmetic interpretation. The algebraic form is precise, general and easier to understand.

Statement1: A number is increased by 12 gives 70.

Algebraic expression: Let number = x,hence$x + 12 = 70$

Statement2: When a number decreased by 20 gives 55.

Let the number be x, hence$x - 20 = 55$

Statement3: A number when multiplied itself, and subtracted by 3 gives 26

Let the number x, hence$x \times x - 3 = 26$

8.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Identify the variables of expressions.
- Identify the expressions.
- Identify the terms of expressions.
- Identify the like and unlike terms.
- Determine the degree of the polynomial.
- Evaluate the value of polynomial if the value of the variable is given.

8.2 VARIABLES AND CONSTANTS:

A symbol having different numerical values is known as a variable. Consider the days or months of a year. Let the months of a year be x and it is clear that x may be any month and it is not fixed. Hence x is called a variable. Let us consider some numbers which can be mentioned in the following expressions

(i) 5, -13x

(ii) $\sqrt{5}x + y$

(iii) $31y^2$,

(iv) 62x

In the above Expressions, 5, -13, $\sqrt{5}$, 31, 62 are real numbers and these real numbers are fixed which are known as constants in the above expressions and -13x, $\sqrt{5}x + y$ $31\,y^2$ and 62x have no fixed values. The values of these expressions depend upon x, y and y^2. Therefore x, y and $y^2$are called variables.The variables or literal numbers have different values where the constant has a fixed value

8.3 ALGEBRAIC EXPRESSION AND POLYNOMIALS

A collection of constants and variables connected with operations +, -, × and ÷ is an algebraicexpression. Hence 4x, 7y+5, $6\,x^2$ +2x+5, $\frac{6x+4}{2x+4,}$ etc. are all algebraic expressions.

Hence, an algebraic expression is a combination of numbers, variables and arithmetic operations. The +ve and –ve sign separate the algebraic expressions into several parts. Each part of the expressions is called the term of the expression the algebraic. The expressions 4x+5y+7 have three terms as 4x, 5y and 7. In the expression, 4 & 5 are coefficients of x and y and 7 is called the constant term and x and y are called variables. The algebraic expression in which the denominator does not occur with the variable is called a polynomial. In other words, no term of a polynomial has a variable in the denominator and in each term of a polynomial, the exponents of the variables are non-negative integers and the numerical coefficient of each term is a real number. Here are some examples of polynomials: 4x , 5x-3y, $\frac{1}{2}x^2 + 4x + \frac{5}{2}$, 4$xy$+5$x^2$. The polynomial 4$x$ is in one variable, the polynomial 5x-3y is in two variables and the polynomial $\frac{1}{2}x^2 + 4x + \frac{5}{2}$, is in one variable

8.3.1 Monomials Expressions

An algebraic expression or a polynomial consisting of only one term is called a monomial. Hence x, $3y$, $6x^2$, $-7xy$ are all monomials.

8.3.2 Binomial Expression

An algebraic expression or polynomial, consisting of only two terms is called a binomial. Hence $5x-7y$, x^2+4x, $3y^2+6$ are all binomials.

8.3.3 Trinomial Expressions

An algebraic expression or a polynomial consisting of only three terms is called trinomial Hence $4x+5y+7$, x^2+6x+5 are the all trinomials.

8.3.4 Like Terms

The terms of a polynomial having the same variable and exponent of the variables are called like terms. This concept is expressed in the following expressions $4x$ and $10x$& $10xy$, $17xy$, $12xy$

8.3.5 Unlike Terms

The terms of a polynomial have different variables and exponents are called the unlike terms. It can be expressed in the following illustrations.

$(4x^2, 5x)$ $(4x^2, 6x, 7y)$

Example 1: Write the variables, coefficients and constant of the expression $4x^2+10y+7$

Solution: Variables: x^2 and y

Coefficients and constants: 4, 7 and 10

Example 2: Write the coefficient and variables in the expression $6x^2+7y^2$

Solution: In $6x^2$, 6 is the coefficient and x is variable

In $7y^2$, 7 is the coefficient and y is the variable.

Example 3: Identify the terms in the expression $4x^2+6y+14y^2$

Solution: $4x^2$, 6y, $14y^2$, are three terms in the expression.

Example 4: Which of the following algebraic expressions are polynomials.

(i) $4x^2+\sqrt{7}$

(ii) $5x^2 + 6\sqrt{x}$

(iii) $\frac{x+1}{x} + \frac{4}{x}$

(iv) $\frac{6x^2}{x} + 16y$

Solution: (i) and (iv) are algebraic polynomials.

Example 5: Identity like terms in the following expressions

(i) $4x+3x^2-6x+xy$

(ii) $\frac{6}{7}x^2+6xy+7x-2xy$

(iii) $2x^3+4x^2+5xy+5x^2$

Solutions:

(i) $4x$, $-6x$

(ii) $6xy$, $-2xy$

(iii) $4x^2$, $5x^2$

CHECK YOUR PROGRESS 8.1

1. Write the variable, coefficients and constants in each of the following expressions:

(i) $4x+5$

(ii) $4x+5y-2$

(iii) $\frac{4}{7}x^2 + \frac{3}{2}y^2 - 4xy$

(iv) $6x^3+5x^2-3x$

(v) $5y + \frac{5}{x}$

2. Identify the like terms in the following expressions:

(i) $4x + 3xy + 5x + 4xy$

(ii) $x^2 + 5xy + 6x^2 - 7xy + 3$

(iii) $6x^2 + 7y^2 - \frac{4}{3}x^2 + \frac{5}{2}y$

3. Identify the polynomials from the following expressions:

(i) $\frac{2}{3}x^2 + 5 + 6xy$

(ii) $2x^{-2} + 5x + \frac{7}{x}$

(iii) $6xy + \sqrt{5}\,x + 4$

(iv) $\sqrt{x} + y + 3$

4. **Identify the following expressions as monomials, binomial or trinomials.**

(i) x^3+5

(ii) $6x^2+2x+5$

(iii) $4x^2y$

(iv) $5x + 7x^2+5y$

(v) $2x^3+4x^2$

(vi) $6x^2$

8.4 DEGREE OF A POLYNOMIALS

The highest exponent of the variable or sum in non-zero coefficient of exponent in case of the product of variable is called its degree. In $4x^2+5x^3-14x$, the highest exponent is 3. Hence the degree of the expressions $4x^2+5x^3-14x$ is 3

Example 6: Write the degree of the expressions $6x^2+5x^3-7x^2y^2$

Solution: The degree of the expression is 2+2 = 4.

A polynomial of degree 2 is called the quadratic polynomial. A polynomial of degree 3 is called the cubic polynomial. The degree of a non-zero constant polynomial is zero, this polynomial is called the zero polynomial. The degree of a zero polynomial is not defined.

8.4.1 Evaluation of a Polynomial

The value of a polynomial can be evaluated for the given value of the variable of the expression. Let p(x) be a polynomial and a is any real number such that $x = a$ in p(x), then p(a) is the value at $x = a$. It can be understood by the following example:

Example7: Find the value of $p(x) = x^3+2x^2-5x+4$ at $x = 1$

Solution: $p(x) = x^3+2x^2-5x + 4$

$p(x) = (1)^3+2(1)^2-5(1) + 4$

$x = 1$, $p(1) = (1)^3+2(1)^2-5(1)+4$

$= 1 + 2 - 5 + 4 = 7 - 5 = 2$

8.5 ZERO OF A POLYNOMIAL

The value(s) of the variable for which the value of a polynomial in one variable having zero is called the zero(s) of the polynomial. In the other words, zero(s) of a polynomial p(x) is a number such that p(a) = 0. This can be understood in the following example:

Example8: Find the zero of the polynomial p(y) = y^3 - $3y^3$ + $3y$ -1

Solution: p(y) = y^3 - $3y^3$ + $3y$ -1

Putting y = 1, p(1) = $1^3 - 3(1)^3 + 3(1) - 1$ = 1 -3 + 3 - 1 = 0

Then p (1) = 0. Hence, y = 1 is the zero of the polynomial.

8.6 ADDITION AND SUBTRACTION OF POLYNOMIALS

Like or unlike terms have been introduced already. In addition and subtraction of polynomials, like terms are identified and added or subtracted according to the requirement.

8.6.1 Addition or Subtraction using Vedic Mathematics

Itisvery simple and interesting.The sum of the coefficient of like termsisobtained in addition.Similarly, the difference in coefficient of like terms is obtained. The addition or subtraction of the coefficient of a like term is obtained in these problems.Using this approach, it is explained in the following way:

Example 9: Solve ($4x^2$ + $5x$ – 6) + ($5x^2$ + $4x$ – 2)

Solution: Using the Vedic Mathematics approach, the variable is written first, then the coefficients are written in the following way

x^2	x	Constant
+ 4	+ 5	-6
+5	+4	-2
$9x^2$	+9x	-8

Coefficients are added and variables are written with the resulted numbers.

Example 10: Add the following expressions

$$4x^2 + 5xy + 3y,^2 \quad -6x^2 + 3xy, \quad -7xy + 10y^2$$

Solution:

x^2	xy	y^2		
4	5	3		
-6	3	$--$	-7	10

$-2x^2 \quad +xy \quad +13y^2$

Example 11: Subtract $5x^3 + 2x^2 + 5x + 4$ from $7x^2 + 4x - 5$

Solution: $x^3 \quad + x^2 \quad + \quad x \quad + \quad Constant$

		+7	+4	-5
± 5	± 2	± 5	± 4	
	- $5x^3$	$+ 5x^2$	$- x$	$- 9$

Example 12: Add the following expressions $4x^3 + 6xy + 2x^2 + 5, 6x^2 + 5xy - 4, \ 3x^3 + 4 + 7xy$

Solution: $x^3 \quad + xy \quad + x^2 \quad + Constant$

4	6	2	5
-	5	6	-4
3	7	-	+4
$7x^3$	$+ 18xy$	$+ 8x^2$	$+ 5$

Example 13: Subtract $6x^2 + 5x + 4$ from the sum of $7x^2 - 4x - 5$ and $6x^2 + 7x - 6$

Solution:

x^2	x	$Constant$
7	-4	-5
6	7	-6
13	3	-11
± 6	± 5	± 4
$7x^2$	$- 2x$	$- 15$

CHECK YOUR PROGRESS 8.2

1. **Write the degree of the following expressions:**

   (i) $4x^2 + 5x^2y + 6y^2 - 3$

   (ii) $6x^2 + 20x^3y^3 + 2xy^7 + 4$

   (iii) $4x^2 + 7x + 10x^3$

   (iv) $10x + 5 - 14x^2 + 2x^5$

2. **Find the value of the following expressions:**

   (i) $4x^2 + 5x - 4$ at $x = 2$

   (ii) $5x - 4xy + 3y^2$ at $x = 1, y = -2$

(iii) $6x^2 - 3xy + 5z^2 - 10,\ at\, x = 1, y = -3, z - 2$

(iv) $7x^2 - 6x - 4\ - 7y^2\ at\, x = -3, y = -5$

3. **Add the following expressions:**

(i) $4x^2 - 5x + 4,,\ 3x^2 - 4x - 3,\ 6x^2 + 5x + 2,$

(ii) $6x^2 + z^2 - 3y,\ 5z^2 + 4z + 7xy,\ \ 6xy - 3z$

(iii) $3y^2 + 5z,\ \ 6z^2 + 5xy + 4,\ \ 7xy + 5$

(iv) $6z^2 + 6z,\ \ 7z^2 + 6xy - 5,\ \ 6z^3 + 5z^2 + 4$

4. **Solve the following expressions:**

(i) Subtract$6x^2 + 5x - 3\ from\ 9x^2 - 7x + 4$

(ii) Subtract$10x^2 + 4xy - 3\ + 5y^2$ from $7x^2 + 6y^2 + 4 - 3xy$

5. **Find the value of the following expression**

(i) Subtract $4x^2 – 5xy + 5y^2$ from the sum of $6x^2 - 4xy + 3y^2$ and $7x^2 - 6xy$

(ii) Subtract $-6x^2 + 4y^2 + 7xy$ from the sum of $7x^2 - 5xy + 6y^2$ and $-8x^2 + 10y^2 + 10xy$

8.7 PRODUCT OF POLYNOMIALS

The ways of obtaining product of the polynomials has been described in the following expressions:

Example 14: Solve $6x^2y \times 7xy^2 \times (-5x^3y^2)$.

Solution: The coefficients are multiplied and exponents of like are added in the following way

$$(6 \times 7 \times -5)\, x^{2+1+3}\, y^{1+2+2}$$

$= –210\ x^6y^5$

Example 15: Find $(4x + 5) \times (6x + 4)$

Solution: Conventional Way

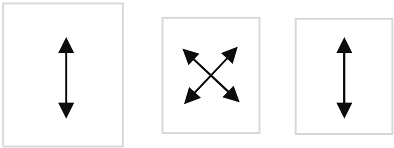
$4x \times (6x + 4) + 5(6x + 4)$

Or $24x^2 + 16x + 30x + 20$

Or $24x^2 + 46x + 20$

Using Vedic Mathematics Sutra

Vertically and crosswise, the product can be obtained in the following way;



1 2 1

In this process, the extreme left terms of both the expressions are multiplied vertically. Then the both the terms of the expressions are multiplied crosswise and then then the extreme left term of the expressions are multiplied in the following way

$4x+\ 5$ Step 1: $5 \times 4 = 20$

$6x+\ 4$ Step 2: $4x + 4\ + 5\ + 6x$

$= 16x + 30x = 46x$

Step 3: $4x \times\ 6x = 24x^2$

The solution is $24x^2 + 46x\ + 20$

Example 16: Obtain the product of $6x^2 +\ 4x - 3$ and $3x^2 -\ 3x\ + 4$

Solution: Conventional Way

$(6x^2 +\ 4x - 3)(3x^2 -\ 3x\ + 4)$

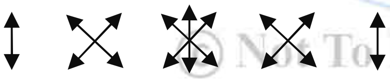
Or $6x^2(3x^2 -\ 3x\ + 4) + 4x(3x^2 -\ 3x\ + 4) - 3(3x^2 -\ 3x\ + 4)$

Or $18x^4 -\ 18x^3\ + 24x^2 + 12x^3 -\ 12x^2\ + 16x -\ 9x^2 + 9x - 12$

Or $18x^4 -\ 18x^3 + 12x^3 + 24x^2 -\ 9x^2 -\ 12x^2 + 16x + 9x - 12$

Or $18x^4 - 6x^3 +\ 3x^2 + 25x - 12$

The product of quadratic expressions can be obtained in the following pattern



Steps: 1 2 3 2 1

$6x^2 +\ 4x - 3$

$3x^2 -\ 3x\ + 4$

From right side

Step 1: $-3 \times 4 = -12$

Step 2: $4x \times 4 + (-3) \times (-3) = 16\,x + 9x = 25x$

Step 3: $6x^2\ \times 4 + (-3) \times 3x^2\ + 4x\ \times\ (-3)x$

$= 24x^2 -\ 9x^2 - 12x^2 = 3x^2$

Step 4: $6x^2\ \times -3\,x + 4x \times\ 3x^2 = -18x^3 + 12x^3$

$$= -6x^3$$

Step 5: $6x^2 \times 3x^2 = 18x^4$

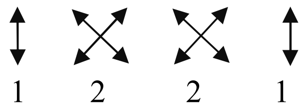
The solution is

$$18x^4 - 6x^3 + 3x^2 + 25x - 12$$

Example 17: Solve the following expression

$(4x^2 + 5x + 2) \times (6x - 3)$

Solution: The solution of the expression can be obtained according to the following way:



When the multiplier is in two terms, the operation can be calculated as indicated in the above expression. Let us solve this problem using this concept.

$4x^2 + 5x + 2$

$\times\ 6x - 3$

From the right side

Step 1: $2 \times -3 = -6$

Step 2: $5x \times (-3) + 2 \times 6x = -15x + 12x = -3x$

Step 3: $4x^2 \times (-3) + 5x \times 6x = -12x^2 + 30x^2 = 18x^2$

Step 4: $4x^2 \times (-6x) = 24x^3$

The complete solution is $24x^3 + 18x^2 - 3x - 6$

8.8 DIVISION OF POLYNOMIALS

8.8.1 Division of Monomials

Division of monomials is very easy. The coefficients of monomials are divided and exponents of some variables are subtracted. It is explained in the following way.

Example 18: Solve $100x^3y^2 \div 25x^2y$

Solution: $\frac{100}{25} \times \frac{x^3}{x^2} \times \frac{y^2}{y^1}$

$= 4\,x^{3-2}y^{2-1} = 4xy$

To divide the polynomial by a monomial, each term is divided by monomials as in the following way:

Example 19: Solve $(4x^3 + 10x^2 - 8x) \div 2x$

Solution: $\frac{4x^3}{2x}+\frac{10x^2}{2x}-\frac{8x}{2x}$

$=2x^2+5x-4$

To divide the polynomials by a binomial has the different approach. It will be explained both the ways conventional as well as Vedic Mathematics approach. The process of division is explained in the following illustrations. Let us understand the concept of algebraic division in the following illustration:

Example 20: Solve $(2x^3 - x^2 - 6x + 2) \div (x-2)$

Solution: Conventional way:

$$2x^2 + 3x$$

$$x-2\sqrt{2x^3 - x^2 - 6x + 2}$$

$$\pm 2x^3 \mp 4x^2$$

$$3x^2 \quad - 6x + 2$$

$$\pm 3x^2 \quad \mp 6x$$

$$+2$$

Step 1: Arrange exponents both of the polynomials in decreasing order.

Step 2: Divide the first term of dividend by the first term of divisor to get the first term of the quotient i.e.

$$\frac{2x^3}{x} = 2x^2$$

And multiply this term to this both terms of the divisor and subtract the result from the dividend.

Step 3: Divide the first term $3x^2$by x i.e.

$$\frac{3x^2}{x} = 3x$$

And multiply both the terms of divisors and subtract from remaining dividend.

Step 4: Remainder '2' has less exponent than the divisor. It cannot be further divided. The quotient is $2x^2 + 3x$

Division using Vedic Mathematics Approach

x-2	$2x^3 +$	$5x^2$	$-6x$	$+2$
+2	2	5	-6	2
		+4	18	24
	2	9	12	26

Step 1: Dividend is bifurcated in two parts. The number of terms in right part depends upon the number in divisor except first term.

Step 2: Except first term of divisor in the rest of terms of divisor, signs are changed. In this illustration, -2 will be converted in +2 and written in the left hand.

Step 3: The co-efficient of dividends is written downwards according to the question.

Step4: The first co-efficient is written downwards as it is and multiplied to converted dividend and written downwards to the second term of the dividend and added, we get 9 which is written in the 3rd row. 9 is multiplied by 2 and the result is written downward to third term and added, we get 12 which has been written in the third row.

Step5: 12 ×2 = 24 which is written down to the right side of the dividend and added. 26 is left on the right side and it is the remainder.

Step6: $2x^2 + 9x + 12$ is the quotient and 26 is the remainder

CHECK YOUR PROGRESS 8.3

1. **Obtain the product of the following monomial:**
   (i) $4x^2,\ 5x,\ 7$
   (ii) $6x^2y,\ 3x^2yz,\ 10xy^2z$
   (iii) $\frac{4}{3}x^2y,\ \frac{5}{8}xy^2,\ \frac{25}{27}x^2y^2$
   (iv) $\frac{5}{7}y,\ \frac{14}{20}x^2y,\ \frac{25}{27}xy^2$
2. **Obtain the product of the following polynomials:**
   (i) $(5x^2 + 2x + 5)\ \&(2x^2 - 4x - 6)$
   (ii) $(3x + 5)\ \&\ (4x + 3)$
   (iii) $(4x - 5)\ \&\ (6x + 4)$
   (iv) $5x^2 + 4x - 5)\ \&(7x^2 - 4x - 6)$
3. **Obtain the quotient of the following questions:**
   (i) $6x^2y^2z^2 \div 2xyz$
   (ii) $36x^4y^3z \div 12xyz$
   (iii) $(16x^3 + 4x^2 + 10x) \div 2x$
   (iv) $(25x^4 + 10x^2 + 40) \div 5$
   (v) $(50m^4 - 24m^3 + 100) \div 10m$

4. Obtain the quotient and remainder of the following questions:

(i) $(4x^3 + 15x^2 + 10x + 7) \div (x + 3)$

(ii) $(20x^2 - 10x - 27) \div (x - 2)$

(iii) $(12x^3 - 10x^2 - 14x + 11) \div (2x - 3)$

(iv) $(6x^4 + 23x^3 + 10x^2 + 5x + 24) \div (2x + 5)$

8.9 REMAINDER THEOREM

When dividend is divided by a divisor, the remainder may be obtained which is less than the divisor. When the degree of r (x) remainder is zero, it means that the divisor is factor of the dividend. Let us understand by the following illustration.

Example21: Find the remainder when p $(y) = 3y^4 - 4y^3 - 3y^2 + 2 + 6y$ is divided by q(y) = $y-1$

Solution: Division Method

$y-1$ $\overline{)3y^4 - 4y^3 - 3y^2 + 6y + 2}($ $3y^3 - y^2 - 4y + 2$

$\pm 3y^4 \mp 3y^3$

$-y^3 - 3y^2 + 6y + 2$

$\mp y^3 \pm y^2$

$-4y^2 + 6y + 2$

$\mp 4y^2 \pm 4y$

$2y + 2$

$2y \mp 2$

4

Remainder by Remainder Theorem:

$p(1) = 3(1)^4 - 4(1)^3 - 3(1) + 2 + 6(1) + 2$

$= 3 - 4 - 3 + 6 + 2 = 4$

8.9.1 Statement of Theorem

Let p (x) be any polynomial of degree greater than or equal to one and a be any number. If p (x) is divided by the linear polynomial $(x - a)$, then the remainder is p (a).

Example 22: Find the remainder when $p(x) = x^4 + x^3 - 3x^2 + x + 2$ is divided by $x - 2$

Solution:

$p(x) = x^4 + x^3 - 3x^2 + 4x + 2$

when $x - 2 = 0$, $x=2$

$p(2) = 2^4 + 2^3 - 3\times2^2 + 4 \times 2 + 2$

$=16 + 8 - 12 + 8 + 2$

$=22$ (remainder)

Example 23: Find the remainder where $p(x) = 4x^3 - 4ax^2 + 3x + a$ is divided by x - a

Solution:

$p(x) = 4x^3 - 4ax^2 + 3x + a$

when $x - a = 0$, $x = a$

$p(a) = 4a^3 - 4a \times a^2 + 3a + a$

$= 4a^3 - 4a^3 + 3a + a$

$= 4a$ (remainder)

8.10 FACTOR THEOREM

When a dividend is completely divided by a divisor, and the remainder is 0, then it can be said that the divisor is factor of dividend. If q(x) divides p(x) it is said that p(x) is divisible by q(x). Hence q(x) is a factor of q(x) this concept can be understood by the following illustration:

Example 24: Using factor theorem show that $y+3$ is a factor of $p(y) = y^4 + y^3 - 7y^2 - y + 6$

Solution: $p(y) = y^4 + y^3 - 7y^2 - y + 6$

when $y + 3 = 0$, $y = -3$

$p(-3) = (-3)^4 + (-3)^3 - 7y^2 - (-3) + 6$

$= 81 - 27 - 63 + 3 + 6$

$= 90 - 90 = 0$

$= 0$

Hence p (-3) =0, $y+3$ is a factor of p(y)

CHECK YOUR PROGRESS 8.4

1. **Obtain the remainder of the following expression when:**
   (i) $p(x) = 2x^3 + 4x^2 - 6x + 7$ is divided by $g(x) = x - 2$
   (ii) $p(x) = 4x^2 - 10x + 25$ is divided by $g(x) = x + 3$
   (iii) $p(x) = 2x^3 - x^2 - 10x - 35$ is divided by $g(x) = x - 4$
   (iv) $p(x) = 40x^3 - 12x^2 + 10$ is divided by $g(x) = 2x - 1$
2. **Using Factor Theorem, show that:**
   (i) $2y - 3$ is a factor of $p(y) = 2y^3 + 4y^2 - 6y + 12$ or not

(ii) x – 1 is a factor of p(y) = x^{20} – 1 and x^{21} – 1 or not

8.11 FACTORIZATION OF POLYNOMIALS

We have introduced the factors of real numbers of arithmetic in the previous classes. In this topic, we shall deal with the factors of algebraic expressions. The factorization of a polynomial is a process of writing the polynomial as a product of two or more polynomials. Each polynomial of the product is a factor of given polynomial. The factors of $4x^2 - 25$ can be expressed in the form of $(2x - 5)(2x + 5)$. A polynomial will be said to be completely factored if none of its factors can be further expressed as a product of two polynomials of lower degree and if the integer coefficient has no common factor other than 1 and -1. The complete factors of $5x + 25$ is $5(x + 5)$. Now we shall discuss the various cases of factorization of the algebraic expressions.

8.11.1 Factorization using Distributive Property

This property is explained in the following examples:

Example25: Factorize $25xy + 10x$

Solution: By taking common $5x$, $5x$ (($5y + 2$)

Hence 5, x, ($5y$+2) are the factors of $25xy + 10x$

8.11.2 Factorization of Perfect Squares

In this pattern, the factorization can be obtained using the identity $x^2 + y^2 + 2xy = (x + y)^2$ and $x^2 + y^2 - 2xy = (x - y)^2$. It can be understood from the following illustrations:

Example 26: Factorize the polynomial $36x^2 + 25y^2 + 60xy$

Solution: The polynomial $36x^2 + 25y^2 + 60xy$ can be expressed as the polynomial $(6x)^2 + (5y)^2$

$+ 2 \times 6x \times 5y$,

Using identity $x^2 + y^2 + 2xy = (x + y)^2$, the given expression $36x^2 + 25y^2 + 60xy = (6x)^2 + (5y)^2 + 2 \times 6x \times 5y$ can be factorized as $(6x + 5y)^2 = (6x + 5y)(6x + 5y)$

Example 27: Factorize the polynomial $121x^2 + 49y^2 - 154xy$

Solution: The polynomial $121x^2 + 49y^2 - 154xy$ can be expressed as the polynomial $(11x)^2 + (7y)^2$

$- 2 \times 11x \times 7y$, Using identity $x^2 + y^2 - 2xy = (x - y)^2$, the expression $(11x)^2 + (7y)^2$

- $2 \times 11x \times 7y$ can be factorized as $(11x - 7y)^2 = (11x - 7y)(11x - 7y)$

8.11.3 Factorization Involving the Differences between Two Squares

The factorization involving the differences between two squares can be obtained using the identity $x^2 - y^2 = (x + y)(x - y)$. Let us understand it by taking some illustrations of the factors.

Example28: Factorize $36x^2 - 25y^2$

Solution: The polynomial $36x^2 - 25y^2$ can be expressed as $(6x)^2 - (5y)^2$

Using identity, it can be expressed as $(6x - 5y)(6x + 5y)$

Example 29: Factorize $x^2 - y^2 + 6y - 9$

Solution: The polynomial $x^2 - y^2 + 6y - 9$ can be expressed as $(x)^2 - (y^2 - 2 \times 3 \times y + 3^2)$

Using identity, it can be expressed as $(x)^2 - (y - 3)^2$

Using identity, it can be expressed as $(x + y - 3)(x - y + 3)$

CHECK YOUR PROGRESS 8.5

1. **Factorize the following polynomials**

(i) $12xy - 48y$

(ii) $22x^3 - 44x^2 + 55x$

(iii) $121 - 9x^2$

(iv) $144x^2 + 120xy + 25y^2$

(v) $81x^2 - 126xy + 49y^2$

(vi) $225 - (144x^2 + 312xy + 169y^2)$

(vii) $(256x^2 - 96xy + 9y^2) - 441b^2$

(viii) $(16x^2 - 40xy + 25y^2) - (81x^2 + 126xy + 49y^2)$

8.11.4 Factorization of Perfect Cube Polynomials

Factorization of perfect cube polynomials can be obtained using identity $x^3 + y^3 + 3x^2y + 3xy^2 = (x + y)^3$ and $x^3 - y^3 - 3x^2y + 3xy^2 = (x - y)^3$. This can be understood by the following illustrations:

Example 30: Factorize $27x^3 + 8y^3 + 54x^2y + 81xy^2$

Solution: The factorization of $27x^3+8y^3 + 54x^2y + 81xy^2$can be obtained using the following identity: $x^3+ y^3 + 3x^2y + 3xy^2 = (x + y)^3$. The polynomial $27x^3+8y^3 + 54x^2y + 81xy^2$ can be expressed in the following way:

$(3x)^3+ (2y)^3 + 3(3x)^2(2y) + 3(3x)(2y)^2$

Hence factors of $27x^3 + 8y^3 + 54x^2y + 81xy^2 = (3x + 2y)^3$

Example 31: Factorize $x^3- 64y^3 -12x^2y + 48xy^2$

Solution: The factorization can be obtained using the following identity: $x^3- y^3 - 3x^2y + 3xy^2 = (x - y)^3$ The polynomial $x^3- 64y^3 -12x^2y + 48xy^2$can be expressed $(x)^3- (4y)^3 - 3(x)^2(4y) + 3(x)(4y)^2$

Hence the factors of $x^3- 64y^3 -12x^2y + 48xy^2= (x - 4y)^3$

8.11.5 Factorization of Polynomials Involving Sum and Differences of Two cubes

Factorization of perfect cube polynomials can be obtained using identity $x^3+ y^3= (x + y)(x^2 +y^2 -xy)$ and $x^3- y^3= = (x - y)(x^2 +y^2 + xy)$. This identity can be understood by the following illustrations:

Example 32: Factorize $27x^3- 64y^3$

Solution: The factorization can be obtained using the following identity:$x^3+ y^3= (x + y)(x^2 +y^2 -xy)$

Hence the factors of the polynomial$27x^3- 64y^3 = (3x)^3- (4y)^3 = (3x - 4y)(9x^2 + 16y^2 +12xy)$

CHECK YOUR PROGRESS8.6

1. **Factorize the following expressions**

(i) $8x^3 + 27y^3$	(ii) $y^3 – 1331$
(iii) $y^3 - 12y^2z + 48yz^2 - 64z^3$	(iv) $8x^3 - 125y^3 - 60x^2y + 150xy^2$
(v) $64m^3 - 144m^2 + 108m – 27$	(vi) $64x^6 - 1$
(vii) $m^2 + m^2n^6$	(viii) $27b^3 - a^3 -3a^2 – 3a - 1$
$(ix)(a + b)^3+ 27c^3$	(x) $16x^7 - 54xy^6$

8.11.6 Factorization of Polynomials by Splitting the Middle Terms

Factorization using the identity $x^2 + (a + b)x + ab = (x + a) (x + b)$.

8.11.7 Factorization by Splitting the Middle Terms

$x^2 + (a+b)x + ab$ is simple and can be understood easily by the learners. Let us understood this concept by the following examples:

Example 33: $x^2 + 10x + 21$

Solution: Conventional Approach to factorize of polynomial $x^2 + 10x + 21$

Step 1: Middle Terms is factorized such that sum of factors will be 10 and their product will be 21. If the expression $x^2 + 10x + 21$ is compared to the general or standard expression $ax^2 + bx + c$ where $a = 1,\ b = 10,\ c = 21$. Now consider two numbers such that the addition of factors is 10 and their product is 21. Clearly, the factors are 7 and 3 whose addition is 10 and the product is 21. This concept can be written as $x^2 + 7x + 3x + 21$

By pairing the first two terms and last two terms$(x^2 + 7x) + (3x + 21)$

Taking common $x(x + 7) + 3(x + 7)$

Again common$(x + 7)(x + 3)$

Using Vedic Mathematical Approach

$$x^2 + 10x + 21$$

First terms and the last terms are factorized in such a way

x 3

x 7

This factor are checked by adding the product of crosswise multiplication

$$7 \times x + 3 \times x = 10x\ (middle\ term)$$

These factors are verified using sutra-Urdhvatiryagbhyam”. Hence the factors are $(x + 3)(x + 7)$

Example 34: Factorize $7x^2 + 26x + 15$

Solution:$7x^2 + 26x + 15$

Where $a = 7, b = 26, c = 15,\ ac = 7 \times 15 = 105\ ,\ b = 26\ (21 + 5)$

Hence $7x^2 + 21x + 5x + 15$

$7x(x + 3) + 5(x + 3)$

$(x + 3)\ (7x + 5)$

Vedic Mathematics Approach

$$7x^2 + 26x + 15$$

$7x$	5
x	3

Using Vedic Mathematics Approach, factors of first term and the last term are obtained. To verify the factors that these factors are correct or not, both the factors are multiplied crosswise and the results are added. If the result of this addition is correct, then the factors are correct. It can be observed in the following examples

$7x\ \times\ 3 + 5 \times\ x = 26\ x$, which is middle term.

Hence the factors are $(7x + 5)(x + 3)$

CHECK YOUR PROGRESS 8.7

1. Factorize the following Expressions

(i) $x^2 - 12x + 35$

(ii) $6x^2 + 10x + 4$

(iii) $2x^2 + 11x + 15$

(iv) $10m^2 + 11m - 6$

(v) $2x^2 + 3x - 9$

(vi) $x^2 - 15x + 54$

(vii) $2a^2 + 11a + 14$

(viii) $(a + b)^2 - (a + b) - 20$

TERMINAL EXERCISE

1. Fill up the following expressions:

(i) Factors of $4a^2 - 25 =$

(ii) Factors of $9a^2 + 30a + 25 =$

(iii) Factors of $25m^2 - 70m + 49 =$

(iv) Factors of $a^3 + 8b^3 =$

(v) Factors of $8b^3 - 64 =$

(vi) Factors of $16x^2 + 32x + 15 =$

2. Factorize the following expressions:

(i) $5m^2 - 8m - 4$

(ii) $27m^6 + 125n^6$

(iii) $a^6 - 64b^6$

(iv) (iv) $x^4 - 8x^2b^3 + 16b^6$

(v) $10a^5 + 25a^3b + 35a^2b$

(vi) (vi) $3a^3 - 81b^3$

(vii) $x^3 + 1$

(viii) $27x^3 - \frac{1}{216}y^3$

(ix) $8x^3 + 27y^3$

(x) $21x^4 - 21x$

ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 8.1

1. (i). Coefficient: 4,Variable: xConstant: 5
   (ii) Coefficient: 4, 5,Variables: x, yConstant: -2
   (iii) Coefficient: $\frac{4}{7}$, $\frac{3}{2}$, -4 Variable: x, y
   (iv) Coefficient: 6, 5, -3 Variable x
   (v) Coefficients: 5, 5 Variables: y, $\frac{1}{x}$

2. (i) $4x$, $5x$& $3xy$, $4xy$
   (ii) x^2, $6x^2$& $5xy$, $7xy$
   (iii) $6x^2$, $\frac{4}{3}x^2$

3. (i) & (iii)

4. (i) binomial (ii) trinomial (iii) monomial
   (iv) trinomial (v) binomial (vi) monomial

CHECK YOUR PROGRESS 8.2

1. (i) 3 (ii) 8 (iii)3 (iv) 5
2. (i) 22 (ii) 25 (iii) 25 (iv)-98
3. (i) $13x^2$-$4x$ +3 (ii) $6x^2$+ $6z^2$ + $13xy - 3y + z$
   (iii) $3y^2 + 5z + 6z^2$+ $12xy$+9 (iv) $6z^3 + 13z^2 + 6z + 6xy$ -1
4. (i) $3x^2$-$12x$ + 7 (ii) -$3x^2$+ y^2 + 7 - $7xy$
5. (i) $9x^2$ - $5xy$ -$2y^2$ (ii) $5x^2$ - $2xy$ + $12y^2$

CHECK YOUR PROGRESS 8.3

1. (i) $140x^3$ (ii) 180 $x^5y^2z^2$ (iii) $\frac{125}{162}x^5y^5$ (iv) $\frac{25}{54}x^3y^4$
2. (i) $10x^4$ - $16x^3$ - $28x^2$ - $32x$-30 (ii) $12x^2$ + $29x$ +15
   (iii) $24x^2$ - $14x$-20 (iv) $35x^4$ + $8x^3$ - $81x^2$ - $4x$+ 30
3. (i) 3 xyz (ii) $3x^3y^2$ (iii) $8x^2$ + $2x$+ 5 (iv) $5x^4$ + $2x^2$ + 8
   (v) $5m^3$ - $2.4m^2$ + 10
4. (i) Q = $4x^2$ + $3x$+1, R = 4, (ii) Q = $20x$ + 30, R = 33, (iii) Q = $6x^2$ + $4x$– 1, R= 8
   (iv)Q = $3x^3$ + $4x^2$ - $5x$ + 15, R = -51

CHECK YOUR PROGRESS 8.4

1. (i) 27 (ii) 91 (iii) -47 (iv) 12
2. (i) No (ii) Yes, Yes

CHECK YOUR PROGRESS 8.5

1.
   (i) $12y(x - 4)$
   (ii) $11x(2x^2$ - $4x$ + 5)
   (iii) (11 - $3x$) (11 + $3x$)
   (iv) ($12x$–$5y$) ($12x$ + $5y$)
   (v) ($9x$ – $7y$) ($9x$ – $7y$)

(vi) $(15 - 12x - 13y)(15 + 12x + 13y)$

(vii) $(16x - 3y - 21b)(16x - 3y + 21b)$

(viii) $(5x + 12y)(13x + 2y)$

CHECK YOUR PROGRESS 8.6

(i) $(2x + 3y)(4x^2 + 9y^2 - 6xy)$

(ii) $(y - 11)(y^2 + 121 + 11y)$

(iii) $(y - 4z)^3$

(iv) $(2x - 5y)^3$

(v) $(4m - 3)^3$

(vi) $(2x - 1)(2x + 1)(16x^4 + 1 + 4x^2)$

(vii) $m^2(1 + n^2)[1 + n^4 - n^2]$

(viii) $(3b - a - 1)[9b^2 + a^2 + 1 + 2a + 3b(a + 1)]$

(ix) $(a + b + 3c)[(a^2 + 2ab + b^2 + ac^2 - 3c(a + b) + 9c^2]$

(x) $2x(2x^2 - 3y^2)(4x^4 + 9y^4 + 6x^2y^2)$

CHECK YOUR PROGRESS 8.7

(i) $(x - 7)(x - 5)$

(ii) $2(3x + 2)(x + 1)$

(iii) $(2x + 5)(x + 3)$

(iv) $(5m - 2)(2m + 3)$

(v) $(x + 3)(2x - 3)$

(vi) $(x - 9)(x - 6)$

(vii) $(2a + 7)(a + 2)$

(viii) $(a - b - 5)$

9 LINEAR EQUATIONS IN TWO VARIABLES

INTRODUCTION

We have learned about linear equations in one variable and their applications. If a and b are two real numbers such that $a \neq 0$ and $ax + b = 0$ is an equation ina variable then, it is an equation in one variable. We have also learned that a value of the variable which satisfies a given linear equation in one variable is known as its solution.

9.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- identify the linear equations in two variables.
- form the linear equations in two variables.
- draw the graph of linear equations in two variables.
- find the solution of a system of linear equations in two variables algebraically.
- find the solution of a system of a linear equation in two variables graphically.

9.2 LINEAR EQUATIONS IN TWO VARIABLES

9.2.1 Standard Form ofLinear Equations in Two Variables

An equation of form $ax + by + c = 0$ where a, b, and c are real numbers, a and b are not both zero is called a linear equation in two variables $x \ and \ y$.

9.2.2 Solution of a linear equation in two variables

A solution of a linear equation means a pair of values one for $x \ and \ other \ for \ y$ which when substituted in the given equation makes its both sides equal.

If $x = m$ and $y = n$ satisfy the equation $ax + by + c = 0$, then $am + bn + c = 0, m$ and n is called a solution of the equation.

9.2.3 Geometrical meaning of a solution

The linear equation $ax + by + c = 0$ always represents a straight line. So, every solution in two variables is a point in the line. Hence, (x, y) as a solution of linear equation in two variables corresponds to a point on the line representing the equation.

9.2.4 A system of Linear equations in two variables

A pair of linear equations in two variables is said to form a system of simultaneous equations i.e.

$$4x - 5y + 7 = 0$$
$$3x + 2y + 5 = 0$$

9.3 ALGEBRAIC METHOD OF A SOLUTION OF A PAIR OF LINEAR EQUATION

There are many methods of solving a system of linear equations in two variables. The two methods of a system of linear equations in two variables are discussed below.

9.3.1 Substitution Method

Using this method, one variable is kept in the left side and second variable is kept in the right side of one equation. Then value of one variable is put in the other equation to convert into one variable. After converting it one variable, the value of one variable is obtained and the value of second variable is evaluated by putting in any given equation. Let us understand it in the following examples.

Example1: Solve the following system of equations by using the method of substitutions

$$x + 2y = -1, 2x - 3y = 12$$

Solution: The given system of equations is

$$x + 2y = -1 \text{ .... (1)}$$

$$2x - 3y = 12 \text{ ....(2)}$$

From the equation (1), we get

$x=-1-2y$

Putting $x=-1-2y$ in equation (2)

$2(-1-2y)-3y=12$

$-2-4y-3y=12$

$-7y=12+2=14$

$y=\frac{14}{-7}=-2$

Putting $y=-2$ in equation $x=-1-2y$

We get, $x=-1-2x-2$

$=-1+4=3$

Hence the solution of the given system of equations is $x=3,\ y=-2$.

Example2: Solve the following system of equations by using the method of substitutions.

$x-y=3, \frac{x}{3}+\frac{y}{2}=6$

Solution: The given system of equations is

$x-y=3;$ ....(1) $\frac{x}{3}+\frac{y}{2}=6$ ....(2)

From equation (1), we get, $x=3+y$ substituting $x=3+y$ in equation (2)

$\frac{x}{3}+\frac{y}{2}=6$

$\frac{3+y}{3}+\frac{y}{2}=6$

$\frac{6+2y+3y}{6}=6$

$5y+6=6\times 6=36$

$5y=36-6=30$

$y=\frac{30}{5}=6$

Putting y=6 in equation (1)

$x-6=3$

$x=3+6=9$

Hence the solution is $x=9,\ y=6$

CHECK YOUR PROGRESS 9.1

1. Solve the following system of equations using the substitution method

(i) $2x-y=2,\ x+3y=15$

(ii) $3x+2y=66x+y=18$

(iii) $x+y=a+b,\ ax+by=a^2+b^2$ ....$(x=a, y=b)$

(iv) $x-3y-7=0,\ 3x-3y-15=0$

(v) $\frac{x}{a}+\frac{y}{b}=a+b,\ \frac{x}{a^2}+\frac{y}{b^2}=2$

9.3.2 Elimination Method

Using this method, one variable is eliminated by equating the coefficient of one variable numerically. Then both the equation is added or subtracted according to the negative and positive coefficient. Now, this concept is understood in the following examples:

Example3: Solve the following system of equations by the elimination method

$2x+3y=11,\ 3x+5y=24$

Solution: The given equations are

$2x+3y=11$ …………….. (1)

$3x+5y=24$ …………….. (2)

Multiplying the equation (1) by 3 and equation (2) by 2, we get

$6x+9y=33$ ……………. (3)

$6x+10y=48$ …………… (4)

Subtraction of the equation (4) from (3)

$$\begin{array}{rl} 6x+9y &= 33 \\ \pm 6x \pm 0y &= \pm 48 \\ \hline -y &= -15 \end{array}$$

$y = 15$

Putting $y = 15$ in equation (1)

$$2x+3y=11$$
$$2x+3\times 15=11$$
$$2x=11-45=-34$$
$$x=\frac{-34}{2}=-17$$

Hence the given solution

$x=-17, \quad y=15$

Example 4: Solve for x *and* y

$$\frac{ax}{b}-\frac{by}{a}=a+b, \quad ax-by=2ab$$

Solution: The equations are

$$\frac{ax}{b}-\frac{by}{a}=a+b \quad \text{…………….. (1)}$$

$$ax-by=2ab \quad \text{........................ (2)}$$

Multiplication the equation (1) by a, we get

$$\frac{a^2x}{b}-by=a^2+ab \quad \text{……………. (3)}$$

Subtracting equation (3) from equation (2)

$$\begin{array}{rl} ax-by &= 2ab \\ \pm\frac{a^2x}{b} \mp by &= \pm a^2 \pm ab \\ \hline \frac{(ab-a^2)x}{b} &= ab-a^2 \end{array}$$

Or $x=b$

Putting the value in equation (2)

Or $a(b) - by = 2ab$

Or $-by = ab$

Or $y = -a$

CHECK YOUR PROGRESS 9.2

1. Solve the following equations using elimination method

(i) $3x - 5y = 4, \quad 5x + 2y = 8$

(ii) $x + 6y = 8, \quad 7x - 2y = 14$

(iii) $6x + y = 20, \quad 4x - 3y = 17$

(iv) $3x - 5y = 24, \quad x + 3y = 15$

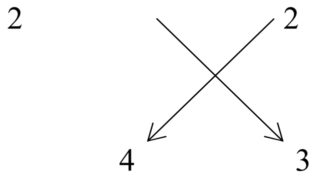
(v) $5x - 2y = 6, \quad 4x + y = 10$

9.4 VEDIC MATHEMATICS METHODS

Vedic Mathematic way to solve the simultaneous equations in two variables is very interesting and unique. Vedic Mathematics Sutras 'Urdhvatiryagbhyam' and 'ParavartyaYojayet' and are applied to solve the equations. If both the sutras are applied, the solution of the simultaneous linear equation can be evaluated in one line easily. To find the value of two variables, the method to evaluate the denominator is common in both cases. Here we first explain the procedure to find the values of the denominator of both variables. Let us understand this concept in the following examples:

Example 5: Solve $2x + 2y = 6$ and $4x + 3y = 5$

Solution: To find the denominator, the coefficients of both the variables are taken firstly and the Vedic Mathematics Sutra 'Urdhvatiryagbhyam' is applied to calculate the denominator



Hence the denominator of x and y = 2 × 3 –2 × 4 = – 2

To evaluate the numerator of the variable x, we take the coefficient of y and constant. Then the Vedic Mathematics Sutra 'Urdhvatiryagbhyam' is applied and the sign is changed.

2 6

3 5

Hence the Numerator of x = – (2 × 5 –3 × 6) = 8

To evaluate the numerator of the variable y, we take the coefficient of x and constant. Then the Vedic Mathematics Sutra 'Urdhvatiryagbhyam' is applied

2 6

4 5

Hence the Numerator of y = (2× 5 –4 × 6) = –14

$x = \frac{8}{-2} = -4$ $y = \frac{-14}{-2} = 7$

Example6: Solve $3x + 2y = 7$ & $4x + (-5)\,y = 3$

Solution:

3 2

4 –5

The denominator of x and $y = 3 \times (-5) - 2 \times 4 = -23$

2 7

– 5 3

The numerator of $x = -(2 \times 3 - 7 \times -5) = -41$

3 7

4 3

The numerator of $y = 3 \times 3 - 7 \times 4 = -19$

$$x = \frac{-41}{-23} = \frac{41}{23} \qquad y = \frac{-19}{-23} = \frac{19}{23}$$

9.5 SPECIAL CASE-I OFSOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

The simultaneous simple equations in two or more variable which is very difficult in solving the equation due to large coefficients of x and y and constant also can be solved using the Vedic Mathematics Sutra 'AnurupyeSunyamManyat' The meaning of this sutra is 'if one is in the ration, the other one is zero'. If the ratio ofthe coefficient of $x$and the constant term in both the equation isthe same, then the value of y will be zero. If the ratio of the coefficient of y and the constant term is zero, then the value of x will be zero. Let'sunderstand this concept in the following way:

Example7: Solve $24x + 63y = 189$ and $31x + 72y = 216$ using Vedic Mathematic concept.

Solution:

The coefficients of both the variables x and y are large and difficult to operate for the students easily. Using Vedic Mathematics Sutra 'AnurupyeSunyamManyat', the solution of such problems can be solved easily. As we observe that the coefficient of y and constant term in both the linear equations 63:189 and 72:216 are the same in ratio i.e. 1:3. And the sutra says that, in such a case, if one is in the same ratio, the other is zero. Hence, $x = 0$. The ratio is 1:3 for y. Hence $x = 0$ and $y = 3$. Here we conclude that if the ratio of one variable and the constant term is the same in both the cases, then the value of the other variable becomes zero and zero is put in one of the linear equations.

Example8: Solve: $52x + 47y = 260$

$45x + 59y = 225$

Solution: In thiscase, the ratio of the coefficient of x and constant term is same i.e. 1:5 and the coefficient of other variable is different. In this case, $y = 0$ and $x = 5$

Example9: Solve: $3ax + by + 2cz = b$

$5bx + ay + 5z = a$

$2ax + 6cy + cz = 6c$

Solution: In this case, the coefficient of y and constant term is 1:1. Hence $x = 0$,

$y = 1$, and $z = 0$

CHECK YOUR PROGRESS 9.3

1. Solve the following questions using Vedic Mathematics Sutra 'Urdhvatiryagbhyam'

(i) $2x + 3y = 15$ & $x + 3y = 4$

(ii) $5x - y = 12$ & $7x - 3y = 5$

(iii) $x + 6y = 20$ & $2x + 3y = 4$

(iv) $64x - 29y = 145$ & $27x - 48y = 240$

(v) $51x - 38y = 153$ & $42x - 63y = 126$

(vi) $\frac{7}{x} + \frac{2}{y} = 5$ & $\frac{5}{x} + \frac{7}{y} = 8$

(vii) $\frac{1}{x} - \frac{5}{y} = 4$ & $\frac{3}{x} + \frac{5}{y} = 4$

9.6 SOLUTION OF A SYSTEM OF LINEAR EQUATION IN TWO VARIABLES USING GRAPHICAL METHOD

Using this method, graphs of both the equations are required to draw and then the solutions are obtained in the following way:

9.6.1 Intersecting Lines

In thistype, the graphs of both the simultaneous equations intersect at a common point. The x-axes and y-axes of the common point of both graphs give the values of x and y.

9.6.2 Parallel Lines

In thistype, the graphs of both the simultaneous equations do not intersect at a common point. If there is no common pointin a system of simultaneous equations, then there is no solution to these equations.

9.6.3 Coincident Lines

In this case, the graphs of a system of linear equations coincide with each other. Hence every common point of such lines will be the solution which means there will be infinitely solutions.

Example10: Solve the following system of equations

$x + 5y = 42x - y = 8,$

Solution:

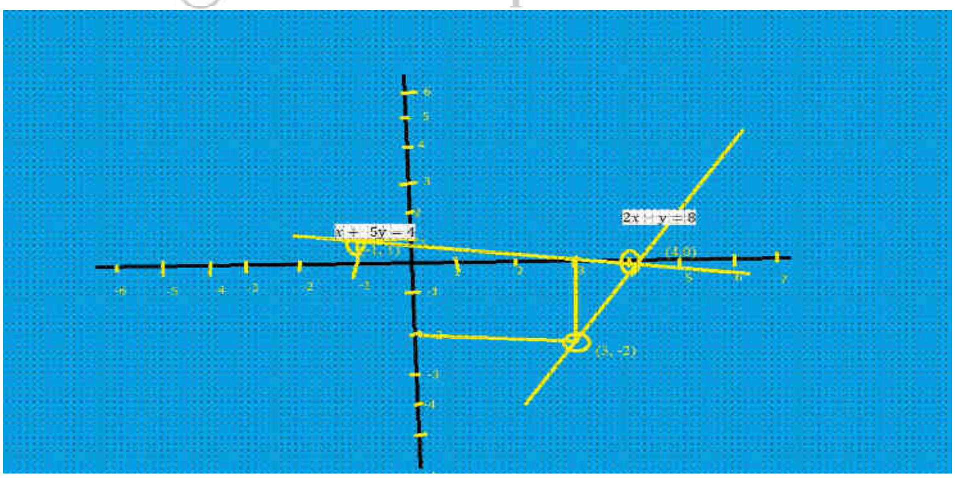
To solve the equations graphically, we need at least two solutions to draw the graphs. To obtain the two solutions for equation $x + 5y = 4, y = 0$, is put and the value of $x = 4$ is obtained. Similarly, if $y = 1$, then $x = -1$.

Hence, the solutions if equations $x + 5y = 4$ are shown in the following table:

x	4	–1
y	0	1

To obtain the two solutions for equation $2x - y = 8, x = 3$, is put and the value of $y = -2$ is obtained. Similarly, if $x = 4$, then $y = 0$, Hence, the solutions if equations $x + 5y = 4$ are shown in the following table:

x	3	4
y	-2	0



In the above figure, both the graphs have been drawn using the solution of equations from the table. Both the graphs intersect at the point (4,0). Hence the interesting point is the solution of a system of equations.

Here are the following features of the solution of a pair of linear equations

(i) A pair of linear equations in two variables, which has a solution, is called a unique solution and has a consistent solution.

(ii) A pair of linear equations which have no solution is called an inconsistent solution.

(iii) A pair of linear equations which are equivalent has many distinct common solutions, such a pair of linear equations is called dependent pair of linear equations in two variables.

Let us consider this concept by a pair of general linear equations in the following way:

A pair of linear equations having two variables are

$a_1x + b_1x = c_1$ and $a_2x + b_2x = c_2$

The lines represented by these equations will intersect at a point when $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(i) The lines represented by these equations will be parallel when $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) The lines represented by these equations will be coincident when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example11: On comparing the coefficient of linear equations $5x + 4y = 12$ and $10x + 8y = 24$, decide whether the lines (i) interesting a point, (ii) coincident lines (iii) Parallel lines?

Solution: On comparing the coefficient of equation $5x + 4y = 12$ and $10x + 8y = 24$

$$\frac{a_1}{a_2} = \frac{5}{10} = \frac{1}{2}$$

$$\frac{b_1}{b_2} = \frac{48}{8} = \frac{1}{2}$$

$$\frac{c_1}{c_2} = \frac{12}{24} = \frac{1}{2}$$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, hence the lines represented by these equations are coincident.

CHECK YOUR PROGRESS 9.4

1. On comparing the coefficient of linear equations, check the following equations about (i) interesting at a point, (ii) coincident lines (iii) Parallel lines

(i) $6x + 3y = 7$ & $12x + 6y = 8$

(ii) $4x + 7y = 7$ &- $8x + 3y = 4$

(iii) $8x + y = 7$ & $32x + 4y = 28$

(iv) $49x + 56y = 36$ & $7x + 8y = 4$

(v) $21x + 16y = 7$ & $7x + 4y = 4$

2. Solve the following questions using graphical method

(i) $x + 3y = 7$ & $2x + 3y = 5$

(ii) $x - y = -3$ & $x - 3y = 1$

(iii) $2x + y = 5$ & $x + y = 3$

(iv) $4x + y = 9$ & $5x + y = 13$

TERMINAL EXERCISE

1. The standard form of linear equations in two variables is

(i) $ax + by - c = 0$

(ii) $ax - by + c = 0$

(iii) $ax + by + c = 0$

(iv) $ax + by = +c$

2. The methods to solve the system of Linear equations in two variables are:

(i) Substitution Method

(ii) Elimination Method

(iii) Vedic Mathematics Method

(iv) All of the Above

3. The solution of $x - y = 2$ and $x + y = 10$

(i) $x = -6 \quad y = 2$

(ii) $x = 2 \quad y = 6$

(iii) $x = 6 \quad y = 4$

(iv) $x = -6 \quad y = -2$

4. Solve the following system of equations using the substitution method

(i) $x - 3y = 6, \; x + 3y = 12$

(ii) $2x + 5y = 6, \; x - 2y = 12$

5. Solve the following equations using the elimination method

(i) $x - 4y = 16, \; x + y = 10$

(ii) $2x - y = 10, \; 4x + y = 20$

6. Solve the following questions using Vedic Mathematics Sutra 'Urdhvatiryagbhyam'

(i) $5x + y = 20, \; 3x + y = 8$

(ii) $2x + 5y = 20, \; 4x + y = 4$

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 9.1

(i) $x = 3, \; y = 4$

(ii) $x = \frac{10}{3}, \; y = -2$

(iii) $x = a, \; y = b$

(iv) $x = 4, \; y = -1$

(v) $x = a^2, \; x = b^2$

CHECK YOUR PROGRESS 9.2

(i) $x = \frac{48}{31},\ y = \frac{4}{31}$

(ii) $x = \frac{25}{11},\ y = \frac{21}{22}$

(iii) $x = \frac{7}{2},\ y = -1$

(iv) $x = \frac{21}{2},\ y = \frac{3}{2}$

(v) $x = 2,\ y = 1$

CHECK YOUR PROGRESS 9.3

(i) $x = 11,\ y = \frac{-7}{3}$

(ii) $x = \frac{31}{8},\ y = \frac{59}{8}$

(iii) $x = -4,\ y = 4$

(iv) $x = 0,\ y = -5$

(v) $x = 3,\ y = 0$

(vi) $x = \frac{39}{19},\ y = \frac{39}{31}$

(vii) $x = \frac{1}{2},\ y = \frac{-5}{2}$

CHECK YOUR PROGRESS 9.4

1.

(i) Parallel lines

(ii) Interesting at point

(iii)Coincident lines

(iv)Parallel lines

(v) Interesting at a point

2.

(i) $x = -2,\ y = 3$

(ii) $x = -5,\ y = -2$

(iii) $x = 2,\ y = 1$

(iv) $x = 4, \ y = -7$

10 QUADRATIC EQUATIONS

INTRODUCTION

The different types of polynomials have been introduced in the previous chapter. In this chapter, the quadratic equations will be dealt with. The standard form of quadratic equations will be introduced and word problems will be solved by the formation of quadratic equations.

10.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- identify the quadratic equations;
- able to write the standard form of quadratic equations;
- solve the quadratic equations by (i) factorization (ii) quadratic formula (iii) Vedic Mathematics Sutra
- know about the nature and roots of quadratic equations.
- solve word problems by formingof quadratic equations

10.2 QUADRATIC EQUATIONS

We have been introduced to the polynomials of degree two. The polynomial consisting of degree two is called the quadratic polynomial. When a quadratic polynomial is equated to zero, then it is called a quadratic equation. In this chapter, quadratic equations with one variable will be dealt with. The standard form of a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$. In fact, Indian Mathematician, *Brahmagupta* (CE598 – 625) gave an explicit formula to solve the quadratic equations of the standard form $ax^2 + bx + c = 0$. One another Indian Mathematician *Sridharacharya* (CE 1025) derived a formula known as the quadratic formula as quoted by *Bhaskara*-II for solving a quadratic equation by the method of completingthe square. The examples of quadratic equations are $5x^2 + 15x + 10 = 0$. As it is observed that the degree of the quadratic equation is 2 and its standard form is in descending order.

Example 1: Which of the following equations are quadratic equations? If so, then write their standard form

(i) $7x+9=0$ (ii) $5x^2+10=0$ (iii) $4x^2+x+7=0$

(vi) $\frac{1}{x}+x=6$ (v) $6+9x^2+x+5=0$ (vi) $(x+5)(x-4)=0$

Solution:

(i) Its degree is not two. Hence it is not a quadratic equation.

(ii) It is a quadratic equation because its degree is two. Its standard form is $5x^2+0x+10=0$

(iii) It is a quadratic equation because its degree is two. Its standard form is $4x^2+x+7=0$

(iv) The equations can be converted in the following way $\frac{1}{x}+x=6$, this can be written as $\frac{1+x^2}{x}=6$ or $1+x^2=6x$ or $x^2-6x+1=0$. It is quadratic equation.

(v) The equation $6+9x^2+x+5=0$ can be written as $9x^2+x+11=0$. Hence it is a quadratic equation since its degree is two.

(vi) The equation $(x+5)(x-4)=0$ can be written as $x^2+5x-4x-20=0$ or $x^2+x-20=0$. Its degree is two, hence it is a quadratic equation.

CHECK YOUR PROGRESS 10.1

I. Which of the following equations are quadratic equations? If so, then write their standard form.

(i) $\frac{x+1}{x}+9=0$ (ii) $x^2+x+\sqrt{5}=0$ (iii) $x+20=0$

(iv) $\frac{1}{x}+\sqrt{x}=34$ (v) $10-4x^2+2x+41=0$

(vi) $(2x+5)(3x+7)=0$

10.3 SOLUTION OF QUADRATIC EQUATIONS

We have been introduced to the zeros of polynomials. A zero of a polynomial is that real number thatmakespolynomial zero which when substituted for the variable. When we discuss the roots of equations, then it is concluded that the value of the left-hand side and right-hand side are equal when the real number is substituted for its variables. Consider the quadratic $x^2+5x-6=0$. If x is replaced by 1 then we get $1^2+5(1)-6=0$. Both the left-hand side and right-hand side are equal. Hence 1 is the root of the equation. It is evident from here that if $x=a$ is put and it satisfies the quadratic equation, then $x=a$ is a solution ofthe quadratic equation. It is evident that the quadratic equations haveat most two roots. There are three types of methods to calculate the roots of quadratic equations. These methods are (i) Factor Method (ii) Quadratic Formula (iii) Vedic Mathematics Sutra

10.3.1 Factor Method

The linear factors of quadratic equations are evaluated to find the roots of the equation. Let us understand this concept by factorization of quadratic equation in the following examples:

Example 2: Solve the equation $x^2+2x-24=0$ by factorization methods

Solution: The factorization can be obtained by splitting the middle terms of the equation or by the Vedic Mathematics Sutra

By splitting the middle term, $x^2+6x-4x-24=0$ or $(x^2+6x)-(4x+24)$ or $x(x+6)-4(x+6)=0$ or $(x+6)(x-4)=0$ or $x=-6,4$

By Vedic Mathematic Sutra

$x^2+2x-24=0$ By factorization of the first term and last term

x 6

x -4

Hence the factors of quadratic equation $x^2+2x-24=0$ are $(x+6)(x-4)=0$ $x=-6,4$ are the solution of the equations

Example 3: Solve the equation $4x^2+12x+9=0$ by factorization methods

Solution: The solution of equation $4x^2+12x+9=0$ by factors by identity $(a+b)^2=a^2+2ab+b^2$ can be evaluated in the following way:

Equation $4x^2+12x+9=0$ can be written as

$(2x)^2+2\times3\times2x+3^2=0$ or $(2x+3)^2=0$

Or $(2x+3)(2x+3)=0$ or $x=\frac{-3}{2},\frac{-3}{2}$ are the solutions of the equations

CHECK YOUR PROGRESS 10.2

1. Find the roots of quadratic equations by factorization method:

(i)	$2x^2+9x+7=0$	(ii)	$25x^2-60x+36=0$
(iii)	$5x^2+16x+12=0$	(iv)	$4x^2+84x+41=0$
(v)	$81x^2+216x+144=0$	(vi)	$5x^2-26x-31$

10.3.2 Quadratic Formula

Now we shall learn a very important formula to find the roots of quadratic equations. To frame the quadratic formula, the standard form of the quadratic equation $ax^2+bx+c=0$ is multiplied by 4*a*, we get

$$4a^2x^2+4abx+4ac=0$$

$4a^2x^2+4abx+4ac+b^2=0+b^2$ [Adding b^2 on both sides]

$(2ax)^2+2(2ax)b+\ b^2=b^2-4ac\ \Rightarrow\ (2ax+b)^2=(\pm\sqrt{(b^2-4ac)})^2$

Or $(2ax+b)=-\ b^2\pm\sqrt{b^2-4ac}$

Or $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

There are two solutions to the quadratic equation $ax^2+bx+c=0$

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}\ \text{and}\ \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The expression $b^2 - 4ac$ is called **'Discriminant'** because it determines the number of solutions or nature of roots of a quadratic equation. The **'Discriminant'** is denoted by D.

Example 4: Find the roots of $4x^2 + 12x + 5 = 0$ using the quadratic formula.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

Where $a = 4$, $b = 12$, $c = 5$

First D = $b^2 - 4ac = 12^2 - 4 \times 4 \times 5 = 144 - 80 = 64$

Hence, the roots are given by

$$x = \frac{-12 \pm \sqrt{64}}{2 \times 4} = \frac{-12 \pm 8}{8}$$

$$x = \frac{-12+8}{8} \text{ or } \frac{-12-8}{8}$$

$$x = \frac{-4}{8} \text{ or } \frac{-20}{8}$$

$$x = \frac{-1}{2} \text{ or } \frac{-5}{2}$$

CHECK YOUR PROGRESS 10.3

1. Find the roots of quadratic equations by quadratic formula:

(i) $3x^2 + 4x + 1 = 0$ (ii) $5x^2 - 6x + 1 = 0$

(iii) $4x^2 + 3x - 12 = 0$ (iv) $x^2 + 11x + 9 = 0$

(v) $x^2 + 5x - 44 = 0$ (vi) $5x^2 - 6x - 31 = 0$

10.3.3 Solution of Quadratic Equation Using Vedic Mathematics Sutra

Vedic Mathematics Sutra- Urdhvatiryagbhyam and ParavartyaYojayate find the roots of quadratic equations. To find the roots of quadratic, the following examples have been solved using this approach:

Example 5: Find the roots of the equation $6x^2 + 52x + 46 = 0$ using Vedic Mathematics approach.

Solution: Using the method of splitting the middle term, the students face difficulties to split themiddle term. Using the concept of Vedic Mathematics, this problem becomes easy for students. Let us understand it by explaining the following steps:

The first term and last terms are factorized. To verify the factors, the addition of cross-product is obtained. If this addition is equal to the coefficient of$x$, then the factors are correct.

$$6x^2+52x+46=0$$

The factors of the first and last terms are

$2x$ $\quad$ 2

$3x$ $\quad$ 23

To verify the factors, the addition of crosswise products is obtained:

$2x\times 23+3x\ \times 2=52x$ which is equal to the middle term.

Hence, the factors are correct and factors of the quadratic equations $6x^2+52x+46=0$ are $(2x+2)(3x+23)=0$, Hence the roots are $x=-1$ or $\frac{-23}{3}$

CHECK YOUR PROGRESS 10.4

1. Find the roots of quadratic equations by using Vedic Mathematics approach:

   (i) $3x^2+16x+21=0$ (ii) $5x^2-27x+28=0$

   (iii) $17x^2+6x-11=0$ (iv) $25x^2+25x+6=0$

   (v) $14x^2+75x-41=0$ (vi) $35x^2+6x-77=0$

10.4 NATURE OF ROOTS

To calculate nature of roots the Discriminant (D)= b^2-4ac is calculated, there will be the following cases of the Discriminant:

(i) If D= $b^2-4ac>0$, then equation has two real distinct roots. The roots will be as

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}$$

(ii) If $D = b^2 - 4ac = 0$, then the equation has tworeal equal roots. The two equal roots are $\frac{-b}{2a}$

(iii) If $D = b^2 - 4ac < 0$, There will be no real roots of the quadratic equations since thesquare root of a negative real number is not a real number.

Example 6: Find the nature of roots of the equation $2x^2 + 16x + 25 = 0$ without calculating actual roots.

Solution: To find the nature of the roots of the equation $2x^2 + 16x + 25 = 0$, this equation is compared with $ax^2 + bx + c = 0$, where$a = 2$, $b = 16$, $c = 25$

$D = b^2 - 4ac = 16^2 - 4\times 2\times 25 = 256 - 200 = 56 > 0$, Hence roots are real and distinct

CHECK YOUR PROGRESS 10.5

1. Find the nature of the roots of the following quadratic equations:

(i) $x^2 + 11x + 2 = 0$	(ii) $10x^2 - 21x + 2 = 0$
(iii) $8x^2 + 7x - 10 = 0$	(iv) $x^2 + 4x + 4 = 0$
(v) $13x^2 + 5x + 4 = 0$	(vi) $9x^2 + 42x + 49 = 0$

10.5 WORD PROBLEMS

Using the formation of quadratic equations, words problems can be solved as explained in the following example:

Example 7: The Product of two consecutive positive integers is 306. Find the integers.

Solution: Let the two consecutive integers numbers be x and$x + 1$

The product of two consecutive integers is$x (x + 1)$

According to Statement

$$x(x+1) = 306$$

$$x^2 + x - 306 = 0$$

$$x^2 + 18x - 17x - 306 = 0$$

$$x(x+18) - 17(x+18) = 0$$

$$(x+18)(x-17) = 0$$

$$x = 17, -18 \text{ (Neglected)}$$

Hence two positive consecutive Integers 17 and 18

CHECK YOUR PROGRESS 10.6

(i) Divide 12 into two parts such that the sum of their squares is 74.

(ii) The sum of two natural numbers is 8 and the sum of their reciprocals is $\frac{18}{5}$. Find the numbers

(iii) The length of a rectangle exceeds its width by 8cm and the area of a rectangle is 240 sq. cm. Find the dimensions of the rectangle.

(iv) The sum of squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.

(v) The numerator of a fraction is one less than its denominator. If 3 is added to its numerator and denominator, the fraction is increased by $\frac{3}{28}$. Find the fraction.

TERMINAL EXERCISE

1. Which of the following are the quadratic equations?

   (i) $(x+1)^2 = (x+3)^2$

   (ii) $(x+5)^2 = 5$

   (iii) $(x-1)^2 = 4(x+3)$

   (iv) $\frac{1}{x} + x = 7x^2$

2. Check whether 2 is a root of the equation $7x^2 - 7x + 6 = 0$.

3. Solve the equation $2x^2 + x - 6 = 0$ by factorization method.

4. Solve the equation $5x^2 - 6x - 2 = 0$ by quadratic formula.

5. If the equation $x^2 + 4x + k = 0$ has real distinct roots, then find the value of k.

6. Find the discriminant of the quadratic equation $36x^2 - 12ax + (a^2 - b^2) = 0$

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 10.1

(i) Yes, $x^2 + 9x + 1 = 0$ (ii) Yes, $x^2 + x + 5 = 0$

(iii) No (iv)No

(v)Yes, $-4x^2 + 2x + 51 = 0$, (vi)Yes, $6x^2 + 29x + 35 = 0$

CHECK YOUR PROGRESS 10.2

(i) $x = -1, \frac{-7}{2}$ (ii) $x = \frac{6}{5}, \frac{6}{5}$ (iii) $x = -2, \frac{-6}{5}$

(iv) $x = \frac{-1}{2}, \frac{-41}{2}$ (v) $x = \frac{-12}{9}, \frac{-12}{9}$ (vi) $x = -1, \frac{31}{5}$

CHECK YOUR PROGRESS 10.3

(i) $\frac{-1}{3} - 1$ (ii) $\frac{1}{5}, 1$ (iii) $\frac{-3 \pm \sqrt{201}}{8}$

(iv) $\frac{-11 \pm \sqrt{85}}{2}$ (v) $\frac{-5 \pm \sqrt{201}}{2}$ (vi) $\frac{3 \pm 2\sqrt{41}}{5}$

CHECK YOUR PROGRESS 10.4

(i) $x = -3, \frac{-7}{3}$ (ii) $x = 4, \frac{7}{5}$ (iii) $x = -1, \frac{11}{17}$

(iv) $x = \frac{-2}{5}, \frac{-3}{5}$ (v) $x = \frac{1}{2}, \frac{-44}{7}$ (vi) $x = \frac{7}{5}, \frac{-11}{7}$

CHECK YOUR PROGRESS 10.5

(i)	Two distinct roots	(ii)	Two distinct roots	(iii)	Two distinct roots
(iv)	Two equal roots	(v)	No real roots	(vi)	Two equal roots

CHECK YOUR PROGRESS 10.6

(i) 5,7 (ii) 3,5 (iii) 20, 12 (iv) 8, 12 (v) $\frac{3}{4}$

11 ARITHMETIC PROGRESSIONS

INTRODUCTION

Patterns play a major role in the solution of problems in all areas of our life.Patterns are found all around us in flowers,plants, arrangement of houses in a city, weather etc. Try to list thingsaround you where you observe a pattern! Finding a pattern is a useful problem-solving strategy in mathematics. Thus, we try to observe patterns in different concepts of mathematics, in particular, the numbers and their arrangements in a sequence. We now extend this idea of patterns in understanding a particular sequence of numbers, called *Arithmetic Progressions.*

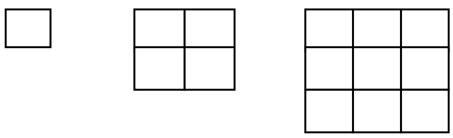
11.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Observe patterns in numbers and extend them
- Know about Arithmetic Progressions (A.P.) from a sequence of numbers.
- Find the General Term of an A.P.
- Add first n terms of an A.P.

11.2 NUMBER PATTERNS

Let us have a look at some situations in our daily life that give rise to some patterns.

1. Geeta works in a software company. Her salary per month in the first year is Rs 35000. Her salaries per month in the next some years are (in Rs.): 35000, 38000, 41000, 44000....
2. A school decides to plant saplings every year in their surroundings. The number of saplings planted every year is: 10,20,40,80,160, ...
3. 

…..

The list of number of squares will be 1,4,9,16, …

Let us take some more lists of numbers –

4. 48, 24, 12, 6, 3, 3/2, …
5. 7,4,1, –2, –5, …
6. 1, 8,27,64,…

In **list 1** above you will find that the numbers increase by 3000.That is,

First number = 35000;

Second number = 35000+ 3000 =38000;

Third number = 38000 + 3000 =41000 and so on. Or,

First number = 35000;

Second number = First number + 3000;

Third number = Second number +3000, …

We say that this list follows a pattern. Why? Because, **any term + 3000 gives the next term**ofthe list**.**

In **list 2** above, you will find that the numbers are increasing,butin a different way.

First term = 10,

Second term = 10 x 2 =20,

Third term = 20 x 2 =40, and so on. Check for the other terms!

That is, First term = 10, Second term = First term x 2, Third term = Second term x 2 ….

So, you can see that except for the first term, the remaining terms can be obtained when the previous term is multiplied by 2. This continues throughout the list.

Thus, in this list you will observe the pattern, **2 x (a term) = next term**

In list 3, the numbers are, $1=1^2$, $4=2^2$, $9=3^2$, $16 = 4^2$,…

Here you will find that first term = 1^2, second term = 2^2, third term= 3^2

Do you see any relation between the term number and the base of the square number? So continuing in this way we can say that the 16th term = 16^2. Thus, this follows a certain pattern.Try to find 25th, 47th terms for yourself !

In all the above lists you will find that if a certain term is given, then you can find the next term.

Try to find patterns in the remaining lists.

The numbers in the above lists are written in general as

$u_1, u_2, u_3, \ldots, u_n \ldots$ *Or,*

$t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$

u_n and t_n represent n^{th} terms of the respective lists.

These are respectively called first term, second term, third term, etc. These lists are also called **sequence of numbers** or **patterns of numbers**.

11.3 ARITHMETIC PROGRESSION

You have seen different patterns above. In all these patterns, different terms are obtained by using different mathematical rules. The numbers in these patterns are called **terms**. Let us see these patterns again,

In (1), (4) and (5) we find that the next term is obtained when we add 3000, –24 and –3 respectively to a term in that list. In other words, the same number in the list is added in a list to get the next number of the list.

However, in (2) to get the next term we have to multiply a term by 2, in (3) none of the operations of addition, subtraction, multiplication or division of a term by a number gives another term.

Let us see the difference in nature of the terms in these lists.

List No.	**2nd term – 1st term**	**3rd term – 2nd term**	**4th term – 3rd term**	**Are differences same throughout? Yes/No**
1	3000	3000	3000	Yes

2	10	20	40	No
3	3	5	7	No
4	– 24	– 12	–6	No
5	– 3	– 3	–3	Yes
6	7	19	37	No

In all the lists the numbers follow a certain pattern. However in lists (1) and (5)the numbers follow a pattern in which, **some term + a fixed number = next term**.

CHECK YOUR PROGRESS 11.1

1. Observe the following lists of numbers and fill in the table given below:

(i) 2, 5, 10, 17, 26, …

(ii) 28,24,20,16, 12, …

(iii) 15,16,17,18,19,…

(iv) 3,7,10,12,10,7,…

(v) –8, –5, –2,1,4,…

List No.	**2nd term – 1st term**	**3rd term – 2nd term**	**4th term – 3rd term**	**5th term – 4th term**	**Are the differences same throughout? Yes/No**
(i)					
(ii)					
(iii)					
(iv)					
(v)					

The list of numbers in which the successive terms are obtained by adding a fixed number to the preceding term except the first term is called an **Arithmetic Progression**(**A.P.**). Thus, the lists

(i) 35000, 38000, 41000, 44000, ...

(ii) 7, 4,1, –2, –5, …

are Arithmetic Progressions. Each number in the A.P. is called a **term.**

The fixed number that is added to each term is called the **common difference** of the A.P. Thus in (i) the common difference is 3000, in (ii) the common difference is –3.

The first term of an AP is normally denoted by a_1, second term by a_2. . .thenth term by a_n and the common difference by ***d***. Then the AP becomes $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$.

Since this is an A.P. so,

$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1} = d$

Remember that common difference can be **positive, negative** or **zero**.

Example 1: Which of the following lists of numbers are in A.P.? In case of A.P. find their first term and common difference.

(i) 145, 146, 147, 148,. . .,

(ii) 2, 8, 32, 128, 512 ....,

(iii) 5,0,–5,–10,–15,…,

Solution:

(i) Here, the successive terms are, $a_1 = 145$, $a_2=146$, $a_3= 147$, $a_4= 148$ . . .,

$a_1 = 145$ is the first term. The common difference can be found as follows.

$a_2 - a_1 = 146 - 145 = 1$, $a_3 - a_2 = 147 - 146 = 1$, $a_4 - a_3 = 148 - 147 = 1$

In all these cases you will find that the differences of the successive terms are same i.e.1.

So the given list of terms forms an A.P.The common difference $d = 1$.

(ii) Let us find the differences of the successive terms.

$a_2 - a_1 = 8 - 2 = 6$, $a_3 - a_2 = 32 - 8 = 24$.

It can be seen that $a_2 - a_1 \neq a_3 - a_2$. So the given list of terms is not in A.P.

(iii) The differences of the successive terms are as follows:

$a_2 - a_1 = 0 - 5 = -5, a_3 - a_2 = -5 - 0 = -5, a_4 - a_3 = -10 - (-5) = -10 + 5 = -5,$

$a_5 - a_4 = -15 - (-10) = 15 + 10 = -5, \ldots$

It can be seen that the successive differences are equal. So, the given list of numbers is an A.P.

The common difference $d = -5$.

CHECK YOUR PROGRESS 11.2

I. Which of these lists are in A.P.? Find the first term and common difference of those which are in A.P.

(i) –3, –1,1,3, …

(ii) 1,2,3,4…

(iii)1,1.1,1.11,1.111, …

(iv)34,32,30,28,26, …

(v) –27, –23, –19, –15, –11,…

11.3.1 Can you tell the difference between these two APs?

(i) 1,3,5,7, 9, ………

(ii) The cash prizes given to the toppers of classes I to XII in a school are as follows:

350,400,450,500,… 900

In (i) there are only a finite number of terms, so (i) is a **finite arithmetic progression**.

In (ii) the number of terms is not finite, it is called an **infinite arithmetic progression.**

We can now generate an A.P. with a as the first term and d as thecommon difference as,

First Term	Second Term	Third Term	Fourth Term	Fifth Term	Sixth Term	…
a	$a + d$	$a + d + d = a + 2d$	$a + 2d + d = a +$	$a + 3d + d = a + 4d$	$a + 4d + d = a + 5d$	…

			3d			

That is, a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, $a + 5d$,…

This is the **general Form of an A.P.**

Example 2: Write an Arithmetic Progression if the (i) first term is 17 and common difference is 5 (ii) first term is –45 and common difference is –7 (iii) first term is 67 and common difference is 0

Solution:

(i) Here, $a = 17$ and $d = 5$. The required A.P. is

$a = 17$, $a + d = 17+5 = 22$, $a+2d = 22+5 = 27$, $a+3d = 27+5 = 32$, …

Thus, the A.P. is, 17, 22, 27, 32, …

(ii) We, have, $a = -45$, $d = -7$

The A.P. is – 45, – 45 + (– 7) = – 52, – 52 + (– 7) = –59, – 59 + (– 7) = – 66

That is, – 45, – 52, – 59, – 66, …

(iii) We have, a = 67, $d = 0$. So A.P. is 67, 67+0=67, 67+0=67, …

That is 67,67,67,67, …

CHECK YOUR PROGRESS 11.3

Write an Arithmetic Progression whose first term and common difference respectively are:

(i) 34; 5

(ii) – 67; 4

(iii) 78; –3

(iv) – 64; –6

(v) 43; 0

11.3.2 General (nth)Termof an AP

In the beginning we saw an example of the salaries (in Rs) of Geeta in successive years as 35000, 38000, 41000, 44000, ...We can see that this is an AP. By looking at this list we can know the salary of Geeta till 4th year. The salaries of 6th or 7th year can be found by adding

3000 successively. But if her salary for 16th year is to be found then 3000 has to added a number of times and it becomes a time taking process. Can we generate a term that helps us to do it conveniently?For this we need to find the General term or nth term of an AP.

Let us consider an AP whose first term is *a*and the common difference is *d*.Suppose t_1, t_2,t_3,t_4,t_5,…. are the successive terms of this AP.

We have, $t_1 = a = a + 0.d = a + (1-1).\ d$

$t_2 = a + d = a + 1.\ d = a + (2-1).\ d$

$t_3 = a + d + d = a + 2.\ d = a + (3-1).\ d$

$t_4 = a + 2.\ d + d = a + 3.d = a + (4-1).\ d$

$t_5 = a + 3.\ d + d = a + 4.d = a + (5-1).\ d$

Looking at the pattern above we can write some terms as,

$t_{12} = a + (12-1)d = a + 11d$

$t_{20} = a + (20-1)\ d = a + 19d$

Thus, in **general n^th^term,** $\mathbf{t_n = a + (n-1)\ d}$

an is a positive integer or a natural number

If we know the general term of an AP, we can find any desired term of that AP.

Example 3: Find the17th term of the AP

21, 15, 9, 3, –3, –9, …

Solution: Let us first find the nth term of this AP.

Here, First term $a = 21$. Since it is given that it is an AP so difference of any two successive terms will give the common difference $d = 15 - 21 = -6$.

So, $t_n = a + (n-1)d = 21 + (n-1)\ x\ (-6)$

$= 21 + (-6n + 6) = 27 - 6n$

$t_n = 27 - 6n$

17th term will be obtained by putting $n = 17$ in t_n.

$t_{17} = 27 - 6 \text{ x } 17 = -75$

Thus the 17th term of the given AP is $t_{17} = -75$

Example 4: Which term of the AP: 1,4,7,10, 13,… is 82?

Solution: The given AP is 1,4,7,10,13, …

Here, first term $a = 1$, common difference $d = 4 - 1 = 3$.

Suppose 82 is the nth term of the AP.

Thus, $t_n = 82$. We have, $t_n = a + (n-1)\,d$

That is, $a + (n-1)\,d = 82$ Or,

$1 + (n-1) \times 3 = 82$

So, $n - 1 = 27$ Or $n = 28$

Thus 82 is 28th term of the given AP.

Example 5: The first term of an AP is 28 and its common difference is –4.

a) Write the first five terms of the AP.

b) Is there any term which is 0 in this AP?

c) Is there any term which is –10 in this AP?

d) Which term of the AP is –12?

Solution: Given that, first term $a = 28$, and common difference $d = -4$

Now nth term $t_n = a + (n-1)\,d$, that is $t_n = 28 + (n-1)(-4) = 28 - 4n + 4 = 32 - 4n$

So, $t_n = 32 - 4n$.

a) First five terms of this AP are, (Put n= 1,2,3,4,5 in t_n)

$t_1 = 32 - 4 \times 1 = 28$

$t_2 = 32 - 4 \times 2 = 24$

$t_3 = 32 - 4 \times 3 = 20$

$t_4 = 32 - 4 \times 4 = 16$

$t_5 = 32 - 4 \times 5 = 12$

The terms are, 28,24,20,16,12

b) Suppose nth term $t_n = 0$. That is, $32 - 4n = 0$. (t_n is found above).

Here n is a natural number.

32 –4n = 0, so n = 8

So, 0 is the 8th term of the AP.

c) Let –10 be the t_n.

Thus, t_n= –10, so 32 – 4n = –10. That is, 32 + 10 = 4n or n = $\frac{42}{4} = 10\frac{2}{4}$ or $10\frac{1}{2}$

Here we do not get n as a natural number. So, –10 is not a term of the AP.

d) Suppose –12 is the nth term of the AP.

Thus, 32 – 4n = –12 Or, n = 11

Hence, –12 is the 11th term of the AP.

Example 6: The first term of an AP is –7 and the 17thterm is–62. Determine its common difference.

Solution: Given that the first term of the AP is a= –7 and the 17th term is t_{17} = –55.

Let the common difference be d.

We have, $t_n = a + (n - 1)\, d$

For n = 17 and a = –7, it becomes, $t_{17} = -7 + (17 - 1)\, d$

Or, – 55 = –7 + 16d (t_{17} = – 55)

16 d = –48 so d= –3

The common difference is –3

Example 7: The common difference of an AP is 4 and its twentieth term is 66. Find its first term.

Solution: Given that the common difference d =4 and the twentieth term t_{20} = 66.

Let the first term be a.

The nth term is $t_n = a + (n - 1)\, d$

For n = 20, $t_{20} = a + (20-1)\times 4$

Therefore, 66 = a+ 76. Or, a = –10.

The first term of the AP is –10.

Example 8: The first term of an AP is 16 and its tenth term is –29.Find its 17th term.

Solution: We have, first term a = 16. Tenth term t_{10} = –29.

To find the 17th term we need to find the common difference d

Now $t_n = a + (n-1)\,d$

For n = 10, it becomes, $t_{10} = 16 + (10-1)\,d$

Or, $-29 = 16 + 9d$. That is, $d = -5$

Now putting n = 17 in t_n, we get, $t_{17} = a + (17-1)(-5)$

Or, $t_{17} = 16 + 16 \times (-5)$. Thus, $t_{17} = -64$

The 17th term of the given AP is –64.

Example 9: Check whether 254 is a term of the list of numbers 7, 12, 17, 22,27,...

Solution: Here,a_1=7, a_2 = 12, a_3 = 17, a_4 = 22, a_5 = 27...

Now,$a_2 - a_1 = 12 - 7 = 5$; $a_3 - a_2 = 17 - 12 = 5$; $a_4 - a_3 = 22 - 17 = 5$,...

In general, $a_{k+1} - a_k$ is same for k= 1, 2, 3, 4,.....

Thus, the given list of numbers is in AP.

Here,first term a = 7, common difference d = 5.

254 be the nth term of the AP. So, $t_n = a + (n-1)\,d$

Or, $254 = 7 + (n-1) \times 5$

Or, $5(n-1) = 247$

Or, $n = \dfrac{247}{5}$

$n = \dfrac{247}{5} + 1$

It can be seen that n is not a positive integer. This means 254 is not a number in the given list.

Example 10: The first four terms of an AP with first term –6 and common difference

(a) –7 (b) 7, are

Solution: (a) Here $a = -6; d = -7$.

Now, $t_n = a + (n-1)\,d = -6 + (n-1) \times (-7) = 1 - 7n$

So $t_1 = 1 - 7x1 = -6$ (Given); $t_2 = 1 - 7\times2 = -13$, $t_3 = 1 - 7\times3 = -20$

$t_4 = 1 - 7\times4 = -27$

The first four terms of this AP are, –6, –13,–20,–27

(b) Here $a = -6$; $d = 7$.

Again, $t_n = a + (n - 1)\,d = -6 + (n-1) \times 7 = -13 + 7n$

Now, $t_1 = -13 + 7\times1 = -6$ (Given) ; $t_2 = -13 + 7\times2 = 1$; $t_3 = -13 + 7\times3 = 8$

$t_4 = -13 + 7\times4 = 15$

Thus the first four terms of the AP are, –6, 1,8,15

Example 11: Saplings are planted in a field in such a way that there are 35 saplings in the first row, 32 in the second, 29 in the third,26 in the fourth row and so on.There are 2 saplings in the last row. How many rows of saplings have been planted?

Solution: The number of saplings row wise are as follows:

1^{st} row	2^{nd} row	Third row	Fourth row	...	Last row
35	32	29	26	...	2

The list becomes 35,32, 29, 26, ...,2

You can see that 32 –35 = 29 –32 =26 –29 =...= –3

So this list forms an AP.

To find the term number of 2.

Suppose it is t_n. Thatis, $t_n = 2$.

Here first term $a = 35$ and common difference $d = -3$

Now, $t_n = a + (n - 1)\,d$

$2 = 35 + (n-1)(-3)$, Or, $-3n + 3 = -33$. Or, $n = 12$

Thus 2 is the 12^{th}term. That is, 12^{th} row is the last row.

The dots in the sequence show that the **same pattern** of numbers will continue till you get the last number 2. That is,

35 –3 = 32, 32 – 3 = 29, 29 –3 = 26, 26 – 3 = 23… till you get 2

Example 12: If p times the p^{th} term of an AP is equal to q times the q^{th} term, prove that its $(p + q)^{th}$ term is zero, provided $p \neq q$.

Solution: For the given AP, let $a$be the first term,d be the common difference and n be the n^{th} term.

We have, p^{th} term, $t_p = a + (p - 1)\,d$; q^{th} term,$t_q = a + (q - 1)d$; $t_{p+q} = a + (p + q - 1)\,d$

Given that, $p.t_p = q.t_q$

$$p\,[a + (p - 1)\,d] = q\,[a + (q - 1)\,d]$$

Or, $ap + p(p - 1)d = aq + q(q - 1)d$

Or, $ap + p^2d - pd = aq + q^2d - qd$

Or, $ap + p^2d - pd - aq - q^2d - qd = 0$

Or, $ap - aq + p^2d - q^2d - pd - qd = 0$

Or, $a\,(p - q) + (p^2 - q^2)\,d - (p - q)\,d = 0$

Or, $a(p - q) + (p - q)\,(p + q)\,d - (p + q)\,d = 0$ ……… $[p^2 - q^2 = (p - q)(p + q)]$

Or, $(p - q)\,[\,a + (p + q)\,d - d] = 0$

As $p \neq q$,so $p - q \neq 0$. And so we can divide both sides of the equation by $p - q$.

We get, $a + (p + q - 1)\,d = 0$

Thus, $t_{p+q} = 0$. That is, the $(p + q)^{th}$ term is zero.

Example 13: If the 3^{rd} term of an AP is 0 and the 7^{th} term is 8, what is its 10^{th} term?

(A) 10 (B) 14 (C) 16 (D) 30

Solution: Let the first term be a, common difference be d and n^{th} term be t_n.

Then, $t_n = a + (n - 1)\,d$.

For n = 3, $t_3 = a + (3 - 1)\,d$. But given that $t_3 = 0$.

So, $a + 2\,d = 0$ …………… (i)

For n = 7, $t_7 = a + 6d$. But $t_7 = 8$

So, $a + 6d = 8$ ……………. (ii)

(ii) – (i) gives $4d = 8$. Thus, $d = 2$.

From (i) then, we get $a + 2 \times 2 = 0$.

This gives, $a = -4$

Hence, 10^{th} term $t_{10} = a + 9d = -4 + 9 \times 2 = -4 + 18 = 14$

Option (B) is the correct option.

Example14: State whether the following statements True or False. Justify your answer.

(i) The common difference of the AP, a +1, a +1, a +1, a +1, a +1,... is 1

(ii) The n^{th} termof the AP $p,p,p,p,p,...$ is p.

(iii) 3, 7, 11, 15, 19, ... does not form an AP.

Solution:

(i) **False**. The difference of two successive terms will be $(a+1) - (a+1) = 0$.So the common difference will be 0.

(ii) **True**. Here first term $a = p$and common difference $= p - p = 0$

Thus, n^{th} term $t_n = p + (n-1) \times 0 = p$

(iii) **False.** It can be seen that, 7–3 = 11 –7 = 15 – 11 = 19 –15 = 4.The given sequence of numbers forms an AP. However it is mentioned that it does not form an AP. Thus this statement is False.

CHECK YOUR PROGRESS 11.4

1. Write an Arithmetic Progression whose first term is 3 and common difference is

   (i) –3 (ii) 4 (iii) 0

2. Write an AP whose common difference is –5 and first term is

   (i) 15 (ii) –15 (iii) 0

3. Find the

   (i) 11th term of the AP –23,–18,–13,–8,...

   (ii) 19^{th} term of the AP 45,52,59,66,...

4. The first term of an AP is 16 and its common difference is – 6.

   (i) Write the first five terms of the AP.

   (ii) Is there any term which is –16 in this AP?

(iii)Is there any term which is –38 in this AP?

(iv) Which term of the AP is –68?

5. The first term of an AP is 27 and the 11thterm is 77. Determine its common difference.
6. The common difference of an AP is –6 and its twentieth term is–136. Find its first term
7. The first term of an AP is –18 and its twelfth term is 48.Find its 21st term.
8. Check whether 101 is a term of the list of numbers 2,9,16,23,30,...
9. Students of a school are standing in rows in the following way: 54 in the first row, 51 in the second row, 48 in the third row,45 in the fourth row and so on,6 in the last row. How many rows have been formed in this arrangement?
10. State whether the following statements are True or False.Justify your answer.

    (i) 0, 5, 0, 5, 0, 5, forms an AP

    (ii) 3, 3, 4, 4, 5, 5, ... does not form an AP

    (iii) $3, 3^2, 3^3, 3^4, ...$ forms an AP.

    (iv) –5,–8,–11,–14, ... does not form an AP.

11. The angles of a triangle are in AP. The greatest angle is twice the least. Find all theangles of the triangle.

11.3.2 Sum of First *n* Terms of an AP

A student of class IX, Amar, saves Rs 25 in his Piggy bank in the first week of the year. He increases his weekly savings by Rs 2.So his savings per week (in Rs) are 25,27,29,31,…..What will be the total amount collected in the piggy bank in 48 weeks?

To do this we will have to add all the numbers as:

25 + 27 + 31 +…. till 48 terms. Will that be easy to calculate manually? No! So we try to find a way for doing it easily.

You can see that the numbers in this list form an AP with common difference 2.

Observe that the numbers given to Gauss (mentioned in the adjoining box) to add were also in AP. Let us apply his method of calculating the sum!

> When Carl Friedrich Gauss, the great German mathematician was in elementary school, his teacher asked the class to find the sum of first 100 natural numbers. While the rest of the class was struggling with the problem, Gauss found the answer within no time. How could he do that? Probably, he did as follows:
> Let S = 1 + 2 + 3 + ... + 99 + 100 ... (1)
> Writing these numbers in reverse order,
> S = 100 + 99 + 98 + ... + 2 + 1 ... (2)
> Adding (1) and (2), term by term, we get
> 2S = 101 + 101 + 101 + ... + 101 + 101 (100 times) = 100 x 101
> Or S = (100 × 101)/2= 5050

Let us see what the 48th term is. Here first term a = 25, common difference $d = 2$.

So, $t_{48} = 25 + (48 – 1) \times 2 = 189$.

Thus, we have to find

S = 25 + 27 + 29 + ...187+ 189.

We write it in the reverse order and add them.

S = 25 + 27 + 29 + ...187+ 189 (48 terms)

+

S = 189 + 187 +... + 27 + 25

2S = 214 + 214 + 214 +... + 214 (48 terms)

So, $S = \frac{214 x\ 48}{2} = 5136$

Based on this method we will evolve a general way to find the Sum of n terms of an AP,

$a,\ a+d,\ a+2d,\ a + 3d, \ldots ,a + (n–1)d$

Let $S_n = a + (a+d)+(a+2d)+(a + 3d) +\ldots + [a + (n–2) d]+ [a + (n–1) d]$…… (i)

So, $S_n = [a + (n–1) d]+ [a + (n–2) d] +\ldots + (a+2d) + (a+d) + a$…………… (ii)

Adding (i) and (ii) we get

$2 S_n = 2a+ (n –1) d$ (n times)

Or $2 S_n = n [2a+ (n –1) d]$

Or, $S_n = \frac{4}{2}[2a + (n –1) d]$

If the last term is given as l, then, $l = t_n = a + (n –1) d)$

Thus, $S_n = \frac{n}{2}(a + l)$

> $S_n = \frac{n}{2}(2a + (n -1) d)$
>
> Or $S_n = \frac{n}{2}(a + l)$,
>
> where l is the last term,

Example 15. Find the sum of the first 12 terms of the following AP

(i) 71, 76, 81, 86,...

(ii) – 125, – 122, – 119, – 116,...

Solution:(i) Here first term a=71, common difference d =5, number of terms to be added n =12

Sum of n terms $Sn = \frac{n}{2}[2a + (n-1)\, d]$

$= \frac{12}{2}\,[2\times 71 + (12-1)\times 5]$

= 6 (197) = 1182

Hence the required sum = 1182

(ii) Here first term a= –125, common difference d = 3, number of terms to be added n = 12

Sum of n terms $Sn = \frac{n}{2}[2a + (n-1)\, d]$

$= \frac{12}{2}[2\times(-125)+(12-1)\times 3]$

= 6(–217) = –1302

Thus, the required sum = –1302

Example 16: How may terms of the AP 3,6,9,12,15,.... are needed to get the sum 630?

Solution: It can be seen that, First term a = 3, common difference d = 3 (Check!)

Let 630 be the sum of n terms of the given AP.

That is, $Sn = \frac{n}{2}[2a+(n-1)\, d]$

Or, $630 = \frac{n}{2}[2\times 3+(n-1)\times 3]$

Or,1260 = n[6 +3n – 3]

Or,$3n^2 + 3n - 1260 = 0$

Or, $n^2 + n - 420 = 0$

Or, (n–20)(n+21) = 0

This gives n = 20 or n = –21

n = –21 is not possible as n is a positive integer.

Hence n = 20. Thus, addition of 20 terms of the given AP gives the sum 630.

Example 17: Find the sum of the first 500 positive integers.

Solution: We have to find S = 1 + 2 + 3 + 4 + ...+ 500

You can see that the first term a = 1; common difference d = 1;number of terms to be added is n = 500 and the last term is given as l = 500.

So we use the formula $Sn = \frac{n}{2}(a + l)$.

So, $S500 = \frac{500}{2}(1 + 500) = 250 \times 501 = 125250$

The sum of the first 500 terms of the given is 125250

Example 18: Find the sum of the first n positive integers.

Solution: Let S_n = 1 + 2 + 3 + 4 + ...+ n

Here first term a = 1; common difference d = 1, number of terms = nand last term l = n.

We use the formula involving the last term, $Sn = \frac{n}{2}(a + l)$.

Thus $Sn = \frac{n}{2}(a + l)$ or $\frac{n}{2}(1 + 4)$ or $\frac{n(n + 1)}{2}$

Thus, the sum of first n positive integers is $\frac{n}{2}(n + 1)$

Example 19: The sum of first ten multiples of 4 is

(A) 40 (B) 220 (C) 60 (D) 120

Solution: Let sum of 10 multiples of 4 be S_{10} = 4 + 8+ 12+ 16+...

Here, first term a = 4; common difference d = 4, number of terms n=10

Thus, $Sn = \frac{n}{2}[2a + (n - 1)\, d]$

$= \frac{10}{2}[2 \text{ x } 4 + (10 - 1) \times 4]$

= 5(44) = 220

Thus the sum of the first 10 multiples of 4 is 220.

Example 20: The sum of any three consecutive terms of an AP is 27 and their product is 504. Findthe three terms of the AP.

Solution: Suppose the three consecutive terms of the AP are: $a - d, a, a + d$

By given condition, $a - d + a + a + d = 27$…… (i)

and$(a - d)a(a + d) = 27$ …………… (ii)

From (i) $3a = 27$ or, $a = 9$

From (ii) $(9-d) \times 9 \times (9 + d) = 504$

Or, $9\,(81 - d^2) = 504$ or $(81 - d^2) = \frac{504}{9}$ $(9 + d)\,(9 + d) = 56$

$81 - d^2 = 56$

$d^2 = 25$

$d = +5$ or -5

Or, $d = 500 - 5$

Thus the terms are 4,9,14 or, 14,9,4

Example 21: If for an AP, $a = 12$; $d = -3$ and $S_n = -204$, find n.

Solution: $Sn = \frac{n}{2}[2a + (n-1)\,d]$

Or, $-204 = \frac{n}{2}[2 \times 12 + (n-1) \times (-3)]$

$-408 = n\,[24 - 3n - 3]$

$n^2 - 9n - 136 = 0$

Or, $(n - 17)(n + 8) = 0$

Since n is a positive integer, $n \neq -8$. Thus, n = 17.

Example 22: Find the sum of first 100 odd natural numbers.

Solution: We have to find S = 1 + 3 + 5 +7 + …+

Here, a = 1, d = 2 (Check it!), n = 100

$$S = \frac{n}{2}\,[2a + (n-1)\,d]$$

$$= \frac{100}{2}\,[2\times 1 + (100-1)\times 2]$$

= 50 x 200 = 10000

Example 23: A sum of Rs2700 is to be used to give nine cash prizes to students of a school for their overall academic performance. If each prize is Rs 50 less than its preceding prize, find the value of each of the prizes.

Solution: Let the total sum be S = 2700.

Since there are nine such prizes, so, n = 9.

It is given that value of each prize is Rs 50 less than its preceding prize. That is, difference between two successive values of prizes is Rs 50.

This means that the sequence of values of the prizes form an AP with common difference d = 50.

If we arrange these values of prices from lower value to higher value, then let the first term of the AP be$a$.

$$\text{Thus, } S = \frac{n}{2}\,[2a + (n-1)\,d]$$

$$2700 = \frac{9}{2}\,[2a + (9-1)\times 50]$$

Solving this we get, a= 100. Thus, the values of prizes (in Rs) are: 100,150,200,250,300,350,400,450,500.

Example 24: Find the sum of all 3–digit numbers which leave the remainder 2, when divided by 6.

Solution: The first three–digit number which when divided by 6 gives remainder 2 is 104.

So, we get an AP with terms 104, 110, 116, 122, …

To calculate the sum of such three-digit numbers we need to know the last such three-digit number.

The last 3-digit number is 999, Dividing this number by 6 gives a remainder 3. So, we can say that last 3-digit number with remainder 2 is 998.

We now find the number of terms to be added i.e n.

Now, $t_n = 998$, first term $a = 104$ and $d = 6$

$t_n = a + (n-1)\,d$

$998 = 104 + (n-1) \times 6$

Or, $n = 150$

Thus, sum of the numbers $S = \frac{n}{2}[2a + (n-1)\,d]$

$S = \frac{150}{2}[2 \times 104 + (150-1)]$

Or, S= 82650

Thus, the required sum of numbers is 82650.

Example 25: If sum of first n terms of an AP is $5n + 2n^2$, find r^{th} term of the A.P.

Solution: Given that $S_n = 5n + 2n^2$

Now $S_r - S_{r-1} = t_r$

Or, $(5r + 2r^2) - [5(r-1) + 2(r-1)^2] = t_r$

Or, $5r + 2r^2 - 5r + 5 - 2r^2 + 4r - 2 = t_r$

Or, $t_r = 4r + 3$

Thus, the r^{th} term of the AP is $4r + 3$.

> S_r = sum of first r terms = $t_1 + t_2 + t_3 + \ldots + t_{r-1} + t_r$
>
> S_{r-1} = sum of first r-1 terms = $t_1 + t_2 + t_3 + \ldots + t_{r-1}$
>
> So, $S_r - S_{r-1} = t_r$

Example 26: If sum of first 8 terms of an AP is 100 and that of the first 18 terms is 495, find the sum of first 10 terms.

Solution: Let a be the first term and d be the common difference of the AP.

Given that $S_8 = 100$, $S_{18} = 495$

Thus, $\frac{8}{2}[2a+(8-1)\ d]=100$

Or, $2a + 7d = 25$ ………. (i)

Also, $\frac{18}{2}[2a + (18-1)\ d] = 495$

Or, $2a+17d = 55$ ……….(ii)

Solving (i) and (ii) we get, $a = 2$ and $d = 3$.

So, $S_{10} = \frac{18}{2}\ [2\times2+(10-1)\times3]$

$= 9(4 + 27) = 279$

Thus, sum of 10 terms of the given AP is 279.

CHECK YOUR PROGRESS 11.5

1. Find the sum of the following APs:

   (i) 4, 7, 10, . . ., to 12 terms.

   (ii) –13, –17, –21, . . ., to 15 terms.

   (iii) –2, 0, 2, 4…, to 20 terms

2. How many terms of the AP:

   (i) –5, –1, 3, 7, ... are needed to get the sum 400?

   (ii) 7, 5,3, 1, … are needed to get the sum – 240?

3. The sum of first twelve multiples of 7 is

   (A) 546 (B) 456 (C) 564 (D) 84

4. The sum of any three consecutive terms of an AP is 33 and their product is 1287. Findthe three terms of the AP.

5. If for an AP, $a = 7$; $d = -3$ and $Sn = -210$, find n.

6. Find the sum of the first 17 multiples of 7.

7. If for an AP, first term a =–7, last term l = –67,number of terms n =13,find common differenced.

8. If the sum of the first n terms of an AP is $n2+ 6n$.Find its 11^{th} term.

9. Pooja saves some amount in the month of January. She increases the saving amount every month by Rs 30. Her saving in the 15th month is Rs 572. What was the amount she saved in the first month?

10. Find the sum of the odd numbers between 10 and 70.

11. In an AP, if $S_n = 4n^2 + 10n$ and $a_k = 142$, find the value of k.[a_k represents k^{th} term]

RECAPITULATION POINTS

- (i) The list of numbers in which the successive terms are obtained by adding a fixed number to the preceding term except the first term is called an **Arithmetic Progression**(**AP**).

  (ii) Each number in the list is called a **term.**

  (iii) The fixed number added to each term is called the **common difference.**

- An AP with a finite number of terms is called a **Finite AP** and an AP with an infinite number of terms is called an **Infinite AP.**
- The general or n^{th} term of an AP is given by, $t_n = a + (n-1)\,d$,where n is a positive integer or a natural number.
- The sum of n terms of an AP is given by $Sn = \frac{n}{2}(2a + (n-1)\,d)$ or $Sn = \frac{n}{2}(a + l)$,where l is the last term.

TERMINAL QUESTIONS

1. Which of the following lists are Arithmetic Progressions?

   (i) 2,5,8,10,13, …

   (ii) –7, –5, –3, –1, …

   (iii) 6,4,2,0,–2, …

   (iv) 7,5,1, –1, …

2. Write the nth term of each of the following Arithmetic Progressions.

   (i) 17,23,29,35,…

(ii) 35,30,25,20,…

(iii) –7,–9,–11,–13,…

3. Finda, b and c such that the following numbers are in AP: *a*, 12, *b*,26, *c*.
4. The12th, 7th and the last term of an AP are –16, –1 and –40, respectively. Find the common difference and the number of terms.
5. Find whether 47 is a term of the AP: –7, –4, –1,... or not. If yes, find which term it is.
6. Find the 7thterm from the end of the AP: 3,–2,–7,–12, ..., –92 .
7. The angles of a triangle are in AP. The greatest angle is thrice the least. Find all theangles of the triangle.
8. Find the sum of last fifteen terms of the AP: 4,7,10,13,...,211
9. The sum of the first six terms of an AP and the sum of the first nine terms of thesame AP is 183. If the sum of the first fifteen terms of this AP is 345, find the sum ofits first twenty five terms.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 11.1

List No.	2^{nd} term – 1^{st} term	3^{rd} term – 2^{nd} term	4^{th} term – 3^{rd} term	5^{th} term – 4^{th} term	Are the differences same throughout? Yes/No
(i)	3	5	7	9	No
(ii)	– 4	– 4	– 4	– 4	Yes
(iii)	1	1	1	1	Yes
(iv)	4	3	2	–2	No
(v)	3	3	3	3	Yes

CHECK YOUR PROGRESS 11.2

(i) Yes; –3 ; 2

(ii) Yes ; 1 ; 1

(iii) No

(iv) Yes; 34; –2

(v) Yes; –27; 4

CHECK YOUR PROGRESS 11.3

(i) 34, 39, 44, 49, ...

(ii) –67, –63, –59, –55,...

(iii) 78, 75, 72, 69, 66, ...

(iv) –64, –70, –76, –82, ...

(v) 43, 43, 43, 43, ...

CHECK YOUR PROGRESS 11.4

1. (i) 3, 0,–3, –6, –9, ... (ii) 3,7,11,15,19,... (iii) 3,3,3,3,3,...

2.(i) 15,10,5,0,–5,... (ii) –15,–20,–25,–30, ... (iii) 0,–5,–10,–15,–29, ...

3.(i) 27 (ii) 171

4.(a) 16, 10, 4, –2, –8,... (b) No (c) Yes, 10^{th} term (d) 15^{th} term

5. 5 6.–22 7.102 8.101 is not a term of the given AP 9.17

10.(i) False (ii) True (iii) False (iv) False

11.40º, 60º,80º

CHECK YOUR PROGRESS 11.5

1. (i) 246 (ii) –615 (iii) 340

2. (i) 16 (ii) 20

3. (A)

4. 9,11,13 Or 13,11,9

5. 15

6. 1071 7. –5 8. 27 9. Rs 152 10.1200 11.17

SUPPLEMENTARY STUDY MATERIAL

1. Mathematics Textbook for class X; NCERT publication

2. Mathematics, Exemplar Problems for class X;NCERT publication.

12 LINES AND ANGLES

INTRODUCTION

Can you imagine a road that never ends? What would a small portion of this hypothetical road look like? What if another road cuts across the imaginary never-ending road?

We have already studied the geometry of lines – never-ending straight geometrical objects. Think about this, can you draw a line in your notebook? Is that a line or a representation of a line?

In this lesson, we will come across some other geometrical elements formed by the play of lines on a plane. Let us dive right in!

12.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- To recognize and identify different geometries arising from the play of lines
- To identify pairs of angles and the relationships between them
- To derive rigorous proofs for theorems.
- To recognize the geometries arising from parallel lines
- To analyze and verify different angle pairs and their relationships
- To reproduce the results for more than two parallel lines
- To apply and manipulate the results for different cases of higher level

12.2 THE BASICS

12.2.1 Lines, Rays, Segments

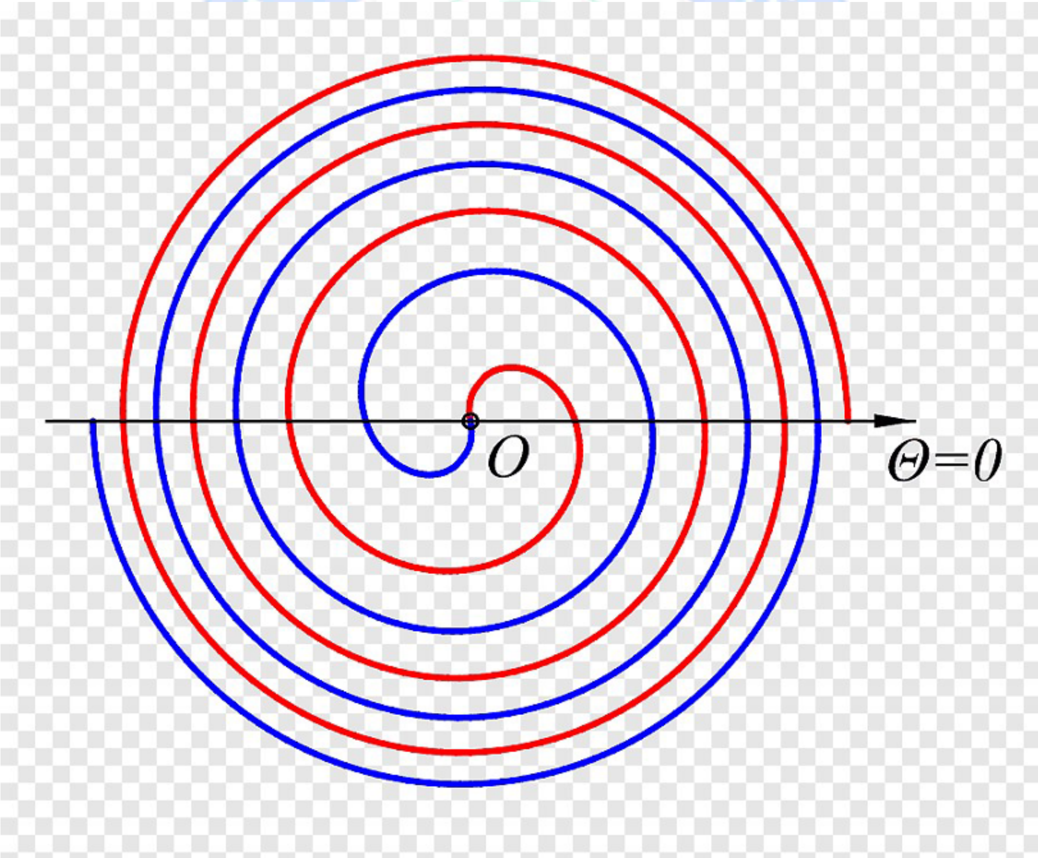
Like earlier, imagine a wire that goes on forever. It does not have end points in either direction. Consider two points on this wire, A and B. This is line AB.

Now imagine that you make a single cut at point A. Now, you have a wire that starts from point A and does not have an endpoint at the other end. This is ray AB.

Next imagine that you make another cut at point B. Now, you have a piece of wire that starts at point A and ends at point B. This is line segment AB.

Just for Fun!

Did you notice how we manipulated the concept of a line to form a ray and a line segment. Did we go from known to unknown? Now, consider the geometrical object given below. Is it similar to a line, a ray or a line segment? In what way?



The important difference between lines, rays and segments is the number of end points (if any). Can you write a sentence that resembles a ray? Does that mean that you would have to write a sentence that never ends? Is it possible?

12.2.2 Points

Take a piece of paper. Draw any three points at random on the paper, as shown below. Can you draw a single line through all three points?

Why or why not?

Can you draw three points such that a single line can pass through all three points? Such points are called collinear points.

Points through which a single line cannot be drawn are called non-collinear points.

Just for Fun!

Why did we take three points when we were trying to define collinear points? Can we take two points that are non-collinear? What about four points? Can four points be non-collinear?

12.2.3 Angles

You have learnt about angles earlier. Imagine two rays superimposed on each other. If one of the rays is rotated anti-clockwise, the resulting figure is called an angle. Thus, an angle is the amount of separation between two rays.

Make an angle with your arms. Did you know that the two rays that make up an angle are also called its arms? And the point at which both rays end is called the vertex of the angle.

Now, let us look at different types of angles.

Close the door of your room. Now, open it slowly.

Type of Angle	Amount of Separation	Interpretation
Acute angle	Greater than 0° but less than 90°	Partial door opening
Right angle	90°	Complete door opening

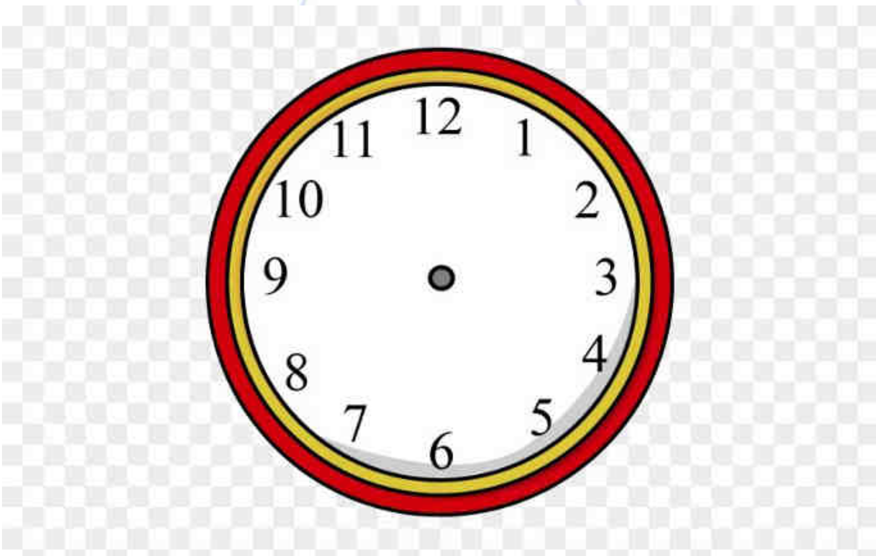
An acute angle is formed when the amount of separation between the arms is less than 90°. A right angle is L-shaped. The amount of separation is exactly 90°.

Draw an acute angle and a right angle.

Now, what would happen if the angle of separation between the arms is greater than 90°? Look at the different types of angles and try drawing them.

Type of Angle	Amount of Separation
Obtuse angle	Greater than 90° but less than 180°
Straight angle	180°
Reflex angle	Greater than 180° but less than 360°

Just for Fun!



Fill up the table below:

Angle	Time
Acute angle	
	3:00pm
	9:15pm
Obtuse angle	
Reflex angle	

Can we have an angle greater than 360°? What is the relationship between the angles 30° and 390°? Can you draw both of them? What do you notice?

12.2.4 Pairs of Angles

Look at the table below:

Sr No.	Pairs of Angles	Characteristics
1	Complementary Angles	Sum is equal to 90°.
2	Supplementary Angles	Sum is equal to 180°.
3	Adjacent Angles	Share a common vertex, a common arm, and their non-common arm lies on opposite sides of the common arm.
4	Linear Pair of Angles	Adjacent angles, with the non-common arms forming a straight line.
5	Vertically Opposite Angles	Formed when two lines intersect, on the opposite sides of the point of intersection.

Can you draw examples of all these angle pairs?

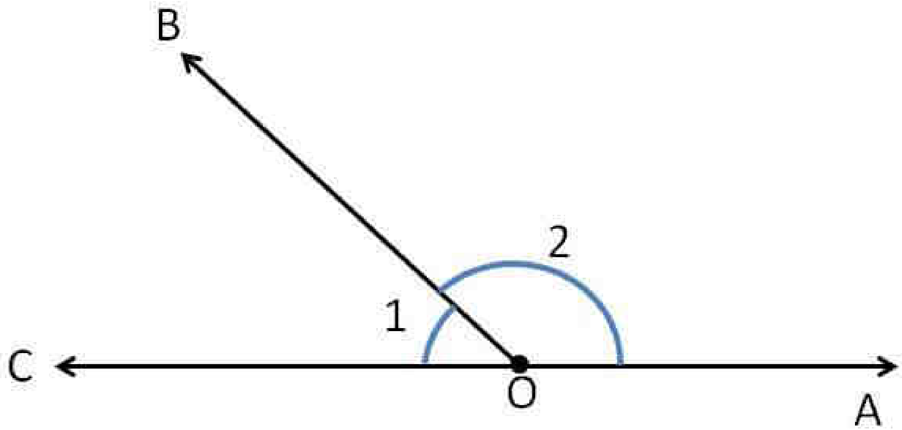
After that, think about the questions given below:

i) Can one (or both) angles in a pair of complementary angles be obtuse? Why or why not?

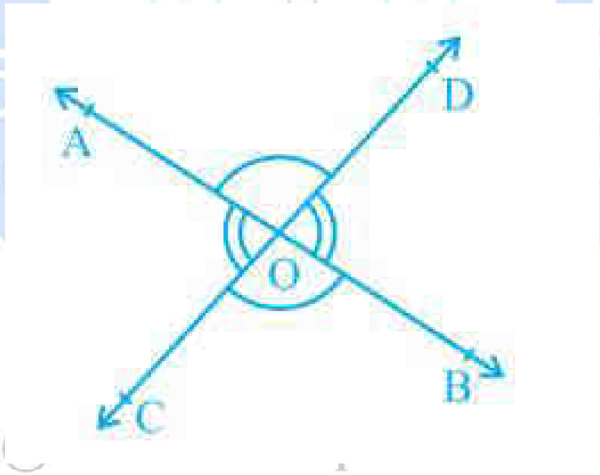
ii) Can both angles in a pair of supplementary angles be right angles? Why or why not?

iii) Both "complement" and "supplement" mean adding extra features to something to enhance its quality. Do these definitions make sense when seen in their mathematical context?

iv) Look at the figure below. What type of angle pair is $\angle AOB$ and $\angle COB$?



v) Look at the figure below. What kinds of angle pairs do you see? Mention the angle pairs and their kinds as a table.

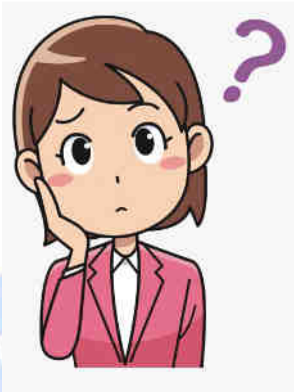


From your answers above, would you agree if we said that a linear pair of angles is a combination of two adjacent angles, which are supplementary? Do you see the relation between the different pairs of angles?

Just for Fun!

India and Pakistan are adjacent countries, since they share a common border. Do you see the relation between adjacent entities, be it angles or countries? Adjacent entities share a functional component. Adjacent angles share an arm. Adjacent countries share a border.

What functional component do you and your neighbours share? Are your flat and your neighbours' flat adjacent? Think about examples of other adjacent entities and the functional components they share.



12.2.5 Intersecting and Non-intersecting Lines

Imagine that you are going on a trip. When you reach the highway, do you see roads having multiple lanes? Do these lanes ever meet? Such lines that do not meet are called parallel (non-intersecting) lines. Non-intersecting lines share a common direction.

Before reaching the highway, did you come across cross-roads? Such lines that intersect at a single point are called intersecting lines.

Draw pairs of lines on a sheet of paper. What do you notice? Is there a pair of lines that is neither parallel nor intersecting?

12.2.6 Relations between Pairs of Angles

Let us take a look at some important axioms and a theorem.

Axiom 1: If a ray stands on a line, then the sum of the two adjacent angles so formed is 180°.

Is this in line with what we have seen before? Does a ray standing on a line form a linear pair of angles? Draw and check! If yes, are the two angles formed supplementary?

Remember your answers to the questions in section 4.2.4!

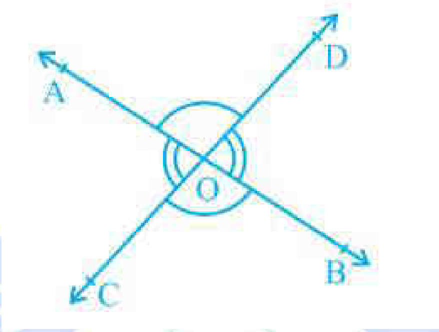
Axiom 2: If the sum of two adjacent angles is 180°, the non-common arms form a line.

Is this also in line with what we have seen before? Try drawing different pairs of adjacent angles and check!

Theorem 1: If two lines intersect each other, then the vertically opposite angles are equal.

Let us prove this theorem.

Proof: Consider two lines AB and CD intersecting at a point O. They form one pair of vertically opposite angles □AOD and □COB, and another pair □AOC and □DOB. We shall now prove that □AOD = □COB. Fill in the blanks below while working through the proof.



□AOD and □DOB are adjacent angles, where the common stands on a line (they are a linear pair).

Thus, □AOB + □DOB = ________ (from Axiom 1)

Now, □DOB and □COB are also adjacent angles, where the common stands on a line (they are a linear pair).

Thus, □DOB + □COB = ________ (from Axiom 1)

Hence, we see that □AOB + □DOB = □COB + □DOB

This is equivalent to saying that a + b = c + b. In this equation, if we remove the quantity "b" from both sides, we get a = c. (Imagine a weighing balance where both pans are level. If we remove the same amount from both pans, won't they still be level?)

Similarly, we can see that □AOB = □COB

Hence, we have proved the theorem. Two vertically opposite angles formed by two intersecting lines are always equal.

Just for Fun!

Do you know the difference between an axiom and a theorem?

Axiom – A statement that is taken to be true, to serve as a starting point for further arguments. An axiom is supported by examples and is yet to be disproved by any counter-example.

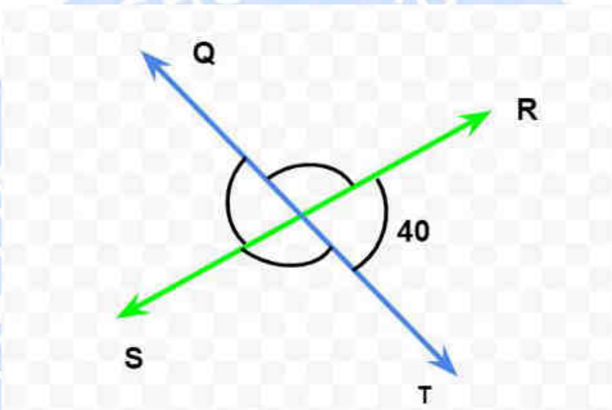
Do you see how Axiom 1 helped us prove Theorem 1? Has Axiom 1 itself been proved rigorously?

Theorem – A statement that is proved, or can be proved to be true.

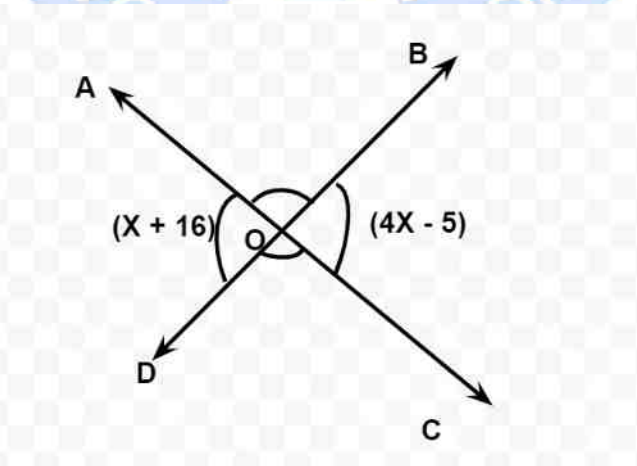
By proving Theorem 1, do you see how we extended the result to every other pair of vertically opposite angles in the universe? Is a theorem a generalized result?

CHECK YOUR PROGRESS 12.1

(i) What is the measure of the other three angles in the figure below?

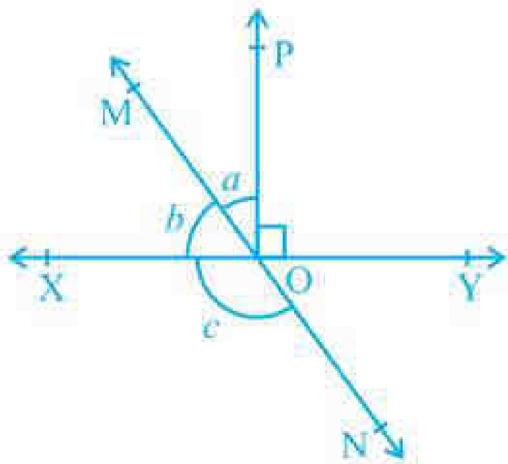


(ii) Find the measures of all four angles in the figure given below.



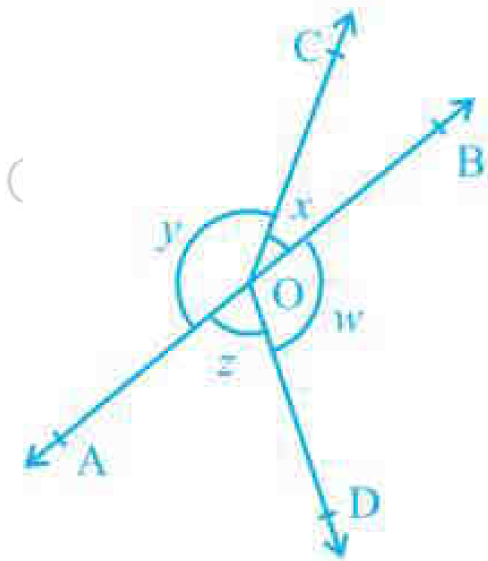
(iii) We have seen that lines can be either parallel or intersecting. But, did you know that this is true only in two dimensions? In three dimensions, you can have lines that are non-parallel and non-intersecting. Such lines are called skew lines, and you can see them in the world around you. Can you try a brief sketch of skew lines?

(iv)In the figure given below, a:b = 1:2. Find the measures of all the five angles.



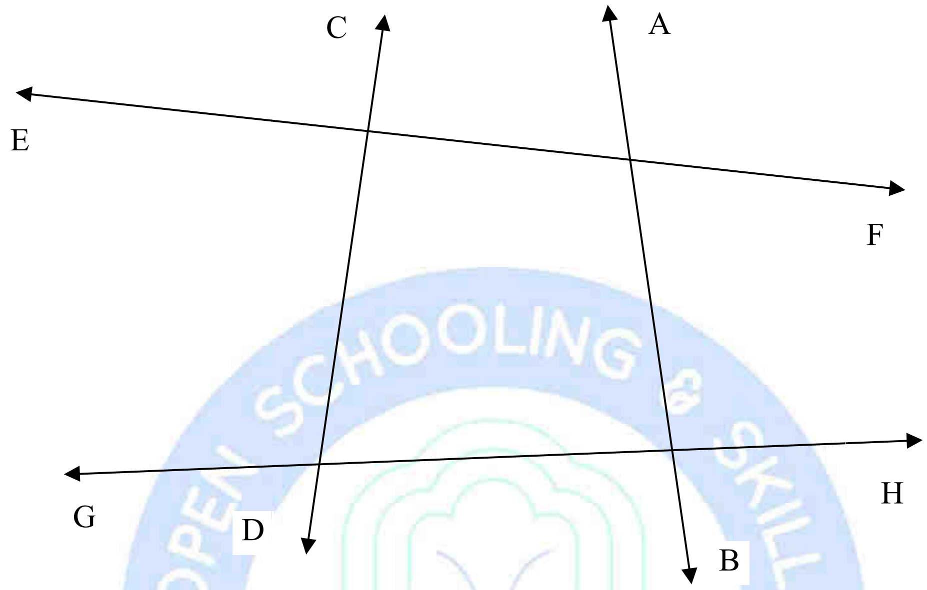
(v) Fill in the table based on the information given.

□AOC (y)	□AOD (z)	□BOD (w)	□BOC (x)
		Insufficient information	20°
		110°	30°
	80°		
140°			

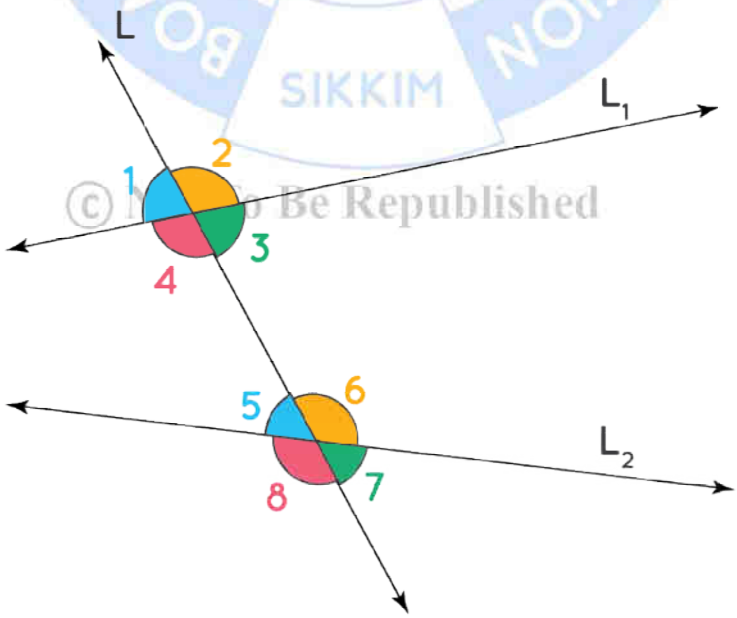


12.3 PARALLEL LINES AND A TRANSVERSAL

A transversal is a line that extends or cuts across two lines. Which is/are the transversal(s) in the following figure?



Consider the figure given below:



Here, we look at the different types of angles formed.

i) Corresponding angles:

□1 and □5

□2 and □6

□3 and □7

□4 and □8

Do you see why they are called corresponding angles?

ii) Alternate interior angles

□3 and □5

□4 and □6

Notice that these angles for a “z” shape (or its mirror image)!

iii) Alternate exterior angles

□1 and □7

□2 and □8

iv) Co-interior angles

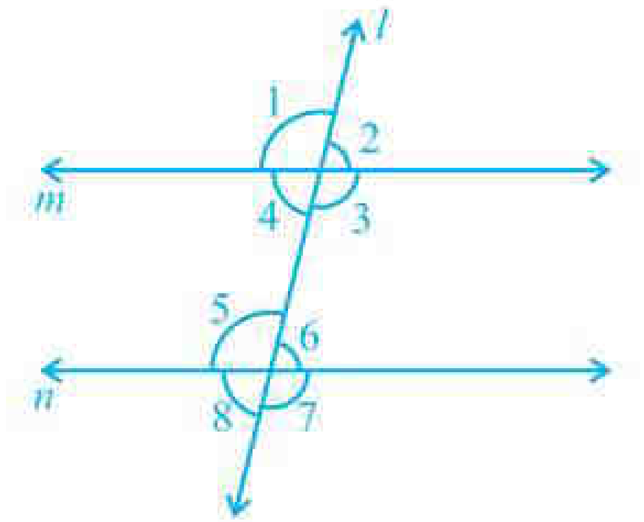
□3 and □6

□4 and □5

Let us now find out the relationships between these different angle pair types.

Just for Fun!

Do you see how the geometry of lines gets complicated with the addition of a transversal?



The lines m and n are parallel. For example, consider that line m represents Jacinda Ardern and line n represents Narendra Modi. Then, the transversal l represents the commonality between the two (being Prime Ministers). The points of intersection then represent New Zealand and India respectively.

Using the analogy given above, fill in the table given below:

Line m	Line n	Transversal l	Points of intersection
Dog	Lion		House and Jungle respectively
Ratan Tata	Rahul Bajaj		Tata Auto and Bajaj Auto respectively
Narendra Modi		Political leaders	BJP and Congress respectively

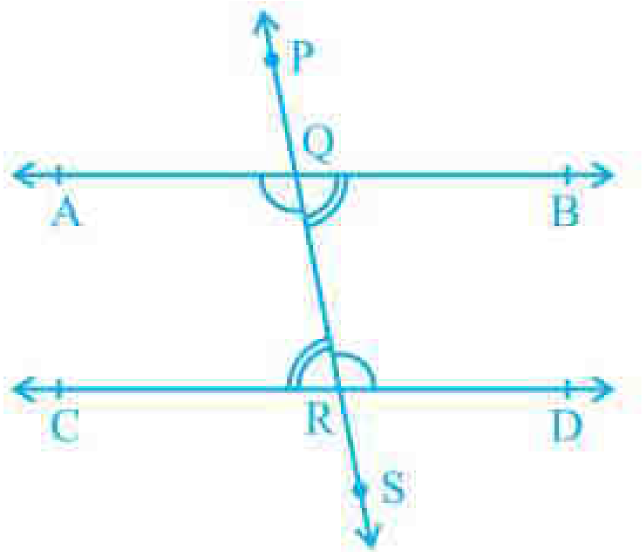
Axiom 3: If a transversal intersects two parallel lines, then each pair of corresponding angles is equal.

This is also called "Corresponding Angles Axiom." Consider the converse of this axiom.

Axiom 4: If a pair of corresponding angles formed when a transversal intersects two lines is equal, then the lines are parallel.

Try verifying these axioms by drawing cases in your notebook.

Theorem 2: If a transversal intersects two parallel lines, then each pair of alternate interior angles is equal.



Now, we know that ∠AQR = ∠CRS (Corresponding Angles Axiom)

But, ∠QRD = ∠CRS (Vertically opposite angles)

Hence, ∠AQR = ∠QRD

Similarly, prove that ∠BQR = ∠QRC.

Theorem 3: If a pair of alternate interior angles formed by a transversal is equal, then the lines are parallel.

Can you prove the above theorem? (Hint: Use Theorem 2 and Axiom 4!)

Theorem 4: If a transversal intersects two parallel lines, then each pair of co-interior angles is supplementary.

Theorem 5: If a pair of co-interior angles formed by a transversal is supplementary, then the lines are parallel.

Can you prove the above theorems using the earlier results?

Can you also state the relation between alternate exterior angles when the lines are parallel? State its converse also.

Just for Fun!

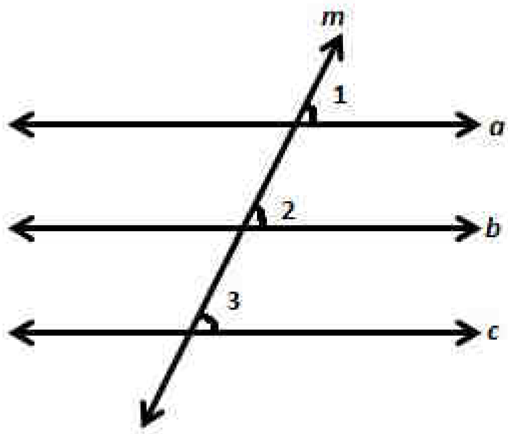
Note that we have only considered parallel lines while finding the relationships between the angle pairs. What happens when the lines are not parallel? Which of the above results would be true and which would be false in such a case?

What happens if a transversal intersects more than two lines? We have one last theorem to look at in this section.

Theorem 6: If two lines are parallel to the same line, they are parallel to each other.

Let us prove this theorem.

Proof:



Consider that line a || line b and line a || line c. We have to prove that line b || line c

Now, □1 = □2 (Corresponding Angles Axiom)

□1 = □3 (Corresponding Angles Axiom)

Thus, □2 = □3.

By the converse of the Corresponding Angles Axiom, we can see that

line b || line c

Notice that the above theorem can be extended to any number of lines. If one line is parallel to x number of lines, all the x lines must be parallel to each other. Thus, lines which are parallel to the same line are parallel to each other.

Just for Fun!

Can you use the analogy we used earlier (the Prime Minister one) for the case of three lines? Try it out!

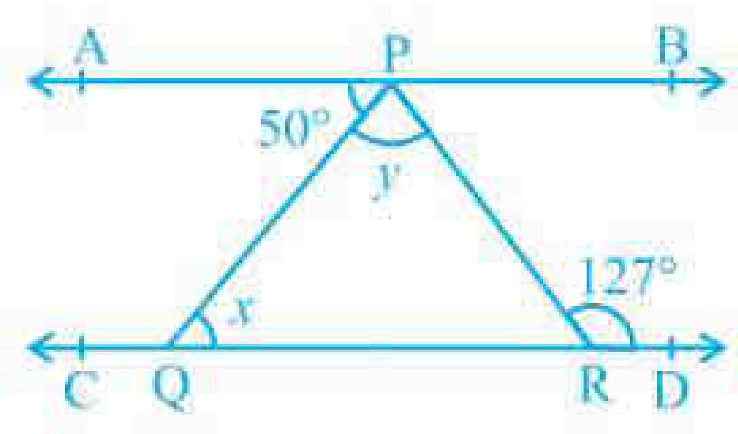
CHECK YOUR PROGRESS 12.2

(i) Imagine that you want to fit a picture onto a page.

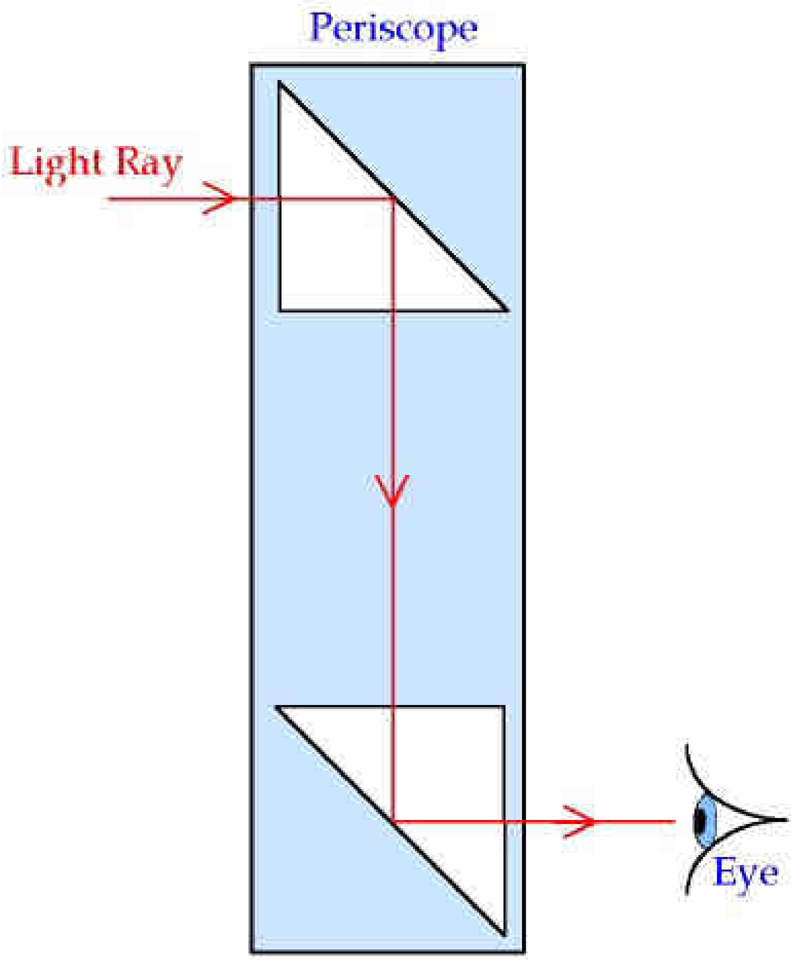
Which lines are parallel in the above pictures? Which are the corresponding angles? Do you see how corresponding angles formed when the lines are parallel can be superimposed upon one another?

(ii) Indian abstract artist Ravindra Pawar paints with a unique sense of geometry. Look at one of his paintings given below. Can you spot parallel lines and any angle pairs? You can discuss with others.

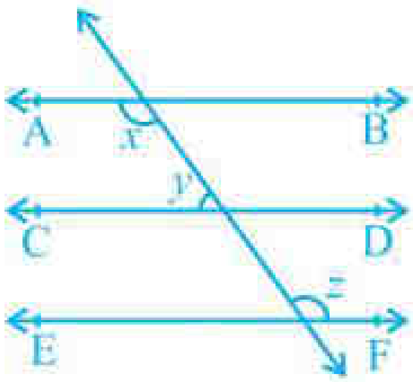
(iii)Look at the figure given below. Find the measures of all the angles that you see.



(iv)Do you know what a periscope is? Try to find out! Look at the diagram given below. Now, if the light ray is horizontal to the ground on entry and is vertical on being reflected from the mirror on top, what is the angle it forms with the first mirror? What is the angle it forms with the second mirror? Do you see the uses of parallel lines/surfaces?



(v) If y:z = 3:7, find x:z and x:y without computing the values of x, y and z. After that, find the values of x, y and z, and all the other angles that you see in the image.



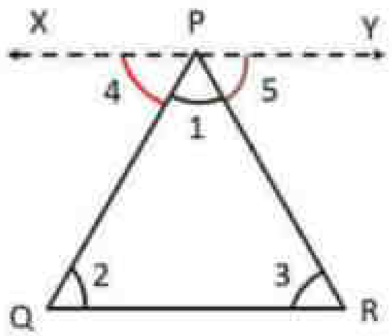
12.4 ANGLE SUM PROPERTY OF A TRIANGLE

You have seen the angle sum property of a triangle before. Let us now prove this as a theorem. Fill in the blanks to derive the proof.

Theorem 7: The sum of the angles of a triangle is 180°

Proof:

Look at ΔPQR below. We have to prove that ∠1 + ∠2 + ∠3 = 180°. Let us draw a line XY parallel to the line segment QR as shown in the figure below.



Now, we know that ∠XPR + ∠PRQ = _____ (Co-interior angles)

∠1 + _____ + ∠3 = _____

Now, ∠4 = ∠2 (Alternate interior angles)

Thus, substituting,

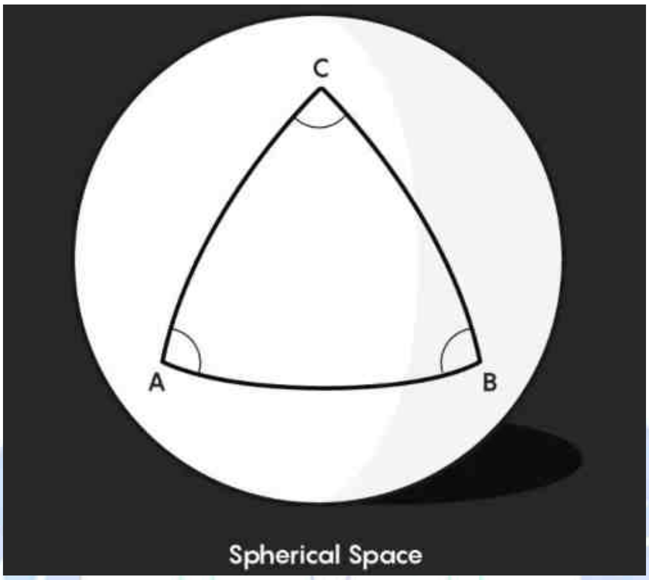
∠1 + ∠2 + ∠3 = 180°.

Hence, proved.

Just for Fun!

The angle sum property is true only for plane surfaces. For example, consider a spherical surface like a globe.

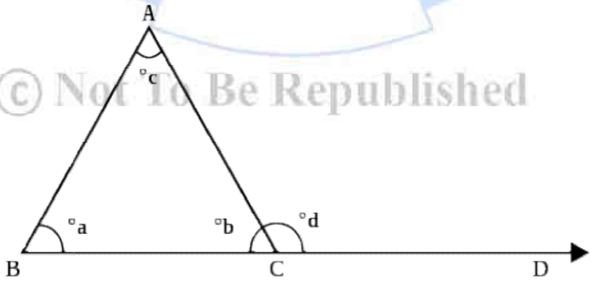
If we wrap a straight line around a globe, it forms a circle. Thus, in a spherical space, a triangle would look as follows:



What would be the sum of the angles A, B and C? Would it still be 180°?

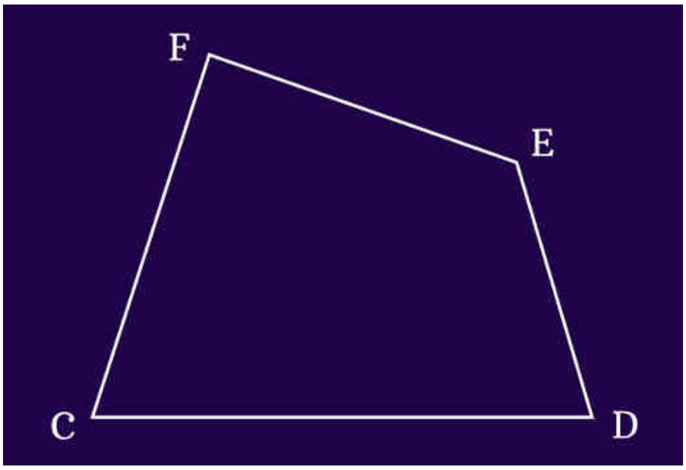
CHECK YOUR PROGRESS 12.3

(i) Look at the image given below. Find the relation between a, c and d.



(ii) In the above figure, if d = 115° and a = 30°, what are the values of b and c?

(iii) We know that the sum of angles of a triangle is 180°. Can you find the sum of the angles of a quadrilateral (a closed shape composed of 4 sides)?

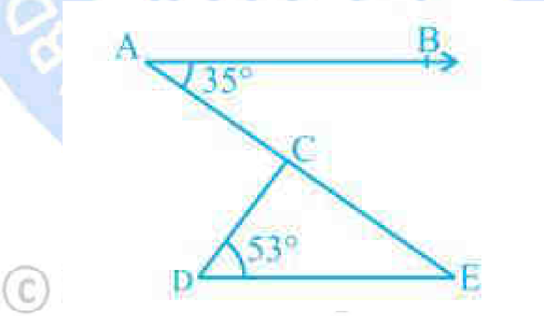


Hint: You might have to manipulate the shape a little bit!

(iv) The result that you found in the above question can be extended to a polygon (a closed shape) with n sides. Sum of angles of a polygon having n sides = (n –2) x 180°.

Construct a convex octagon (polygon with 8 sides) with 4 right angles. Note that in a convex polygon, all the interior angles are less than or equal to 180°.

(v) Look at the figure given below. Can you find the measures of all the angles you can see? Also, extend the line segment DC such that it intersects ray AB. What are the new angles formed by this manipulation? Can you find their measures?



RECAPITULATION POINT

- We learnt about lines, rays and line segments.
- We learnt about different pairs of angles and their relationships.
- If a ray stands on a line, then the sum of the two adjacent angles so formed is 180° and vice versa. This property is called as the Linear pair axiom.
- If two lines intersect each other, then the vertically opposite angles are equal.

- If a transversal intersects two parallel lines, then
  - each pair of corresponding angles is equal
  - each pair of alternate interior angles is equal,
  - each pair of interior angles on the same side of the transversal is supplementary.

  The converse of all the three statements above is true.

- Lines which are parallel to the same line are parallel to each other.

- The angle sum property of a triangle is proved, that is, the sum of the angles of a triangle is 180°.

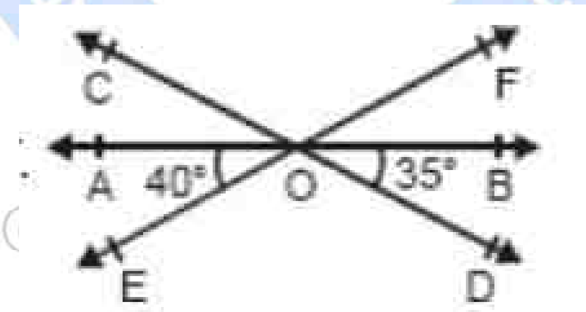
- We also learnt an analogy to interpret the concept of parallel lines and a transversal.

- We also learnt the difference between an axiom and a theorem.

TERMINAL EXERCISE

(i) The exterior angles formed by producing the base of a triangle both ways are 140° and 120°. Find the measures of all the angles of the triangle.

(ii) Find the measures of all the unknown angles.



(iii)In ΔABC,□A – □B = 20°, and □C – □B = 20°. Find the values of all three angles of the triangle.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 12.1 (Hints)

(i) Identify the vertically opposite angles and the linear pairs in the given figure.

(ii) Like the first question, this question also requires you to find the vertically opposite angles and the linear pairs and solve for x.

(iii) Think of a cube. Are there any skew lines in this 3-D object?

(iv) The key is to identify the vertically opposite angles and the linear pairs in the given figure. Remember that the angle pairs are formed by the intersection of *lines*.

(v) The key is to identify the vertically opposite angles and the linear pairs in the given figure. Remember that the angle pairs are formed by the intersection of *lines*.

CHECK YOUR PROGRESS 12.2(Hints)

(i) Label the vertices in the given figures to find the parallel sides and the corresponding angles.

(ii) For one example, you can look at the bottom left corner. Remember to find others!

(iii)Use all the results to find the required solution.

(iv)The light ray is reflected perpendicular to its initial path due to the first mirror. Using this information, find the required solution.

(v) Use the given information to find partial solutions first. After that, solve for the complete solution. Sometimes, you do not need to calculate the final answer to arrive at a conclusion!

CHECK YOUR PROGRESS 12.3(Hints)

(i) Use the angle sum property and the concept of linear pair to find the relationship. This is also an important theorem.

(ii) You can simply apply the formula you have derived earlier.

(iii) Try drawing one diagonal of the quadrilateral.

(iv) With four angles being right angles, the other four angles must together measure [(8 – 2) x 180] –4 x 90 = 1080 –360 = 720°. On average, each angle must then measure 180°. Is this possible for a convex polygon?

(v) Use the angle sum property and the exterior angle property that you derived in Q11 to find the measures of the other angles. Draw the figure clearly in your notebook while manipulating it to form other angles.

SUPPLEMENTARY STUDY MATERIAL

Thales visualized the Pythagorean theorem before Pythagoras. Read about his discoveries and thought process here: **https://bigthink.com/thinking/pythagorean-theorem-renamed-thalean-theorem/**

Read more about the geometry of the universe: **https://www.quantamagazine.org/what-is-the-geometry-of-the-universe-20200316/**

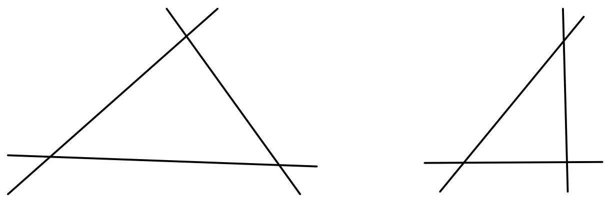
Photography is converting 3-D reality into 2-D photos. What happens to parallel lines in 3-D reality during this conversion. Read more about perspectives here: **https://fstoppers.com/landscapes/how-apply-different-perspective-techniques-compelling-photography-595363**

Watch videos and do the exercises: **https://www.khanacademy.org/math/in-in-grade-9-ncert/xfd53e0255cd302f8:lines-and-angles**

13 TRIANGLES

INTRODUCTION

In class, teacher asked the students to draw the closed shape with the three intersecting lines. Student A and Student B draw the following shapes:



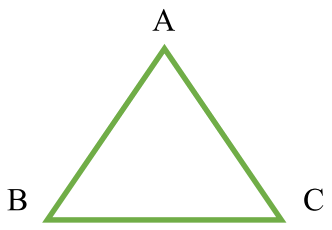
These closed shapes with the three intersecting lines are called triangles.

Triangle (Tri means three)is closed figure having three sides, three angles and three vertices.

Three sides – AB, BC and CA

Three angles - $\angle A$, $\angle B$ and $\angle C$

Three vertices – A, B and C



13.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Define the types and properties of triangles.
- Define the relation between the angles of triangles
- Define the congruence of triangles.
- Define the similarity of triangles.
- Differentiate the congruence of triangles and the similarity of triangles.

- Define the relationship between the sides of right triangle.

Expected background knowledge:

- Lines and Angles
- Vertices.

Some used Symbols:

(i) ≅ : ‘Congruence’

(ii) ∼ : ‘similar’

(iii) ⊥ : 'perpendicular'

(iv) Δ : 'triangle'

(v) ∠ : 'angle'

(vi) i.e.: 'that is'

13.2 ITS TYPES AND PROPERTIES

Classification of triangles based on sides and angles

Classification of triangles based on sides	Classification of triangles based on angles
1. **Scalene triangle**:- length of all three sides of triangle are different	**Acute-angled triangle**:- measure of any one of the angle of triangle is less than 90^o
2. **Isosceles triangle**:- length of any two sides are equal	**Obtuse-angled triangle**:- measure of any one of the angle of triangle is more than 90^o
3. **Equilateral triangle**:- length of all three sides are equal	**Right-angled triangle**:- measure of any one angle of triangle is exactly equal to 90^o

13.2.1 Some properties of triangle

1. Angles opposite to equal sides of an isosceles triangle are equal.
2. The sides opposite to equal angles of a triangle are equal.
3. If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).

4. In any triangle, the side opposite to the larger (greater) angle is longer.

5. The sum of any two sides of a triangle is greater than the third side.

13.2.2 Angle-sum-property of triangle

Theorem 13.1(Angle-sum-property of triangle): Sum of the interior angle of triangle is 180^o.

Proof: Take a ΔABC (fig. 13.2). Draw a line XY through A. now XAY is a straight line.

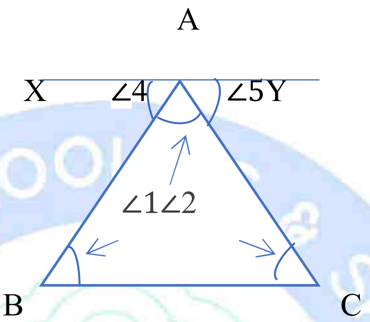


Fig. 13.2

$\angle 4 + \angle 1 + \angle 5 = \angle 180^o$ ……….(1)

(Because straight angle = 180^o)

XY//BC and AB, AC are transversal lines

Therefore, $\angle 4 = \angle 2$ and $\angle 5 = \angle 3$ (alternative interior angles)

Put the values in (1),

Therefore $\angle 2 + \angle 1 + \angle 3 = \angle 180^o$

or $\angle 1 + \angle 2 + \angle 3 = \angle 180^o$.

13.3 CONGRUENCE OF TRIANGLE

Observe the following figures as shown in fig. 13.3

Fig. 13.3

They are identical (both shapes and sizes are same). You may observe that on placing a figure on another figure they cover each other completely. They are called congruent figures.

Let us try in case of triangles.

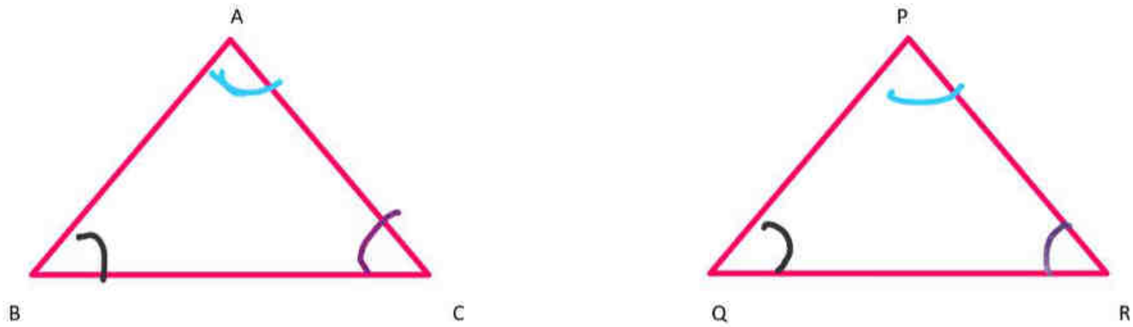


Fig. 11.4

Two triangles are congruence if

1. The sides of one triangle are equal to corresponding sides of other triangles.
2. The angles of one triangle are equal to corresponding angles of other triangles.

If ΔABC is congruent to ΔPQR (that is $\Delta ABC \cong \Delta PQR$)

That is, on placing ΔABC on ΔPQR ,

AB =PQ, BC=QR, CA=RP and $\angle A = \angle P,\ \angle B = \angle Q$ and $\angle C = \angle R$

That is, A corresponds to $P(A \leftrightarrow P)$, B corresponds to $Q(B \leftrightarrow Q)$, and C corresponds to $R(C \leftrightarrow R)$

So, $\Delta ABC \cong \Delta PQR$ is the correct symbolic form for writing the congruence of triangles (but $\Delta BCA \cong \Delta PQR$ is not correct).

13.3.1 Criteria for Congruence of Triangles

Observe that given two triangles are not congruent (fig. 13.5)

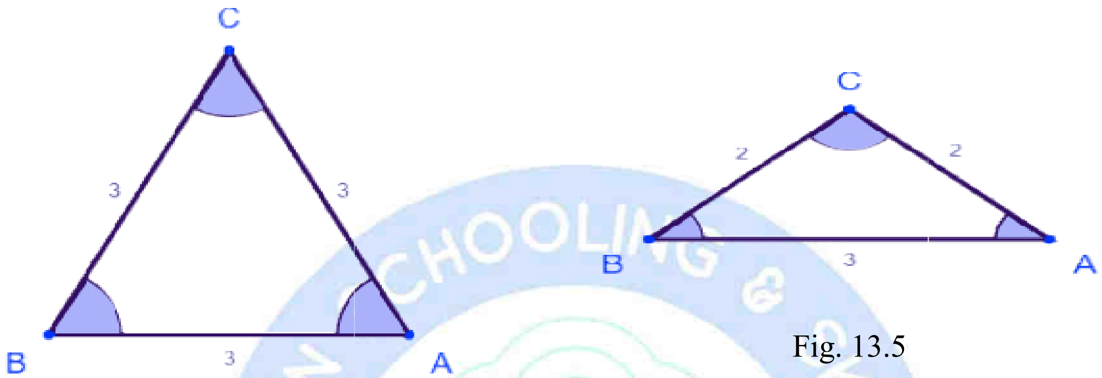


Fig. 13.5

Rules for congruence of triangles

1. SAS(side-angle-side) congruence rule	Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.
2. ASA(angle-side-angle) and AAS(angle-angle-side) congruence rule	Two triangles are congruent if two angles and one side of one triangle are equal to two angles and corresponding side of other triangle. (i.e., two triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal).
3. SSS(side-side-side) congruence rule	If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent
4. RHS(right angle-hypotenuse-side) congruence rule	If in two right angled triangles the hypotenuseand one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.

Example 1: In given figure, AO = OD and OC = OB. Show that $\Delta AOB \cong \Delta DOC$

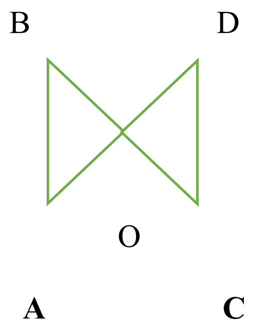


Fig. 13.6

Solution: In ΔAOB and ΔDOC

AO = OD and OC = OB (given)

□AOB = □DOC (Vertically opposite angles)

So, by SAS congruence rule

$\Delta AOB \cong \Delta DOC$.

Example 2:In given figure, AB//CD and AO = OD. Show that $\Delta AOB \cong \Delta DOC$.

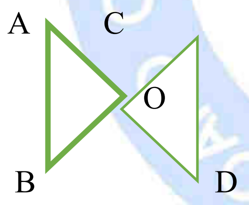


Fig. 13.7

Solution: In ΔAOB and ΔDOC

AO = OD (given)

□AOB = □DOC (Vertically opposite angles)

□ABO = □DCO (Alternate interior angles, because AB//CD and BC transversal line)

So, by AAS congruence rule

$\Delta AOB \cong \Delta DOC$.

Example 3: In given figure,PQRS is a parallelogram, then prove that $\Delta PQR \cong \Delta RSP$.

Solution: In ΔPQR and ΔRSP

PQ = SR and PS = QR (PQRS Parallelogram)

PR = PR (Common)

So, by SSS congruence rule,

$\Delta PQR \cong \Delta RSP$

Fig. 13.8

Example4:In given figure, QS ⊥ PR, RT ⊥ PQ and QS=RT. Prove that $\Delta QRT \cong \Delta RQS$.

Solution: In ΔQRT and ΔRQS

□QTR = □RSQ (Each 90^0)

QR = QR (Common hypotenuse)

QS = RT (Given)

So, by RHS congruence rule, $\Delta QRT \cong \Delta RQS$

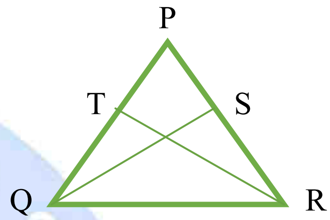


Fig. 13.9

CHECK YOUR PROGRESS 13.1

(i) Which of the following is not a criterion for congruency of triangles?

(a) SSS (b) RHS (c) SAS (d) SSA

(ii) If $\Delta ABC \cong \Delta PQR$, than

(a) BC = PR (b) AC = QR (c) BA = QP (d) PQ = CB

(iii)Sum of the three angles of triangle

(iv)isequal to___________.

(v) If AB = EF, BC = FD and AC = ED, then

(a) $\Delta ABC \cong \Delta DEF$ (b) $\Delta CBA \cong \Delta DFE$

(c) $\Delta BAC \cong \Delta FDE$ (d) $\Delta BCA \cong \Delta DEF$

(vi)In ΔPQR, PQ = PR and □R = 50^o. Then □Q is equal to

(a) 50^o (b) 40^o (c) 80^o (d) 100^o

(vii) Two figures are congruent if they have the ___________shape and same _________.

(viii) Two circles are congruent if they have same radii.(true/false)

(ix) Which is the largest angle in ΔABC ?

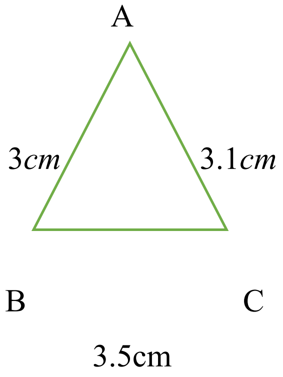
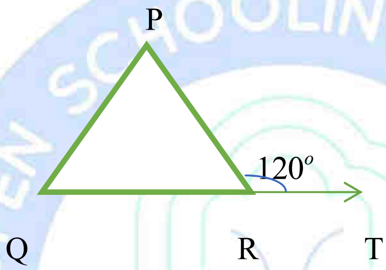


Fig. 13.10

(x) In given figure PQ = PR and ∠PRT. Find ∠P

Fig. 13.11

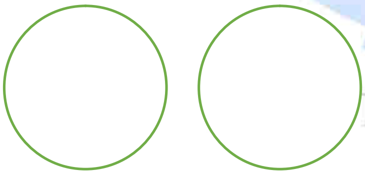


13.4 SIMILARITY OF TRIANGLES

Consider two circles. Are they congruent?

Yes, because they have same shape and same size (radius).

Now consider another two circles.



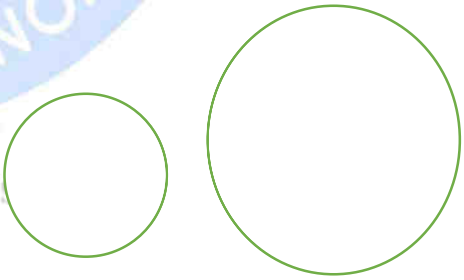


Fig. 13.12 Fig. 13.13

Are they congruent? No, why? Check condition for congruent (both shapes and sizes are same),

(i) Same shapes – yes

(ii) Same sizes (radius) - No.

So, two figures having the same shape but not necessarily the same size are called ***similar figures.***

Think about some other figures like two or more squares? You find that they are also similar.

So from above discussion we conclude that all congruent figures are similar but the similar figures may or may not be congruent.

Activity 1 takes a picture in mobile. Now zoom (enlarge) it. You observe that when you zoom it all the corresponding sides are enlarged in the same ratio (or proportion).so, the both the pictures (original and zoom picture) are similar (same shape but of different sizes).

You observe that any two shapes (polygons of the same number of sides) are similar, if (i) all the corresponding angles are equal and (ii) all the corresponding sides are in the same ratio.

Now consider any two triangles. Are they similar?

From above discussion we observe that two triangles are similar

If (i) their corresponding angles are equal and

(iii) Their corresponding sides are in the same ratio.

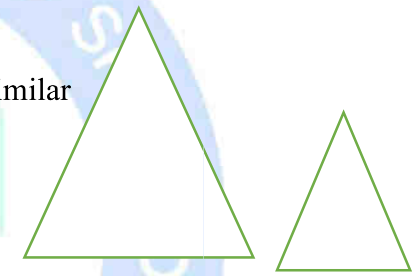


Fig. 13.14

That is, $\Delta ABC \sim \Delta PQR$, if

(i) □A=□P, □B=□Q and □C=□R and

(ii) $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CA}{RP}$

(**Note:** symbol□ stands for 'is similar to')

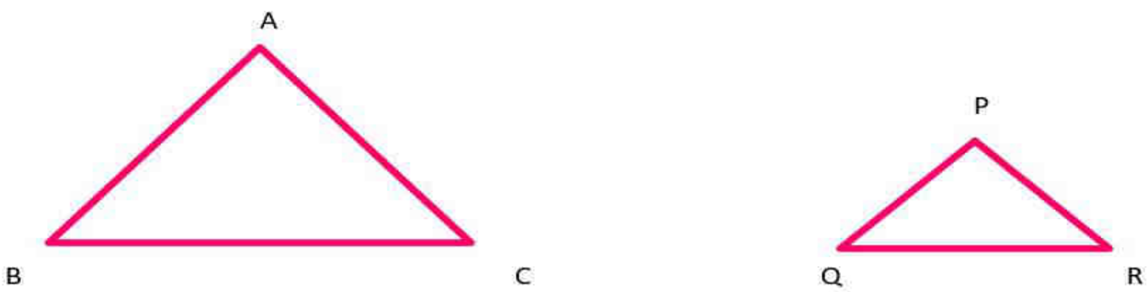


Fig. 13.15

Theorem 13.2:Basic Proportionality Theorem (Thales Theorem)

If a line is drawn parallel to one side of a triangle and intersect the other two sides in distinct points, the other two sides are divided in the same ratio.

Proof: Consider a triangle PQR in which a line XY parallel to QR intersect other two sides PQ and PR at X and Y respectively.

To Prove $-\dfrac{PX}{XQ}=\dfrac{PY}{YR}$

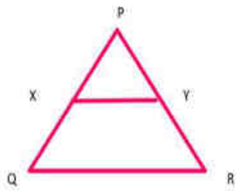


Fig. 13.16

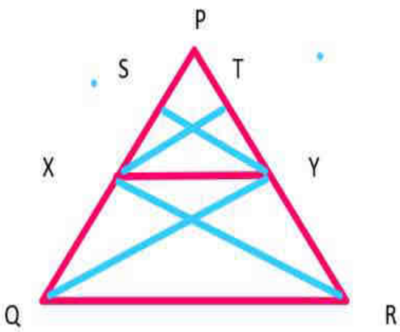


Fig. 13.17

Join XR and QY and draw XT perpendicular PR and YS perpendicular PQ.

Area of $\Delta PXY=\frac{1}{2}\times PX\times SY$ (because area of triangle $=\frac{1}{2}\times$ base $\times$ height)

Similarly Area of $\Delta QXY=\frac{1}{2}\times XQ\times SY$

Area of $\Delta PXY=\frac{1}{2}\times PY\times XT$ and Area of $\Delta XYR=\frac{1}{2}\times YR\times XT$.

So, $\dfrac{\text{area of }\Delta\text{PXY}}{\text{area of }\Delta\text{QXY}}=\dfrac{\frac{1}{2}\times PX\times SY}{\frac{1}{2}\times XQ\times SY}=\dfrac{PX}{XQ}=\dfrac{PX}{XQ}$ …..(1) and

$\dfrac{\text{area of }\Delta\text{PXY}}{\text{area of }\Delta\text{XYR}}=\dfrac{\frac{1}{2}\times PY\times XT}{\frac{1}{2}\times YR\times XT}=\dfrac{PY}{YR}$ …….(2)

But Area of $\Delta QXY=$ Area of ΔXYR …….(3)

(Because ΔQXY and ΔXYR are on the same base XY and between the same parallels QRand XY).

Therefore from (1),(2) and (3) we get $\frac{PX}{XQ} = \frac{PY}{YR}$

Converse of the theorem is also true that is if a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side. (Why?)

Rules for similarity of triangles	
1. SAS(side-angle-side) similarity rule	Two triangles are similar if one angle of a triangle is equal to the one angle of other triangle and the sides includes these angles are in the same ratio.
2. AAA(angle-angle-angle) or AA (angle-angle)similarity rule	Ifthree (or two) angles of one triangle is equal to the three (or two) angles of the other triangle, then their corresponding sides are in the same ratio and hence the two triangles are similar. (Remark: if two angles of a triangle are equal to two angles of another triangle, then their third angle will also be equal.) (Why?).
3. SSS(side-side-side) congruence rule	If three sides of one triangle are in the same ratio (proportional) tothe three sides of another triangle, then their corresponding angles are equal and hence the two triangles are similar.

Example 5: Observe that twotriangles ΔABC and ΔPQR are similar, AB=5 cm, BC=7 cm, CA=5.2 cm, PQ= 2.5 cm, QR= 3.5 and RP= 2.6 cm and ∠B=50^{o}, and then find □Q.

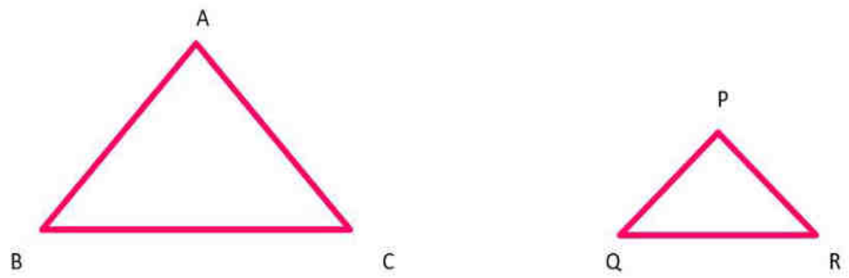


Fig. 13.18

Solution: In ΔABC and ΔPQR

$$\frac{AB}{PQ}=\frac{5}{2.5}=\frac{2}{1},\frac{BC}{QR}=\frac{7}{3.5}=\frac{2}{1}=\frac{CA}{RP}=\frac{5.2}{2.6}=\frac{2}{1}$$

i.e., $\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CA}{RP}$

Therefore, $\Delta ABC \sim \Delta PQR$ (by SSS similarity rule)

$\Rightarrow \angle B = \angle Q$ (Corresponding angles of similar triangles)

So, $\angle Q = 50^o$

Example 6: A street light bulb is fixed on a pole 8 m above the level of the street. If a woman of height 2 m casts a shadow of 4 m, find how far she is away from the base of the pole.

Solution: Let AB=8m denote the height of pole and DE=2 mthe height of woman and EC=4 is the shadow of the woman. Let BE = x metre (far from the base of the pole).

Note that in ΔABC and ΔDEC,

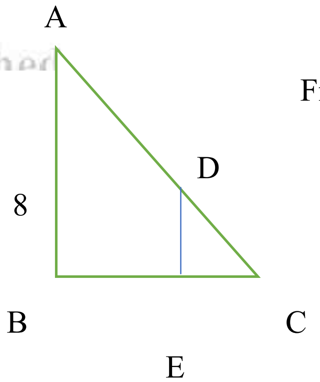


Fig. 13.19

□B = □E (Each 90°)

□C = □C(Same angle)

So, $\Delta ABC \sim \Delta DEC$ (by AA similarity rule)

Therefore, $\frac{AB}{DE}=\frac{BC}{EC}$

$\Rightarrow \frac{8}{2}=\frac{BE+4}{4}4$

$\Rightarrow 4\times 4 = x+4$

$\Rightarrow 16 - 4 = x$

$\Rightarrow x = 12$

So, she is 12 m far away from the base of the pole.

Example 7: In given figure, $\frac{AO}{OD} = \frac{BO}{OC}$, then prove that $\Delta AOB \sim \Delta DOC$

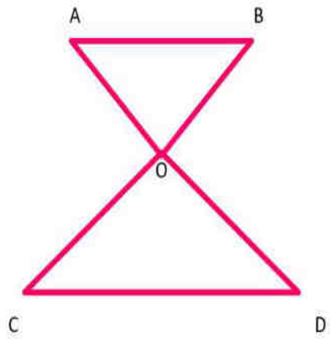


Fig. 13.20

Solution: In ΔAOB and ΔDOC

$\frac{AO}{OD} = \frac{BO}{OC}$ (Given)

$\angle AOB = \angle DOC$ (Vertically opposite angles)

Therefore, $\Delta AOB \sim \Delta DOC$ (by SAS similarity rule)

Example 8: In given figure, DE//BC and $\angle ABC = \angle AED$. Show that $\Delta ABC \sim \Delta AED$

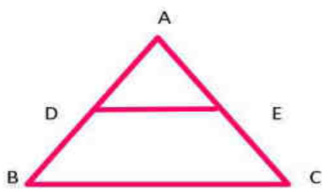


Fig. 13.21

Solution: In ΔABC & ΔAED

$\angle ABC = \angle AED$ (given)

$\angle A = \angle A$ (common)

Therefore, $\Delta ABC \sim \Delta AED$ (by AA similarity rule)

Example 9: If $\Delta ABC \sim \Delta PQR$, AX and PY are medians of ΔABC and ΔPQR, respectively, then prove that $\frac{AB}{PQ} = \frac{AX}{PY}$

As $\Delta ABC \sim \Delta PQR$, Hence $\angle B = \angle Q$ and

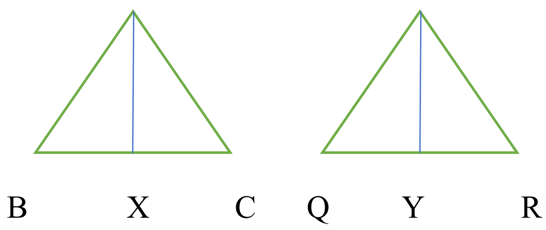


Fig. 13.22

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{\frac{1}{2}BC}{\frac{1}{2}QR} = \frac{BX}{QY}$$

In ΔABX and ΔPQY

$\frac{AB}{PQ} = \frac{BX}{QY}$ and $\angle B = \angle Q$

So, $\Delta ABX \sim \Delta PQY$ (by SAS Similarity rule)

Hence, $\frac{AB}{PQ} = \frac{AX}{PY}$

CHECK YOUR PROGRESS 13.2

(i) All equilateral triangles are similar. (true/false)

(ii) All circles are similar. (true/false)

(iii) If a line is drawn parallel to one side of a triangle to intersect the other two sidesin distinct points, the other two sides are divided in the __________ ratio.

(iv) If $\Delta ABC \sim \Delta PQR$, then $\frac{AB}{\_\_\_} = \frac{\_\_\_}{QR}$.

(v) All the congruent figures are always similar. (true/false) (vi)In the given figure $\Delta ABC \sim \Delta ADE$ and $\frac{AD}{AE} = \frac{AE}{AC} = \frac{1}{3}$, then

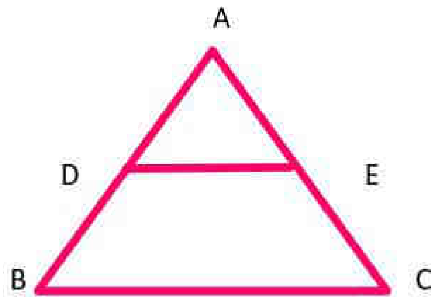


Fig. 13.23

(a) DE = BC (b) DE = $\frac{1}{2}$ BC (c) DE = $\frac{1}{3}$ BC (d) DE=$\frac{1}{4}$BC

(vii)In the given figure, if $\Delta ABC \sim \Delta PQR$, find the value of x?

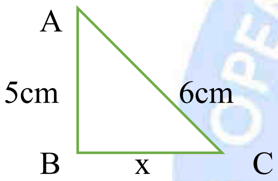
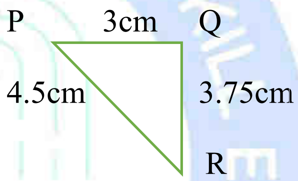


Fig. 13.24



13.5 AREAS OF SIMILAR TRIANGLES

Theorem 13.3: The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

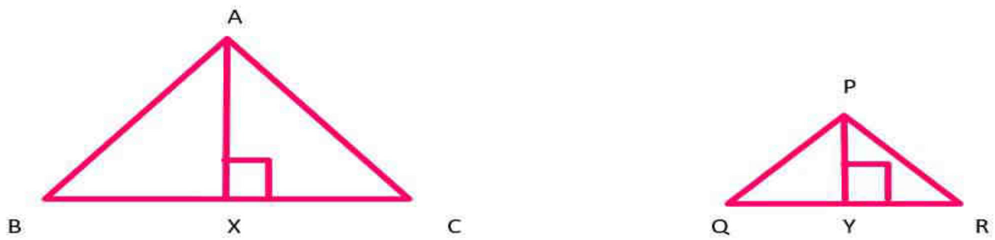


Fig. 13.25

Proof: Given $\Delta ABC \sim \Delta PQR$

To prove – $\frac{ar(ABC)}{ar(PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$

Note: Area of ΔABC is written as *ar*(ABC) and Area of ΔPQR is written as *ar*(PQR).

$$ar(ABC) = \frac{1}{2} \times BC \times AX$$

$$ar(PQR) = \frac{1}{2} \times QR \times PY$$

$$\frac{ar(ABC)}{ar(PQR)} = \frac{\frac{1}{2} \times BC \times AX}{\frac{1}{2} \times QR \times PY} = \frac{BC \times AX}{QR \times PY} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

In ΔABX and ΔPQY,

□B=□Q (because ΔABC~ΔPQR)

□X=□Y (each 90^o)

Therefore, ΔABX □ ΔPQY, (by AA similarity rule)

$$\Rightarrow \frac{AX}{PX} = \frac{AB}{PQ} \quad \ldots\ldots\ldots (2)$$

But $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ (As $\Delta ABC \sim \Delta PQR$) ………(3)

From (1) and (3) $- \frac{ar(ABC)}{ar(PQR)} = \frac{AB \times AX}{PQ \times PY}$

$$\frac{AB \times AB}{PQ \times PQ} = \left(\frac{AB}{PQ}\right)^2 \quad \text{[from (2)]}$$

So, by using (3), we get $\frac{ar(ABC)}{ar(PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$

Example 10: Sides of two similar triangles are in the ratio 3:5. Find the ratio of the areas of these triangles.

Solution: let $\Delta ABC \sim \Delta PQR$ and $\frac{AB}{PQ} = \frac{3}{5}$

$\frac{ar(ABC)}{ar(PQR)} = \left(\frac{AB}{PQ}\right)^2$ (From theorem 13.3)

So, $\frac{ar(ABC)}{ar(PQR)} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

Therefore the ratio of the areas of these triangles is 9:25.

CHECK YOUR PROGRESS 13.3

(i) If the sides of two similar triangles are in the ratio 5:10, then the areas of thesetriangles are in the ratio 25:100. (true/false)

(ii) If the areas of two similar triangles are in the ratio 16:81, then the sides of these triangles are in the ratio 4:8. (true/false)

(iii)Let $\Delta ABC \sim \Delta PQR$ and their areas be, 25 cm^2 and 49 cm^2, respectively. If PQ=14 cm, find AB.

(iv)Perimeter of two equilateral triangles ΔABC and ΔXYZ are 144 m and 96 m.Findar(Δ ABC) :ar(ΔXYZ).

13.6 PERPENDICULAR ON THE TRIANGLE

Theorem 13.4: if a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

Given: a right triangle PQR, rightangled at Q.
Let QS be the perpendicular to the hypotenuse PR (Fig.).

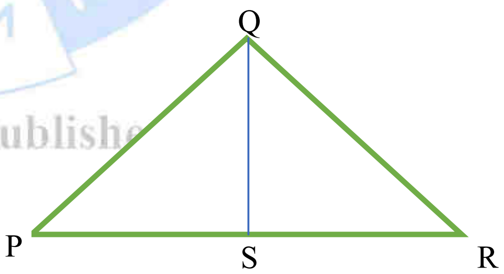


Fig. 13.26

To Prove: $\Delta PSQ \sim \Delta PQR$

$\Delta QSR \sim \Delta PQR$ and $\Delta PSQ \sim \Delta QSR$

Proof: in ΔPSQ and ΔPQR

$\angle P = \angle P$

and $\angle PSQ = \angle PQR$ (each 90°)

So, $\Delta PSQ \sim \Delta PQR$ (by AAsimilarity rule) ……….(1)

Similarly, in ΔQSR and ΔPQR

$\angle R = \angle R$ and $\angle QSR = \angle PQR$ (each 90^o)

$\Delta QSR \sim \Delta PQR$ (by AAsimilarity rule) ……..(2)

So, from (1) and (2), triangles on both sides of the perpendicular QS are similar to the whole triangle PQR.

So, $\Delta PSQ \sim \Delta QSR$

Pythagoras Theorem 13.5: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

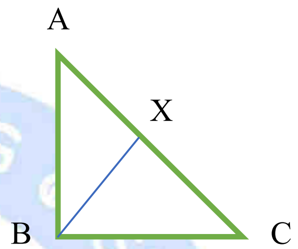


Fig. 13.27

Proof:

Given a right triangle ABC right angled at B.

To prove: $AC^2 = AB^2 + BC^2$

Draw BX ⊥ AC

$\Delta ABX \sim \Delta ABC$ (From theorem 11.4)

$\frac{AX}{AB} = \frac{AB}{AC}$ (Corresponding sides are proportional)

$AX \times AC = AB^2$ ……..(1)

$\Delta BXC \sim \Delta ABC$ (From theorem 11.4)

$\frac{CX}{BC} = \frac{BC}{AC}$ (Corresponding sides are proportional)

$CX \times AC = BC^2$ ……..(2)

Adding (1) and (2)

$$AX \times AC + CX \times AC = AB^2 + BC^2$$
$$AC(AX + CX) = AB^2 + BC^2$$
$$AC \times AC = AB^2 + BC^2$$
$$AC^2 = AB^2 + BC^2$$

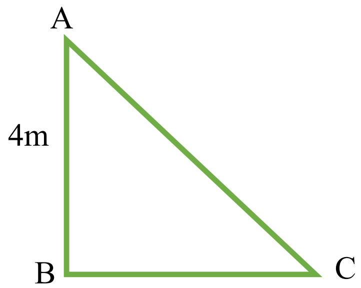
The above theorem is also called Baudhayan theorem.

Converse of this theorem is also true. That is, in a triangle if square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a 90^o.

Example 11: A ladder is placed against a wall such that its foot is at a distance of 3 metre from the wall and its top reaches a window 4 metre above the ground. Find the length of the ladder.

Solution: let AC be the ladder and BA be the wall withthe window at A.by Pythagoras Theorem

$AC^2 = AB^2 + BC^2$
$AC^2 = (4)^2 + (3)^2$
$AC^2 = 16 + 9 = 25$
$AC = 5$



Therefore, the length of ladder is 5 metre.

Fig. 13.28

CHECK YOUR PROGRESS 13.4

(i) 5 m, 12 m, 13 m are the sides of triangle. It is right triangle. (true/false)

(ii) 4 m, 6 m, 8 m are the sides of triangle. Is it right angled triangle?(yes/no)

(iii) Pythagoras theorem is valid for any triangle. (true/false)

(iv) Pythagoras theorem is also called Baudhayan theorem. (true/false)

(v) If the three sides of a triangle are a, $\sqrt{2}a$ and $\sqrt{3}a$ then the measure of theangle opposite to longest side is

(*a*) 45° (*b*) 30°

(*c*) 60° (*d*) 90°

(vi)A vertical pole of length 3 m casts a shadow of 7 m and a tower casts a shadowof 28 m at the same time. Find the height of tower.

(vii) A street light bulb is fixed on a pole 6 m above the level of the street. If a woman ofheight 1.5 m casts a shadow of 3 m, find how far she is away from the base of the pole.

RECAPITULATION POINTS

- Two figures having the same shape and size are called congruent figures.

- Two circles of same radii are congruent.
- Two triangles are congruence if
  - (i) The sides of one triangle are equal to corresponding sides of other triangles.
  - (ii) The angles of one triangle are equal to corresponding angles of other triangles.

  That is, symbolically $\Delta ABC \cong \Delta PQR$ if AB=PQ, BC=QR, CA=RP and □A=□P, □B=□Q and □C=□R. That is, A corresponds to $P(A \leftrightarrow P)$, B corresponds to $Q(B \leftrightarrow Q)$, and C corresponds to $R(C \leftrightarrow R)$
- Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.(SAS congruence rule)
- Two triangles are congruent if two angles and one side of one triangle are equal to two angles and corresponding side of other triangle.(AAS congruence rule)
- If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent. (SSS congruence rule)
- If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent. (RHS congruence rule)
- Angles opposite to equal sides of an isosceles triangle are equal.
- The sides opposite to equal angles of a triangle are equal.
- If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).

- In any triangle, the side opposite to the larger (greater) angle is longer.
- The sum of any two sides of a triangle is greater than the third side.
- Two triangles are said to be similar if their corresponding anglesare equal and their corresponding sides are proportional.
- Two triangles are similar if one angle of a triangle is equal to the one angle of other triangle and the sides includes these angles are in the same ratio. (SAS similarity rule)

- If three (or two) angles of one triangle is equal to the three (or two) angles of the other triangle, then their corresponding sides are in the same ratio and hence the two triangles are similar. (AAA or (AA) similarity rule)
- If three sides of one triangle are in the same ratio (proportional) to the three sides of another triangle, then their corresponding angles are equal and hence the two triangles are similar. (SSS similarity rule)
- If a line is drawn parallel to one side of atriangle to intersect the other sides in distinct points, the other two sides aredivided in the same ratio.
- The ratio of areas of two similar triangles is equal to the square of the ratio of theircorresponding sides.
- Pythagoras Theorem:In a right triangles the square of the hypotenuse is equalto the sum of the squares of the other two sides.
- In a triangle, if the square of one side isequal to the sum of squares of other sides then the angle opposite to the first sideis a right angle.

TERMINAL EXERCISE

(i) Find the measure of each interior angle of equilateral triangle.

(ii) Which of the following is not a criterion for congruence of triangles?

(a) SSS (b) SAS (c) ASS (d) ASA

(iii) The vertical angle of isosceles triangle is 80^o. Find its base angles.

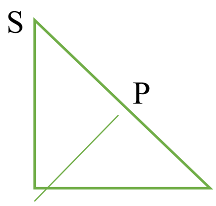
(iv) If sides of two similar triangles are in the ratio of 4:5, then areas of these trianglesare in the ratio ___________ .

(v) ΔABC is an isosceles right triangle, right angled at B, then AC^2 =........... .

(a) AB^2 (b) $2AB^2$ (c)$3AB^2$ (d)$4AB^2$

(vi) A man goes 3 m towards West and then 4 m towards North. How far is hefrom the starting point?

(vii) In the given figure PQ = PR and PQ = PS. Prove that $\angle$QRS = 90^o.



R Q

Fig. 13.29

(viii) State and prove Basic Proportionality Theorem.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 13.1

(i) (d) SSA

(ii) (c) BA = QP

(iii) 180^o

(iv) (b) $\Delta CBA \cong \Delta DFE$

(v) (a) 50^o

(vi) Same size

(vii) True

(viii) ∠A

(ix) ∠P = 60^o

CHECK YOUR PROGRESS 13.2

(i) True

(ii) True

(iii) Same

(iv) $\frac{AB}{PQ} = \frac{BC}{QR}$

(v) True

(vi) (c) $DE = \frac{1}{3} BC$

(vii) $x = 4cm$

CHECK YOUR PROGRESS 13.3

(i) True

(ii) False

(iii) AB = 10

(iv) $ar(ABC) : ar(XYZ) = 9 : 4$

CHECK YOUR PROGRESS 13.4

(i) True

(ii) No

(iii) False

(iv) True

(v) 90^o

(vi) 12 m

(vii) 9 m

SUPPLEMENTARY STUDY MATERIAL

- https://diksha.gov.in
- https://ncert.nic.in
- https://mathworld.wolfram.com/topics/Triangles.html
- https://www.mathopenref.com/similartriangles.html
- Pythagorean Theorem water demo on YouTube

14 QUADRILATERALS

INTRODUCTION

Look around you and you will find so many closed figures with four sides, four angles and four vertices. Such figures are called quadrilaterals.

In Quadrilateral PQRS,

Four sides – PQ, QR, RS and SP

Four angles – ∠P, ∠Q, ∠R and ∠S

Four vertices – P, Q, R and S

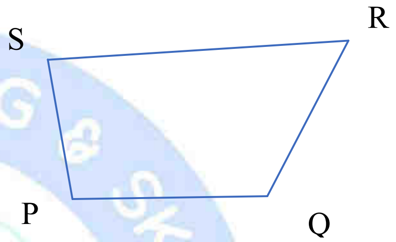
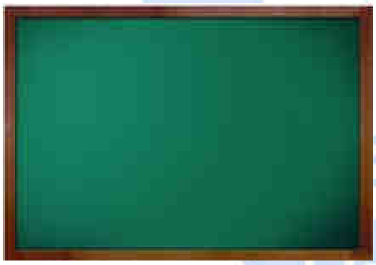


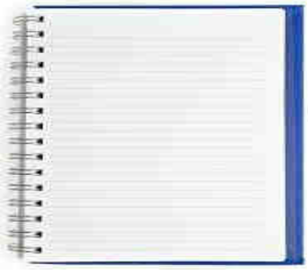
Fig. 14.1

You see manysuch objects which are in the shape of a quadrilateral-

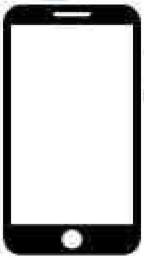


Green board

TV Screen



Pages of notebook



mobile

Fig 14.2

14.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Define the types and properties of quadrilaterals.
- Define the relation between the angles of a quadrilateral.

- Define the area of the parallelograms.
- Define the area of the triangles.
- Define the relation between the areas of the Parallelograms on the same base and between the same parallels.
- Define the relation between the areas of the two triangles on the same base and between the same parallels.

Expected background knowledge:

- Lines and Angles
- Vertices.

Some used Symbols:

(i) $\cong$: ′Congruence′

(ii) ~ : ′similar′

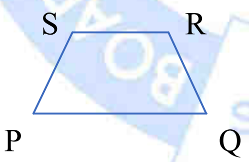
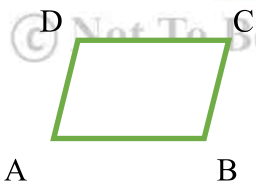
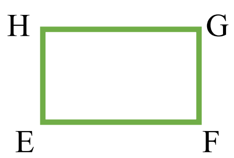
(iii) $\perp$: 'perpendicular'

(iv) Δ : ‘ triangle’

(v) □ : 'angle'

(vi) i.e.: 'that is'

14.2 TYPES OF QUADRILATERALS

1. Trapezium 	In quadrilateral PQRS, One pair of opposite sides (PQ and RS) are parallel is called trapezium.
2. Parallelogram 	In quadrilateral ABCD, Both pair of opposite sides are parallel (i.e., AB//DC and BC//AD) is called parallelogram.
3. Rectangle 	In parallelogram EFGH (why?), one of its angle is 90^o. Such a parallelogram is called rectangle.

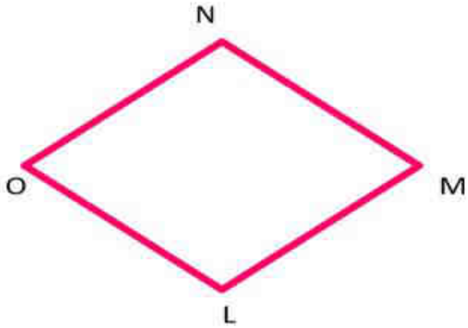
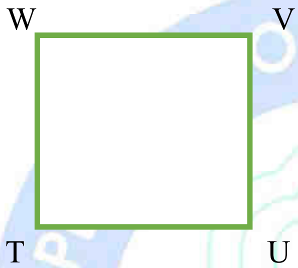
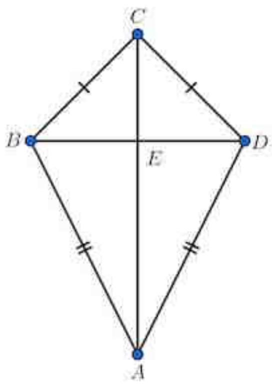
4.Rhombus 	In parallelogram LMNO, all the sides are equal (i.e., LM=MN=NO=OL). It is called a rhombus.
5. Square 	In parallelogram TUVW, one of its angle is 90°and all sides equal is called a square.
6. Kite 	In quadrilateral ABCD, two pairs of adjacent sides are equal i.e., BC = CD and AB = AD. It is called a kite.

Fig 14.3

Note: Square, rectangle and rhombus are all parallelograms.

14.3 PROPERTIES OF DIFFERENT TYPES OF QUADRILATERALS

Activity 1

Draw a parallelogram PQRS on a cardboard sheet and draw a diagonal PR.

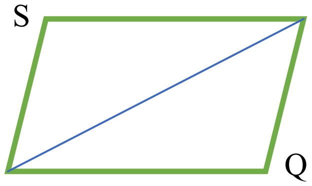


Fig. 14.4

Next, cut it along diagonal PR. You obtain two triangles.They are identical (both shapes and sizes are same).You may observe that on placing a triangle on another triangle they cover each other completely. They are congruent to each other. So, you will observe that a diagonal of parallelogram divides it into two congruent triangles.

14.3.1 Properties of different types of quadrilaterals

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram, opposite sides are equal.
3. If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.
4. In a parallelogram, opposite angles are equal.
5. If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.
6. The diagonals of a parallelogram bisect each other.
7. Diagonals of a rectangle bisect each other and are equal.
8. Diagonals of a rhombus bisect each other at right angles.
9. Diagonals of a square bisect each other at right angles and are equal.
10. If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
11. A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel.

14.3.2 Angle Sum Property of a Quadrilateral

The sum of the angles of a quadrilateral is 360°.

Let PQRS be a quadrilateral and PR be a diagonal

and dividing the quadrilateral PQRS into two triangles

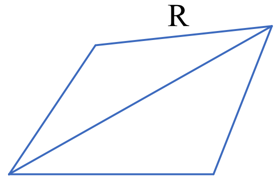


Fig.14.5

In ΔPQR

$\angle Q + \angle QRP + \angle RPQ = \angle 180^o$ …….(1) (why?) (Recall angle sum property of a triangle)

Similarly, in ΔPSR

$\angle S + \angle SRP + \angle RPS = \angle 180^o$ ……..(2)

Adding (i) and (ii), we get

$\angle Q + \angle QRP + \angle RPQ + \angle S + \angle SRP + \angle RPS = \angle 180^o + \angle 180^o = \angle 360^o$

So, $\angle P + \angle Q + \angle R + \angle S = \angle 360^o$ (because □QRP + □SRP = □R and □RPQ + □RPS = □P)

Therefore, the sum of the angles of a quadrilateral is 360°.

Example 1: Three angles of a quadrilateral are 85^o, 80^o, 90^o then find the fourth angle.

Solution: Let the fourth angle be x^o

We know, sum of the angles of a quadrilateral is 360°

Therefore, $85^o + 80^o + 90^o + x^o = 360^o$

So, $x^o = 360^o - 255^o = 105^o$

14.4 DIAGONAL OF A PARALLELOGRAM AND RELATION TO THE AREA

Let PQRS be a parallelogram and PT ⊥ RS

Let b = RS be the base of a parallelogram

and h = PT be its height (Altitude)

then, area of a parallelogram = base × altitude = $b \times h$

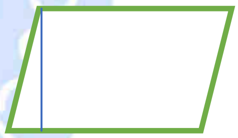


Fig. 14.6

Theorem 14.1:A diagonal of a parallelogram divides it into two congruent triangles.

Proof: Let PQRS be a parallelogram and PR be a diagonal. Observe that the diagonal PR divides parallelogram PQRS into two triangles, ΔPQR and ΔRSP

To prove: $-\Delta\ PQR\ \cong \Delta\ RSP$

In ΔPQR and ΔRSP, QR || PS and PR is a transversal.

So,□QRP = □SPR(alternate interior angles)

Also, PQ || SR and PR is a transversal.

So,□QPR = □SRP(alternate interior angles)

PR = PR(Common)

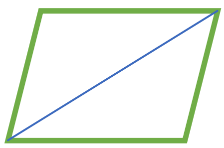


Fig. 14.7

So, $\Delta PQR \cong \Delta RSP$ (by ASA congruence rule)

i.e., diagonal PR divides parallelogram PQRS into two congruent triangles, ΔPQR and ΔRSP.

Example 2: In given figure, PQRS be a parallelogram and PR is a diagonal, area of Δ PQR is $42cm^2$ then, find the area of Δ PSR.

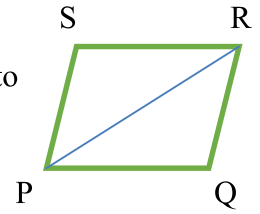


Fig. 14.8

Solution: We know, diagonal PR divides parallelogram PQRS into two congruent triangles, Δ PQR and Δ PSR

So, area of Δ PQR = area of Δ PSR

Therefore, area of Δ PSR = $42cm^2$

14.5 PARALLELOGRAMS AND TRIANGLES BETWEEN THE SAME PARALLEL

Activity 2: Draw two parallelograms PQRS and AQRB on Geoboard

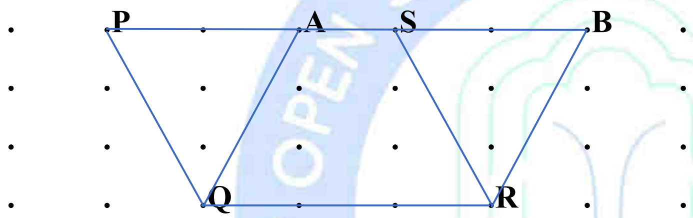


Fig. 14.9

You observe that two parallelograms are on same base QR and between the same parallel PB and QR. Counting the squares, you will see that area of these two parallelograms is equal.

Theorem 14.2: Parallelograms on the same base and between the same parallels are equal in area.

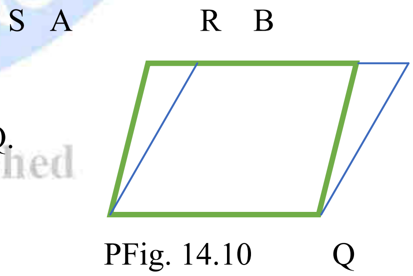


Fig. 14.10

Proof: Given – Two parallelograms PQRS and ABQP, on the same base PQ and between the same parallelsSBand PQ.

To prove: area of PQRS = area of ABQP.

In Δ SPA and Δ RQB,

∠PSA = ∠QRB(Corresponding angles, because SP || RQ and SB transversal line) ………(1)

∠SAP= ∠RBQ(Corresponding angles, because AP || BQ and SB transversal line)..…….(2)

∠SPA = ∠RQB(using Angle sum property of a triangle)…………… (3)

Also, SP = RQ (Opposite sides of the parallelogram PQRS)……….(4)

Therefore, $\Delta SPA \cong \Delta RQB$ (By ASA congruence rule, using (1), (3), and (4))

So, areaΔSPA = areaΔRQB (Congruent figures have equal areas)…… (5)

Now, area PQRS = area Δ SPA + area APQR

= area Δ RQB + areaAPQR [using (5)]= area ABQP

So, parallelograms PQRS and ABQP are equal in area.

Example 3: In given figure, PQRS and ABQP are parallelograms, if area of PQRS = 27 cm^{2}and area PQRA = 19 cm^2 then find area ΔQRB.

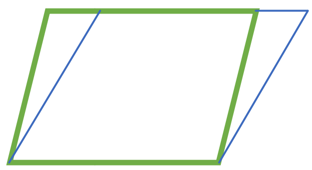


Fig. 14.11

Solution:area PQRS = area ABQP (parallelograms on same base and between the same parallel)area PQRS = area PQRA + area Δ QRB

$\Rightarrow$ 27 = 19 + area Δ QRB

Therefore,area Δ QRB = 27 – 19 = 8 cm^2

14.6 TRIANGLES ON THE SAME BASE AND BETWEEN THE SAME PARALLEL:

Theorem 14.3: Two triangles on the same base and between thesame parallels are equal in area.

Proof: Given, PQRSand ABRS are parallelogramswhose one of the diagonals are PR and AR respectively.

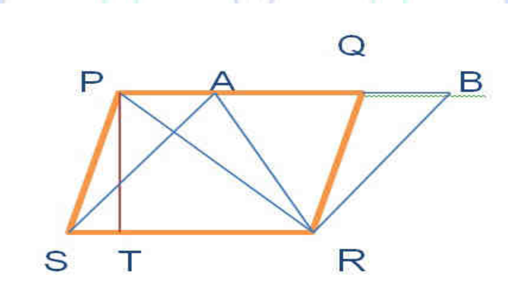


Fig. 14.12

To Prove:area Δ PSR = area Δ ASR

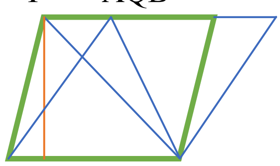
We get, area of PQRS = area of ABRS(theorem 14.1)

$\Delta PSR \cong RQP$ and $\Delta ASR\ \Delta RBA$ (Why?)

So, area Δ PSR $\frac{1}{2} = \times$ *area PQRS* and area$\Delta ASR = \frac{1}{2} \times$ area$ABRS$

Therefore, area Δ PSR = area Δ ASR

Converse of theorem is also true i.e.,Two triangles having the same baseand equal areas lie between the same parallels.



Let PT $\perp$ SR.

$\Delta PSR \cong \Delta RQP$ (both shapes and sizes are same)

So, area Δ PSR = area RQP

Therefore, area Δ PSR $= \frac{1}{2} \times$ *area PQRS*

Fig.14.13

$\frac{1}{2} \times PT \times SR$ (area of parallelogram PQRS)

So, area of Δ PSR $= \frac{1}{2} \times$ (base) $\times$ (altitude)

You can observe that two triangles with same base and equal areas will have equal corresponding altitudes. For having equal corresponding altitudes, the triangles must lie between the sameparallels.

Example 4: In Δ PQR, PS be its median and area of Δ PQS = $21cm^2$, then find the area of Δ PRS.

Solution:

In Δ PQR

draw PT$\perp$ QR.

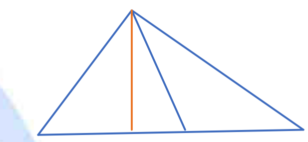


Fig. 14.14

Now area of Δ PQS $= \frac{1}{2} \times QS \times PT$ (area of $\Delta = \frac{1}{2} \times$ base $\times$ altitude)

$= \frac{1}{2} \times SR \times PT$ (as QS = SR)

= area of Δ PRS = $21cm^2$

Example 5: In given figure, the point S divides the side QR of Δ PQR in the ratio 2:3. Prove that area of Δ PQS: area of Δ PSR = 2:3

Solution: Let PT be altitude at QR

Area of Δ PQR $= \frac{1}{2} \times PT \times QR$

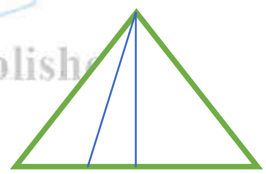


Fig. 14.15

Similarly, Area of Δ PSR $= \frac{1}{2} \times PT \times SR$

andArea of Δ PQS $= \frac{1}{2} \times PT \times QS$

But QS $= \frac{2}{5} \times QR$ and SR $= \frac{3}{5} \times QR$

Therefore, $\frac{\text{Area of } \Delta\text{PQS}}{\text{Area of } \Delta\text{PSR}} = \frac{\frac{1}{2} \times PT \times \frac{2}{5} \times QR}{\frac{1}{2} \times PT \times \frac{3}{5} \times QR} = \frac{2}{3}$

Example6: P,Q,R and S are the mid-points of the sides of a parallelogram ABCD. Show that the area $PQRS = \frac{1}{2} \times \text{area } ABCD$

Solution: Draw the figure and Construct diagonals AC,BD and SQ

In ΔADC, S and R are the midpoints of AD and CD

SR||AC (using midpoint theorem)

In ΔABC, P and Q are the midpoints of AB and BCS

PQ||AC (using midpoint theorem)

Therefore, PQ□AC□SR

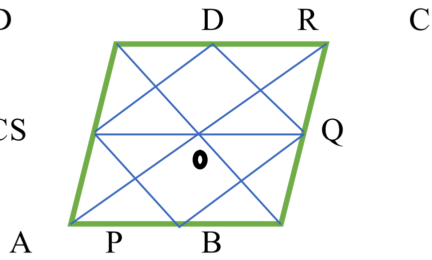


Fig. 14.16

So, we get, PQ□SR

Similarly, we get SP□RQ

Therefore, PQRS is a parallelogram

In ΔABD, O is the midpoint of BD and S is the midpoint of AD

OS||AB (using midpoint theorem)

In ΔABC

OQ||AB (using midpoint theorem)

So, SQ||AB

And AS||BQ (why?)

Therefore, ABQS is a parallelogram

So, Area of $\Delta SPQ = \frac{1}{2}$(Area of parallelogram ABQS) ......(1)

Similarly, Area of $\Delta SRQ = \frac{1}{2}$(Area of parallelogram SQCD) ......(2)

By adding (1) & (2), we get

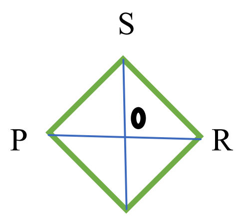
Areaof ΔSPQ +Area of $\Delta SRQ = \frac{1}{2}$(Area of parallelogram ABQS+Area of parallelogram SQCD)

So, Area of parallelogram $PQRS = \frac{1}{2}$(Area of parallelogram ABCD)

Example 7: Diagonals of a rhombus bisect each other at right angles.

Solution:

Let PQRS be a rhombus.



So, PQ=QR=RS=SP (All sides are equal in rhombus)

In ΔPOS and ΔROS

Q

Fig. 14.17

OP=OR(Diagonals of rhombus bisect each other)

OS=OS (Common)

PS=RS (Why?)

Therefore, $\Delta POS \cong \Delta ROS$ (By SSS congruence rule)

$\Rightarrow \angle POS = \angle ROS$

$\angle POS + \angle ROS = 180^\circ$ (Linear pair)

$\Rightarrow 2\angle POS = 180^\circ$

Therefore, $\angle POS = 90^o$

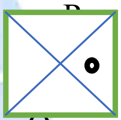
Hence, the diagonals of a rhombus bisect each other at right angle.

Example 8: Diagonals of a square bisect each other at right angles and are equal.

Solution: Let PQRS be a square.

To prove: PR and QS bisect each other at right angles and PR=QS.

S R



P Q

Fig. 14.18

In ΔPQR and ΔQPS,

PQ = PQ (common)

QR = PS (why?)

$\angle PQR = \angle QPS$ (each 90°)

$\Delta PQR \cong \Delta QPS$ (By SAS rule)

Therefore, PR=QS

In a ΔOPS and ΔORQ,

PS=RQ (why?)

$\angle OPS = \angle ORQ$ (Transversal line PR)

$\angle OSP = \angle OQR$ (Transversal line QS)

$\Delta OPS \cong \Delta ORQ$ (by ASA rule)

Therefore, OP=OR ……..(1)

and OQ=OS ……. (2)

From (1) and (2) PR and QS bisect each other.

In ΔOQP and ΔOSP

OQ=OS (from (1))

QP=SP (why?)

OP=OP (common)

$\Delta POQ \cong \Delta POS$ (SSS rule)

$\angle POQ = \angle POS$ (why?)

$\angle POQ + \angle POS = 180°$ (linear pair)

$2\angle POQ = 180°$

So, $\angle POQ = \angle POS = 90°$

Therefore, PR and QS bisect each other at right angles.

Example 9: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Solution: LetPQRS be a quadrilateral with PR and QS are diagonals intersecting at O.

OP=OR and OQ=OS (given)

In ΔPOS and ΔROQ,

OP=OR

∠POS=∠ROQ (Vertically opposite angles)

OS=OQ

Therefore $\Delta POS \cong \Delta ROQ$ (by SAS Congruence rule)

Therefore, ∠OPS=∠ORQ

Similarly, $\Delta POQ \cong \Delta ROS$

Therefore,∠PQO=∠RSO

So, for lines PQ and RS with transversal QS,

PQ||RS(because ∠PQO and ∠RSO are alternate angles and are equal)

Similarly, for lines PS and QR, with transversal PR,

PS||QR(because ∠OPS and ∠ORQare alternate angles and are equal).

Thus, in PQRS, both pairs of opposite sides are parallel.

Therefore, PQRS is a parallelogram.

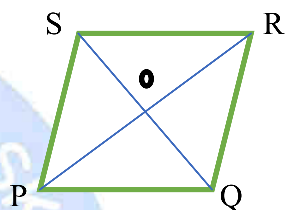


Fig. 14.19

Example 10: If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

Solution: Let ΔABE and parallelogram ABCD be on the same base AB and between the same parallels AB and EC.

To prove: area (ΔEAB) $= \frac{1}{2}\, ar(ABCD)$

Draw BF || AE to obtain another parallelogram ABFE.

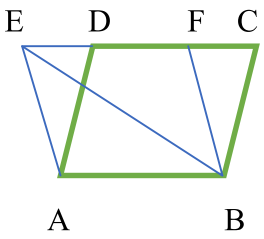


Fig. 14.20

So, ar (ABFE) = ar (ABCD)…..(1) (becauseParallelograms on the same base and between the same parallels are equal in area)

But $\Delta EAB \cong \Delta BFE$ (Diagonal EB divides parallelogram ABFE into two congruent triangles.)

So, area (Δ EAB) = area (Δ BFE)…….. (2)

Therefore, area (Δ EAB) $= \frac{1}{2}\ ar(\text{ABFE})$ [from (2)]

$\Rightarrow$ area (ΔEAB) $= \frac{1}{2}\ ar(\text{ABCD})$ [from (1)]

Example 11: A median of a triangle divides it into two triangles of equal areas.

Solution: Let PQR be a triangle and Let PS be one of its medians.

In Δ PQS and Δ PSR the vertex is common and these bases QS and SR are equal.

QS=SR

Draw PT $\perp$ QR.

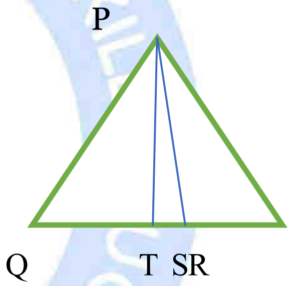


Fig. 14.21

So, area(Δ PQS) $= \frac{1}{2} \times$QS$\times$PT

area(Δ PQS) $= \frac{1}{2} \times$SR$\times$PT (QS=SR)

Therefore, area(Δ PQS)=area(Δ PRS) (why?)

Hence the median of a triangle divides it into two triangles of equal areas.

CHECK YOUR PROGRESS 14.1

(i) Three angles of a quadrilateral are $85^0, 85^0, 90^0$ the fourth angle is

(a) 85^o (b) 110^o

(c) 90^o (d) 100^o

(ii) In a parallelogram diagonals are unequal. (true/false)

(iii)The diagonals of a parallelogram bisect each other. (true/false)

(iv) The area of parallelogram is the product of its base and its corresponding…………… (altitude/side)

(v) The area of a parallelogram on the same base and between the same parallels are ………… (equal/unequal)

(vi) Area of triangle $=\frac{1}{2}\times(\text{base})\times(\ldots\ldots\ldots..)$ (altitude/side)

(vii) The median of the triangle divides it into two triangles of equal area. (true/false)

(viii) In a parallelogram, opposite sides are ………….. (Equal/unequal)

(ix) In a parallelogram, opposite angles are ……….. (Equal/unequal)

(x) If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram. (true/false)

(xi) ΔPQS and parallelogram PQRS are on the same base PQ. If base and altitude of the parallelogram are 14 cm and 10 cm, respectively. Find the area of the parallelogram.

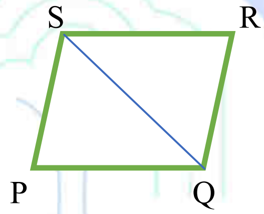


Fig. 14.22

RECAPITULATION POINTS

- Sum of the angles of a quadrilateral is 360°.
- A diagonal of a parallelogram divides it into two congruent triangles.
- In a parallelogram, opposite sides are equal.
- If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.
- In a parallelogram, opposite angles are equal.
- If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.
- The diagonals of a parallelogram bisect each other.
- Diagonals of a rectangle bisect each other and are equal.
- Diagonals of a rhombus bisect each other at right angles.
- Diagonals of a square bisect each other at right angles and are equal.
- If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
- A quadrilateral is a parallelogram if a pair of opposite sides is equal and parallel
- Parallelograms on the same base and between the same parallels are equal in area.

- Area of a parallelogram is the product of its base and the corresponding altitude.
- If a parallelogram and a triangle are on the same base and between the same parallels, thenarea of the triangle is half the area of the parallelogram.
- Twotriangles on the same base and between the same parallels are equal in area.
- Area of a triangle is half the product of its base and the corresponding altitude.
- A median of a triangle divides it into two triangles of equal areas.

TERMINAL EXERCISE

(i) Sum of the angle of a quadrilateral is…………

(ii) Diagonals of a rectangle bisect each other and are equal. (true/false)

(iii) A diagonal of a parallelogram divides it into two congruent triangles. (true/false)

(iv) Twotriangles on the same base and between the same parallels are equal in area.(true/false)

(v) The angle of a quadrilateral PQRS are in ratio 2:3:5:8. Find the measure of largest angle.

(vi) The quadrilateral formed by joining the mid-point of the sides of square is also a square. (true/false)

(vii) If the area of a parallelogram PQRS is $72cm^2$, ST $\perp$ PQ and ST = 6 cm, then find PQ.

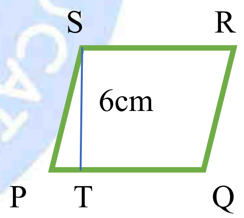


Fig. 14.23

(viii) If the area of a parallelogram PQRS is $50cm^2$,find the area of ΔTPQ

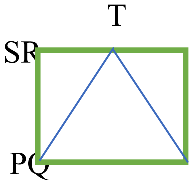


Fig. 14.24

(ix) ABCD and PQCD are parallelogramson the same base CD and between the sameparallels AQ and CD, if area of ABCD = $27cm^2$, then find area of PQCD.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 14.1

(i) (d) 100^o
(ii) False
(iii) True
(iv) Altitude
(v) Equal
(vi) Altitude
(vii) True
(viii) Equal
(ix) Equal
(x) True
(xi) $70cm^2$

SUPPLEMENTARY STUDY MATERIAL

- https://diksha.gov.in
- https://ncert.nic.in
- https://mathworld.wolfram.com/Quadrilateral.html
- https://www.mathopenref.com/tocs/quadrilateraltoc.html
- https://www.mathopenref.com/parallelogram.html
- https://mathworld.wolfram.com/Parallelogram.html

15 CIRCLE

INTRODUCTION

The most important invention was ofwheel.This invention accelerated many inventionsfurther. We have read about points, lines, angles, triangles and quadrilaterals. In this lesson, we shall study about circles, its parts and some important results about circles.

15.1 LEARNINGOBJECTIVES

Aftercompletingthislesson,youwillbeableto:

- The angle subtended by an arc at the centre isdoubletheanglesubtended by it atthe remaining part of circle.
- Anglesinthesamesegmentofacircleareequal.
- Studyaboutconcyclicpoints.
- Thesumofoppositeangleofacyclicquadrilateralis180^{o}.
- Ifthesumofapairofoppositeanglesofaquadrilateralis180^{o},thequadrilateralis cyclic.
- Useofpropertiesofcyclicquadrilateral.
- Solveproblemsbasedonaboveproperties.
- Solvenumericalproblemsbasedonanglebyanarcandcyclicquadrilateral

Expectedbackground

1. Anglesofatriangle.
2. Quadrilateralanditstypes
3. Arc,chordandcircumferenceofacircle

15.2 CIRCLESANDITSRELATEDTERMSANDCONCEPTS

Inourday-to-day life,wecomeacrossmanyobjectsofthefollowingtype –

Ring, wheel of a cycle, watch, bangles, buttons, coins of one rupee denomination etc. (see fig below). All these objects are circular in shape.

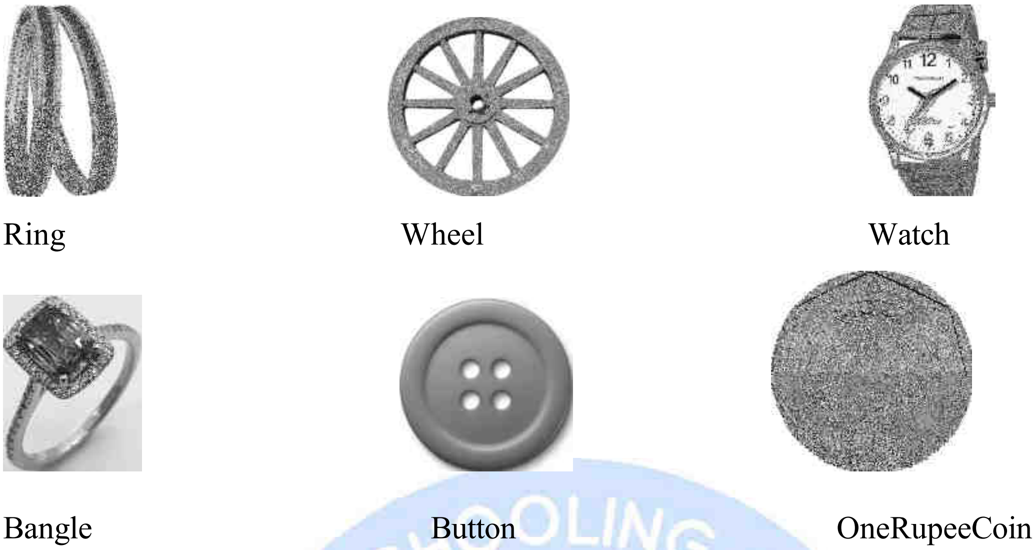


Fig 15.1

Fig 15.2

Acirclecanbedrawnonapaperwiththehelp ofacompass by keeping the pointed leg of compass fixed and rotating the pencil holding edge – around in one complete revolution as shown in Fig 15.2. From the above we are led to the following definition of a circle.

A circle is the locus of a point in a plane which moves in such a way that its distance from a fixed given point in the same plane always remains constant.

15.2.1 Centre and Radius

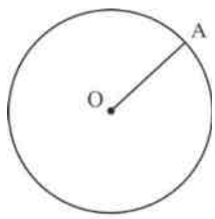


Fig 15.3

The fixed point is called the centre of the circle and fixed distance is called radius of the circles. Thus, in Fig 15.3, O is **centre** of the circle and OA is **radius** of the circle.

Itmaybenotedthatthetermradiushasbeenusedintwosenses,

(i) Thelinesegmentfromcentretoanypointonthecircle.

(ii) InthesenseoflengthOA(Saytheradiusofcircle is 4 cm)

Like a triangle and quadrilateral, a circle is also a closed figure. In Fig 15.4, the circle divides the plane, on which it lies into three parts:

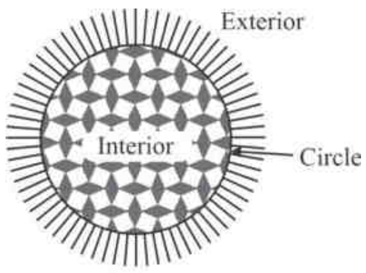


Fig 15.4

(i) Interiorofacircle
(ii) Circleitself
(iii) Exteriorofacircle.

Thecircleanditsinteriortogetherdeterminethecircular region.

15.2.2 Interiorofcircle,exteriorofacircle

To decide whether a point lies with interior, exterior of the circle on the circle, we go through the following–

LetObecentreofthecircleofradius*r*andP$_1$,P$_2$andP$_3$ bethethreepointsontheplaneof the circle.

(i) When OP$_1$<r, the radius of the circle, all such points P$_1$lie in the interior of the circle. Thus,allpointslyingintheinteriorofacircle,theirdistancefromthecentreisless than the radius of the circle (Fig 15.5).

(ii) Where OP$_2$ = r, the radius of the circle then all such points P$_2$ lie on the circle.

Thus, for all points P$_2$ lying on the circle their distance from centre of circle equals to the radius of circle.

(iii) When OP$_3$>r, the radius of the circle, all such points P$_3$ lie in the exterior of the circle. See Fig 15.5.

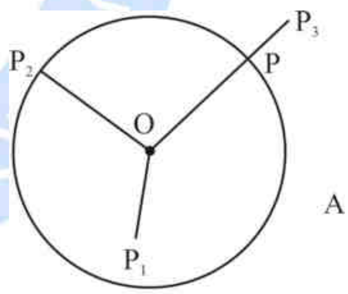


Fig 15.5

Thus, for all points lying in the exterior of the circle, their distance from centre of the circle is greater than radius of the circle.

Again, we can seethat when apoint lies with exterior of thecircle, and is joined withthe centre of the circle, it must cross the circle at some point P of the circle (see Fig 15.5).

15.2.2.1 Chord of a circle

If there are two points P and Q on the circle, then the segment PQ is called a chord of the circle. Thus, a line segment whose

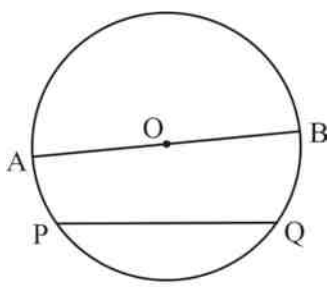


Fig 15.6

end points lie on a circle, is called a chord of the circle (Fig 15.6)

15.2.2.2 Diameterofacircle

A chord which passes through thecentreof acircleis calledadiameterofthecircle.Thus,inFig15.6AOBisdiameterof a circle.

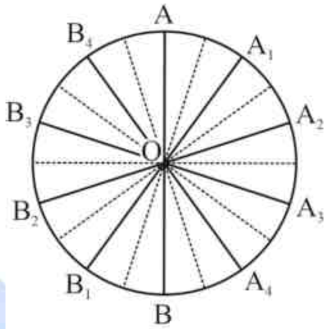


Fig 15.7

Howmanydiameterscan you draw for a circle?

You can see from Fig 15.7 that any number of diameters of a circle can be drawn. As in the case of radius, the diameter is also used in both senses – segment and length.

15.2.2.3 Arcofacircle

Ifwetaketwo points P and Qon a circle,then thepartsof thecirclebetweenpointsPandQiscalledanarcPQofthecircle(seeFig15.8).ArcPQ isdenotedbythesymbol $\overset{\frown}{PQ}$.

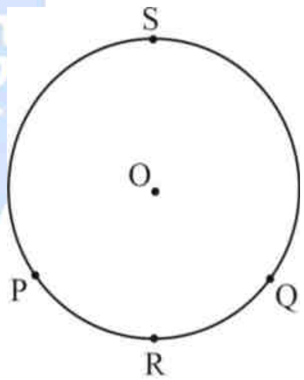


Fig 15.8

Fromfig15.8,youcanseethattherearetwoarcs.

ArcPRQ ($\overset{\frown}{PRQ}$) andarcPSQ ($\overset{\frown}{PSQ}$).Arc $\overset{\frown}{PQR}$ issmallerthansemi-circleandiscalleda**minor**arcand greaterthansemi-circleiscalled**major**arcofthecircle.

15.2.2.4 Semi-circle

WhenPandQaretheendpointsofadiameter,thenarcs $\overset{\frown}{PRQ}$ and $\overset{\frown}{PSQ}$ areequalandeachiscalledsemi-circle(see Fig 15.9).

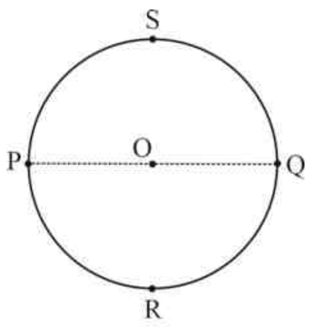


Fig 15.9

15.2.2.5 Circumferenceofacircle

Thelengthofcompletecircleiscalleditscircumference.

Thus,inFig15.9,thelengthofarc $PRQSP$ iscalledthe circumference of the circle.

15.2.2.6 Segmentofacircle

Theareaenclosedbyanarcanditscorrespondingchordiscalledasegmentofthe circle. The area enclosed by a minor arc and its corresponding chord is called the **minor** segment.Thesegmentismajorifarcismajor.InFig15.10,themajorsegmentisshown by the

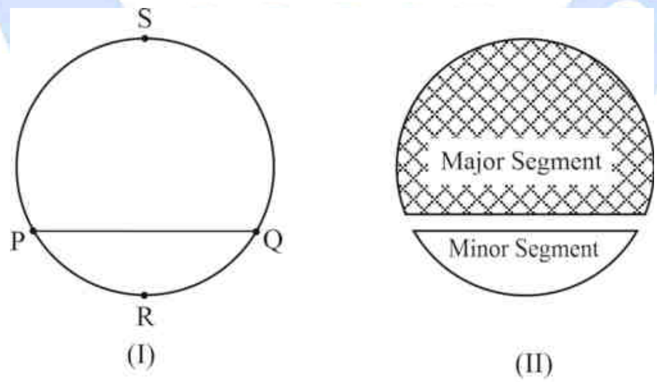


Fig 15.10

shaded region.

15.2.2.7 Sectorofacircle

The area of circle enclosed by an arc and two radii joining the centre to end points of the arc is called a sector of circle.

As in the case of a segment of a circle, the sector correspondingtotheminorarciscalledtheminorsector

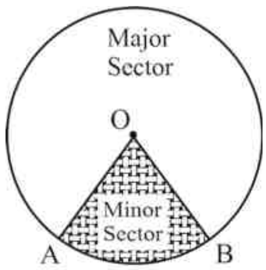


Fig 15.11

andtheonecorrespondingtomajorarcofthecircleis called major sector of the circle.

Infig15.11,theminorsectorisshaded.

When arcs are equal then each is semi-circle, then both segment and both sectors are equal and each is called a semi-circular region.

CHECKYOURPROGRESS15.1

I. Classifythefollowingstatementsastrue/false–

(i) Acirclehasnoequalchords.

(ii) Ifacircleisdividedintofourequalarcs,eachisaminorarc.

(iii) Thelengthofachordisalwaysequaltoradiusofthecircle.

(iv) Sectorisaregionbetweenthechordanditscorrespondingarc.

(v) Thelengthofadiameteris4timesitsradius.

(vi) Thelargestchordofacircleistwicetheradiusofthecircle.

(vii) Acircledividesaplane,inwhichitlies,intofourequalparts.

(viii) Majorarcisgreaterthansemicircle.

II. Fillinthefollowingblankstomakeeachofthefollowingatruestatement–

(i) Minorsegmentofacircleistheareaenclosedbycorrespondingarcandaofthecircle.

(ii) Anarcisa…………whenitsendpointslieattheendof…….

(iii) A .... isthelargestchordofthecircle.

(iv) Apointwhosedistancefromthecentreislessthantheradiusofthecircle lies in of circle.

(v) Asectoristheareaofthecircleenclosedbyanarcandtwo………….. joiningthecentretotheendpointsofthearc.

(vi) Achordofacirclecannotbegreaterthanits…………

(vii) Thelinesegmentjoiningthecentretoanypointonthecircleiscalleda…….. ofthecircle.

(viii) Thecentreofacircleliesinthe..........................ofthecircle.

15.2.2.8 Concentriccircles

Take a point O on a piece of paper and draw circles ofdifferentradiusi.e.,1cm,2cm,3cm,4cmtaking Oas centre. We can draw as many circles as we wish fig (15.12). It gives us family of circles with the same centre. Such circlesare called concentric circles. In fig (15.12) are four concentric circleswithcommoncentreOandradiusOA,OB,OCand OD.

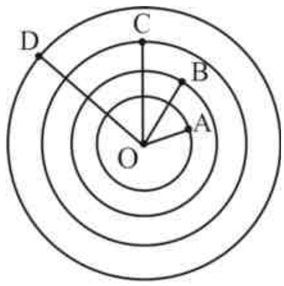


Fig 15.12

Thus, a family of circles is said to be concentric if they havethe same. centre.

15.2.2.9 Congruentcirclesorarcs

Two circles are said to be congruent if they have the same radius i.e., radius of one circle is equal to radius of other circle. Suchcirclecoverseachotherperfectlyiftheyareplacedoneover other. Similarly, two arcs are said to be congruent if after placing them one over the other the arcs cover each other perfectly.

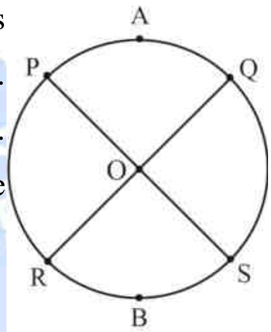


Fig 15.13

Infigure(15.13)

If arc$\widehat{PAQ}$ = arc $\widehat{RBC}$

then ∠POQ = ∠ROS

conversely if∠POQ = ∠ROS then

arc $\widehat{PAQ}$ = arc $\widehat{RBS}$

Twoarcsofacirclearecongruentifandonlyiftheanglessubtendedbythematthe centre are equal

Infigure(15.14)

if arc $\widehat{PAQ}$ = arc $\widehat{RBS}$ then

∠POQ = ∠ROS and conversely if

∠POQ = ∠ROS then

arc $\widehat{PAQ}$ = arc $\widehat{RBS}$

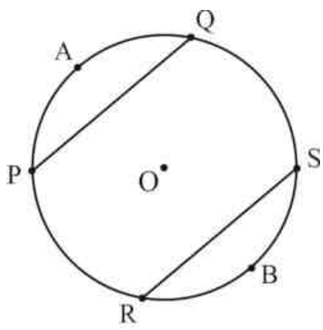


Fig 15.14

Two arcs of a circle are congruent if and only if their corresponding chords are equal

Theorem (prove)

Equal chords of a circle subtend equal angles at the centre.

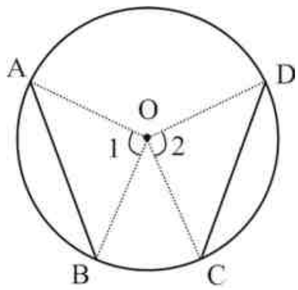


Fig 15.15

Given: A circle with centre O in which

$\overline{AB}=\overline{DC}$

To prove:∠ AOB = ∠COD

Construction: Join AO, BO, CO and OD

Proof: In triangle's AOB and COD

OA = OC (Radius of same circle) OB=OD(Radiusofsamecircle)

AB=CD(given)

ΔAOB≅ΔCOD(SSSRule)

∠AOB=∠CODor∠1=∠2

Thus,wehaveprovedthatequalchordsofacirclesubtendequalanglesatthecentre.

Converselywhataboutttwochordsofacirclewhichsubtendequalanglesatthecentre? Are they equal?

Forthatwedothefollowingactivity:

Drawacircleonthepaperwith centreO.Draw ∠AOBatthecentre,whereA and B arepointsonthecircle.Make∠POQatthecentreofthecirclesuchthat∠POQ=∠AOB.Cutthesectors PRQOandACBOwherepointsRandCareonminorarcs PQandABrespectively.PutthesectorPRQOonthesectorACBO.Youwillfindthat $\widehat{PRQ} = \widehat{BCA}$i.e.,thearcsalongwithsegmentsPQandABareequal.

⇒AB=PQ

Thus,wesaythattwochordsofacirclewhichsubtendequalanglesatthecentreare equal.

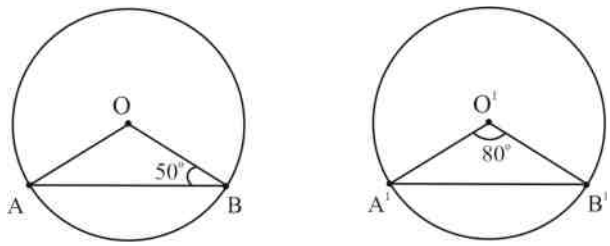
Theorem: If two chordssubtend equal angles at thecentrethen thechords are equal. This is converse of the theorem above

Note: Theaboveresultalsoholdsgoodincaseofcongruentcircles

Wetakeafewexamplesusingtheaboveproperties.

Example1:

Infigure15.16aretwoequalcircleswithcentresOandOin(i)∠OBA=50ºandin(ii)∠A′O′B′=80º.Provethat AB=A′B′



(i) Fig 15.16 (ii)

Solutions: Incircle (i)∠OAB=∠OBA=50º(given)

∠AOB=180º–(∠OBA+∠OAB)

=180º–(50º+50º)

= 80º

In circle (ii) ∠A′O′B′= 80º Chord AB = chord A′B′

Example2: Infig15.17aregiventwocircleswithcentreOandO′.ChordABandchordPQofthecircles arelength$4\sqrt{2}$arecongruent each.If∠AOB=90º=∠PO′Qprovethattwocircle are congruent.

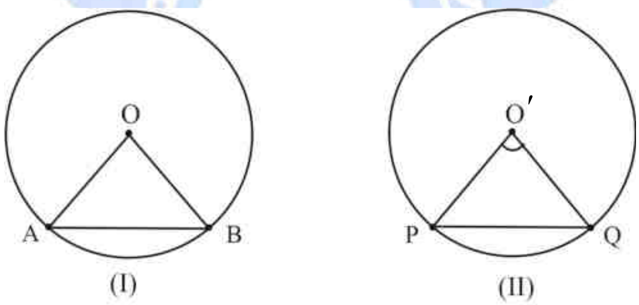


Fig 15.17

Solution: Incircle(i)OA=OBradiusofcircle …………(I)

∠AOB=90º

$\therefore AB^2=OA^2+OB^2$

$=OA^2+OA^2$

$\Rightarrow (4\sqrt{2})^2 = 2OA^2$

$\Rightarrow OA^2 = \frac{32}{2}$

$= 16 = (4)^2$

OA=4cm,OB=4cm

Similarly, incircle(ii)OP=OQradiusofcircle …………(II)

$\angle PO'Q = 90^o$

$\therefore PQ^2 = O'P^2 + O'Q^2$

$= O'P^2 + O'P^2$

$\Rightarrow (4\sqrt{2})^2 = 2O'P^2$

$\Rightarrow O'P^2 = \frac{32}{2}$

$= 16 = (4)^2$

O′P=4cm,O′Q=4cm

From (I) and (II) –

The radius of two circles is equal. Two circles are congruent.

Example3:

Infig15.18ChordAB=ChordCDshowthatChordAC=ChordBD

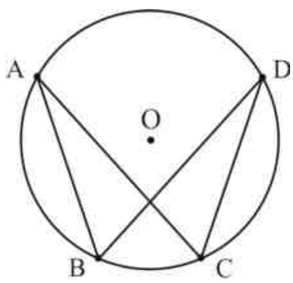


Fig 15.18

Solution: Two arcs corresponding to equal chord AB and CD are equal Add to each arc the arc BC

arcAB+arcBC=ArcCD+arcBC

= arc ABC = arc DCB ChordAC = Chord BD

Example4: Infig15.19ArcPQ=ArcQR,$\angle POQ = 20^o$and$\angle POS = 80^o$.Find$\angle ROS$.

Solution: Sincearc PQ=arc QR

$\angle POQ = \angle QOR$

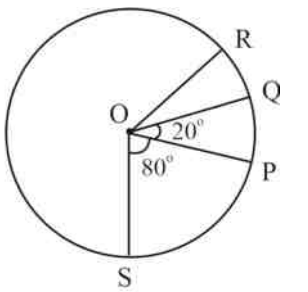


Fig 15.19

(Equalarcssubtendequalanglesatthecentre)

$\angle QOR = 20^{o}$

Now $\angle ROS = \angle ROQ + \angle QOP + \angle POS$

$= 20^{o} + 20^{o} + 80^{o}$

$= 120^{o}$

Activity1:(doityourself)

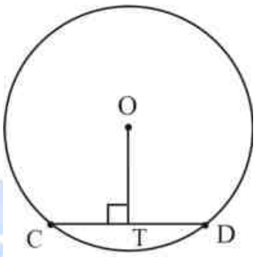


Fig 15.20 (A)

(i) DrawacirclewithcentreO(seefig 15.20A)

(ii) DrawachordCD

(iii) FromOdrawOT ⊥ CD

(iv) Measure CT and TD You willsee that CT = TD

NoteI–Theperpendiculardrawnfromthecentreofacircletoachordbisectsthechord.

ActivityII:(doityourself)

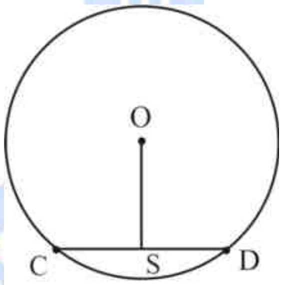


Fig 15.20 (B)

(i) DrawacirclewithcentreO(seefig 15.20B)

(ii) DrawachordCD

(iii) TakeSasmidpintofCD

(iv) JoinOandS

(v) Measure $\angle OSC$ and $\angle OSD$

You find that $\angle OSC = \angle OSD = 90^{o}$

Note II – The line joining the centre of a circle to the mid-point of a chord is perpendicular tothe chord. (NoteII-isconverseofNoteI)

15.2.2.10 Circlethroughthreepoints

Recall that two points are sufficient to determine a line uniquely. The question arises, "how many points are necessary to determine a circle uniquely?"

Youcanseefromfig15.21thatfromapointP,youcandrawasmanycirclesasyoulike.

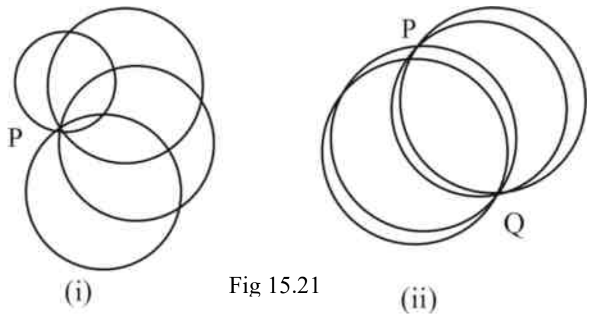


Fig 15.21

NowtaketwopointsP&Qyou can againseethat youcandraw many circle asyoulike passing through these two points P, Q (see fig 15.21(ii)) imagine what happens when there are three points P, Q & R? of courseif these points are collinear, you cannot draw a circle passing through three points (see fig 15.22a)

LetusseethecaseinwhichP,QandRarenotcollinear.JoinPQandQAanddrawrightbisectofsegment PQandQRrespectivelyintersectingatO.

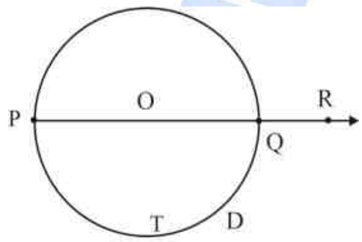


Fig 15.22 (a)

Fig 15.22 (b)

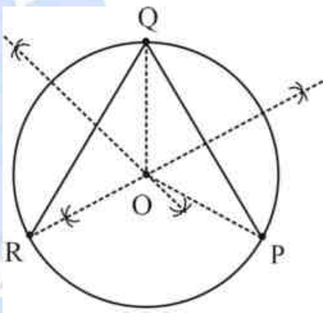


Fig 15.22 (c)

OliesontherightbisectorofPQ⇒OP=OQ …(i)

Similarly, O lies on the right bisector of QR⇒OQ = OR …(ii)

From (i) and (ii)

OP=OQ=OR

It indicates that O is equidistance from P, Q & R. Therefore, if we draw a circle with centre O andradiusOP,itwillpassthroughQandRalso.Thus,wecandrawacirclepassingfrom three points P, Q & R.

We know that perpendicular line can intersect only at onepoint, therefore, thepoint O is unique and hence the circle is unique. Thus, there is one and one circle passing through three given points (not collinear).

Thethreenoncollinearpointsdetermineatriangleandauniquecirclepassesthrough itsvertices.Thecircleiscalledcircumcircleofthetriangle.Thecentreandradiusof the circle are respectively called circumcentre and circum radius.

ActivityIII

(i) DrawacirclewithcentreO(seefig 15.23a)

(ii) DrawtwocordsXYandRTofthecircle

(iii) DrawOZ⊥XYandOS⊥RT

(iv) MeasureOXandOS

(v) WefindOZ=OS

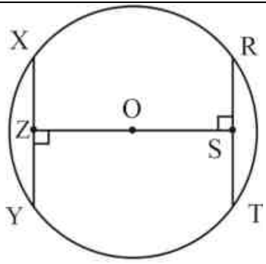


Fig 15.23 (a)

Equalchordsofacircleareequidistantfromthecentreinfig (15.23b), OZ = OS

Measure and observe

that XY = RT

Chords,thatareequidistantfromthecentreofacircle,areequal.

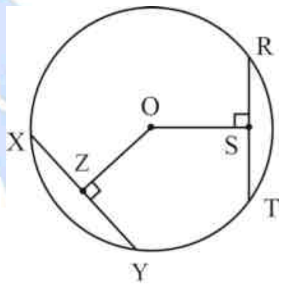


Fig 15.23 (b)

Theaboveresultsaretrueforcongruentcirclesalsowetakeafewexamplesusingthese properties of circles.

CHECKYOURPROGRESS15.2

Inquestion1to5fillinblankstomakeeachofthestatementtrue. Refer fig 15.24.

(i) ArcAPBis.......... arcofthecircle

(ii) Mis‘mid-pointofAB.OMis............toABifOiscentre

(iii) CDisthelongest…….ofthecircle

(iv) Theratioofcircumferencetothediameterofthecircleisalways ………..

(v) Themaximumnumberofcommonpointsoftwointersectingcircle is ……….

(vi) Find the circumference of circle whose radius is (a)10.5 cm (b) 3.5 cm (use $\pi = 22/7$)

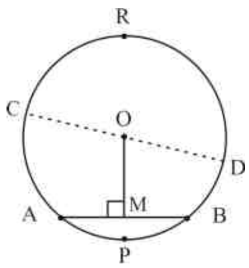


Fig 15.24

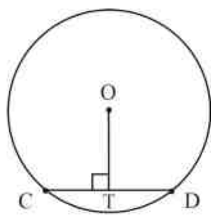


Fig 15.25

(vii) Infig15.25,theradiusofcircleis7.5cmandOT⊥CD ifOT = 4.5 cm, find the length of chord CD

(viii) Inthefig15.26,APBisanarcofthecircle.Howtolocateits centre.

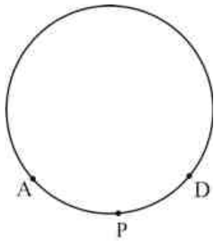


Fig 15.26

15.3 ANGLESINACIRCLEANDCYCLICQUADRILATERAL

We studied different kind of angles. These angles were angles of parallel lines, triangles and quadrilaterals. We shall study theangles made by arc and chords in a circle. Weshall alsostudy about a cyclic quadrilateral.

15.3.1 Angles in a Circle

15.3.1.2 Centralangle

Theanglemadebyanarcofthecircleatthecentreofacircleis called central angle in fig (15.27) are ACB subtends ∠AOB at thecentreO. So ∠AOB is called central angle subtend by an arc ACB or chord AB.

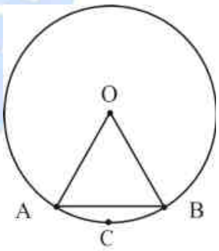


Fig 15.27

Themeasurementof∠AOBdependsonthelengthofthearcorchord.

15.3.1.3 Degreemeasureofanarc

Thelengthofanarcisclearlylinkedtothecentralanglesubtended by at the arc.

Thedegreemeasureofaminorarcofacircleismeasureof corresponding angle.

Infigure(15.28)degreemeasureof arcACB=x^o

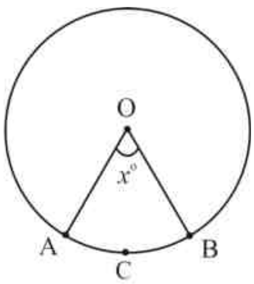


Fig 15.28

So,wewill concludethatdegreemeasureofasemicircleis180º.

Thedegreeofmajorarcis360ºminusdegreemeasureofcorrespondingminorarc.

15.3.1.3 Relationbetweenlengthofarcanditsdegreemeasure

$$Length of an arc = circumference \times \frac{degree measure of arc}{360°}$$

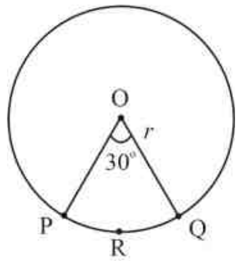


Fig.15.29

Example 5: Findthelengthofarcwhodegreemeasureis30º

Solution:

$$Length of an arc = circumference \times \frac{degree measure of arc}{360°}$$

$$= \frac{30}{360} \times 2\pi r = \frac{\pi r}{6}\ length$$

15.3.1.4 Inscribed angle

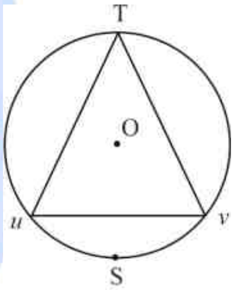


Fig 15. 30

The angle made by an arc (or chord) on any pointontheremainingpartofthecircleiscalledinscribedangle.Infig(15.30) ∠UTVistheangleinscribedbyarcUSVatpointTofthe remaining part of the circle or by the Chord UV at point T.

ActivityIV

Draw a circle with centre A.Let PSQ be an arc and R any point onthe remaining part up circle.

Measure central angle PAQ and inscribed ∠PRQ by arc PSQ at centre A and at R at the remaining part of the circle, we find that

∠PAQ=2∠PRQ

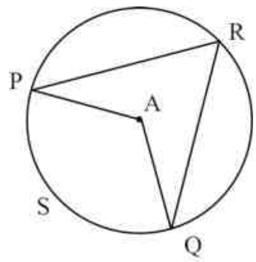
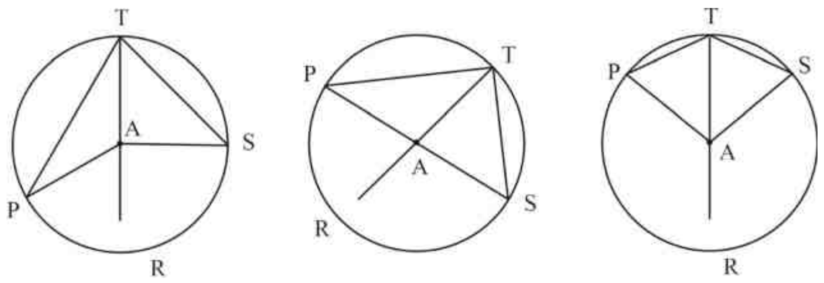


Fig 15.31

This activity may be performed two or three times by taking circles of different radii. The arcs in different circle are to be taken of different size. Every time the result is that:

Central angle = 2 (inscribed angle) Thus we conclude that the angle subtended by an arc at the centre is double the angle subtendedby it at theremaining part of the circle.



(i) (ii) (iii)

Fig 15. 32

Infig15.32(i),fig15.32 (ii)andfig15.32 (iii),wefindcentralangleslessthan180ºinfig15.32 (i);two right angles in fig 15.32 (ii) and reflex angle in fig 15.32 (iii).

Wecometothesameconclusionthatcentralanglemadebyanarcisdoubletheinscribed angle on the remaining part of the circle.

Again,infig15.32 (ii)weobservethatanglePASbyarcPRSatcentreAistworightangleand angle PTS at point T on the remaining part of circle is 90º.

Wegetthefollowingresult.

15.4 ANGLEINASAMECIRCLEISARIGHTANGLE

Theorem: Angleinthesamesegmentofacircleareequal

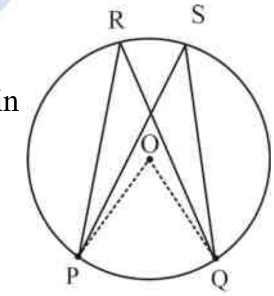


Fig 15.33

Given: A circle with centre O and angles ∠PRQ and ∠PSQ are in thesame segment formed by the chord PQ (or arc PAQ)

Toprove:∠PRQ=∠PSQ

Construction: JoinOPandOQ

Proof: Anglesubtendedbyanarcatthecentreisdoubletheanglesubtendedbyitatany point on the remaining part of circle. So, we have

∠POQ=2∠PRQ............................(i)

∠POQ=2∠PSQ.............................(ii)

find(i)and(ii)weget

2∠PRQ=2∠PSQ

⇒∠PRQ=2∠PSQ

Theconverseofthetheoremistrue.Itisasfollows:

Ifalinesegmentjoiningtwopointssubtendsequalanglesattwootherpointsonthesame side of the line containing the segment the four points lie on a circle.

Forverificationofabovestatement,wedrawasegmentPQ(sayof6cm).FindtwopointsR and S on the same side of PQ such that ∠PRQ = ∠PSQ.

NowdrawacirclethroughthreenoncollinearpointP,R,Q.

We know that one and only one circle passes through non collinear points. A circle passes through three points P, R and Q. ThiscirclealsopassesthroughS.

AllthefourpointsP,Q,RandSareconcyclic(lyingononecircle)

Wecanrepeatthesameactivitytakinganothersegment.Everytimeyouwillfindthatfour points lie on the circle.

Thisverifiestheresult.

Letustakesomeexamples.

Example6: Infigure(15.34)Oiscentreofthecircle.Findthemeasureof∠BAC

Solution:

∠BOC=360º–(∠AOB+∠AOC)

=360º–(120º+110º)

=360º–230º

∴∠BOC=130º

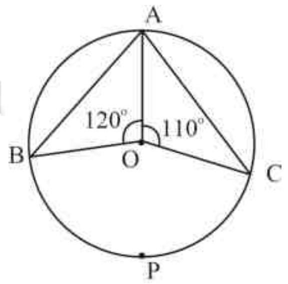


Fig 15.34

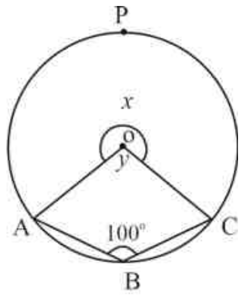
∠BAC=1/2∠BOC

(AnglesbyarcBPAatthecentreandonremainingpartof circle)

$\angle BAC = \frac{1}{2}$(130°)=65°

Example7: Infigure(15.35)Oiscentreofthecircleand∠ABC= 100°.Find ∠*y*.

Solution: Fromfig15.35itisclearthat∠*x*iscentralangleofmajor arc APC and ∠ABC on a point B at remaining part of circle



∠*x*=2∠ABC

=2(100)=200°

∠*y*=360–∠*x*

=360°–200=160°

Example8: Infigure15.36,acirclewithcentreO. ∠BAC=50º.Find*x*

Solution:

∠BAC=50°

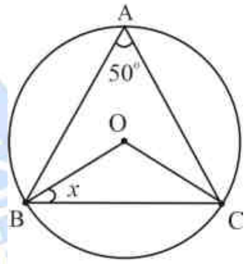


Fig 15.36

∠BOC=2∠BAC=100°

InΔOBC,OB=OC(radiiofcircle)

∠OBC = ∠OCB In ΔBOC

∠OBC + ∠OCB + ∠BOC = 180° 2*x* + ∠BOC = 180°

2*x*+100=180°

2*x*=180°–100°=80°

x=40°

Example9: Infig15.37,A,B andCarefourpointsonacirclewithcentreOsuchthat∠ABC=110º.Findthemeasureof∠OAC

Solution:

Reflex∠AOC=2∠ABC

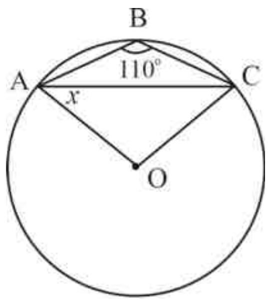


Fig 15.37

=2×110º

=220º

∠AOC=360º–220º

=140º

∠OAC+∠ACO+∠AOC=180º(AnglesofaΔlet∠OAC=*x*°)

$x^o + x^o + 140^o = 180^o$

$x = 20^o$

∠OAC=20º

Example10: Infig15.38,OiscentreofthecircleandADbisects∠BAC.Find∠BCD

Solution:

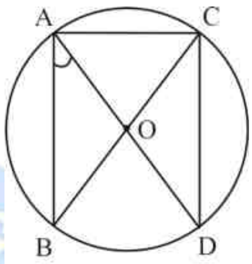


Fig 15.38

BCisdiameterofthecircle

∠BAC=90º

(Angle in a semi-circle) As AD bisects ∠BAC

∠BAD=$\frac{1}{2}$

(90º)=45º

But∠BCD=∠BAD(Angleinthesamesegment)

∠BCD=45º

Example11: TwocirclesintersecteachotheratPandQ.FromP,diameterPSandPRare drawn to two circles. Prove that R, Q and S are collinear. (Fig 15.39)

Solution:

JoinP&Q

PQRissemi-circle.

∠PQR=90º.. (i)

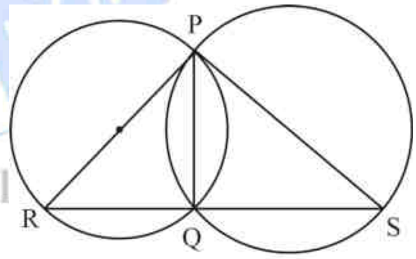


Fig 15.39

Similarly,∠PQS=90º................................. (ii)

Adding(i)&(ii)weget

∠PQR+∠PQS=180º

R,Q&Sarecollinear.

Example12: TheverticesofatrapeziumABCDlieonacirclewithcentreOsuchthat AD||BC and AD is the diameter, prove thatAB=DC(seefig15.40)

Solution:

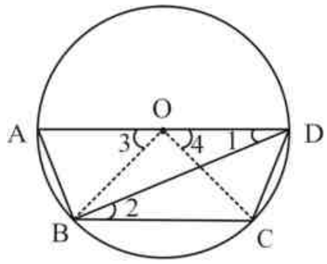


Fig 15.40

AD||BCandBDisatransversal

∠1=∠2(alternate∠s) ……….(i)

∠3=2∠1(anglesbyarcABatcentreOandotherpart of circumference) ……………………….(ii)

∠4=2∠2 ……………………….(iii)

From(i),(ii)and(iii)weget

∠3=∠4

AB=CD(chordsoppositetobeequalangles)

CHECKYOURPROGRESS15.3

I. Shortanswerquestions

(i) Namethelargestchordofthecircle

(ii) Writeareaofasectorofacircleofradius$r$,subtendingangleatthecentreofcircle

(iii) Find the numberofdiametersthatcanbedrawntoacircle

(iv) In figure15.41,O is centre ofthe circle OC ⊥ chord AB the radius ofthe circle is 5 cm.If OC = 3cm find the length of chord AB

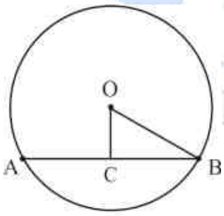


Fig 15.41

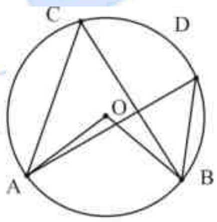


Fig 15.42

(v) Infig15.42,statetherelationbetween∠ACBand∠ADB.

(vi) Findmeasureof∠ACBwhere∠B=40^o(fig15.43)

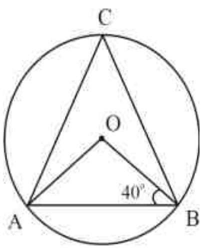


Fig 15.43

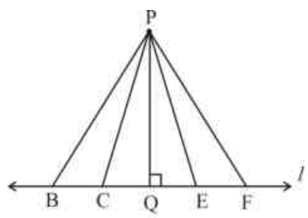


Fig 15.44

(vii) Infig15.44,thelinesegmentPB,PC,PQ,PE&PFaredrawnfrompointPtotheline. Write the line of segment whose distance from P is shortest.

(viii) Infig15.45,OiscentreofcirclePQ||AB.If∠POQ=120^o.Find∠POA

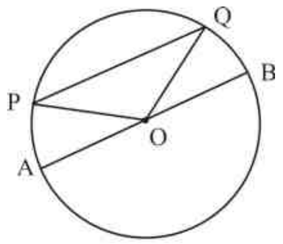


Fig 15.46

(ix) IfOiscentreofthecircle,find*x*inthefig(15.47)andfig(15.48)

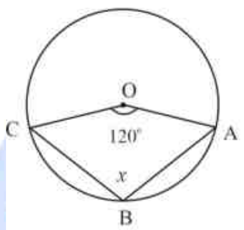


Fig 15.47

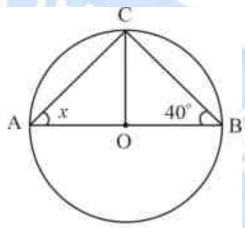


Fig 15.48

(x) Infig(15.49),AB||CDwhere∠Disadiameter.if ∠AED=70^o,find∠DAB.

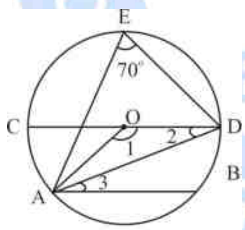


Fig 15.49

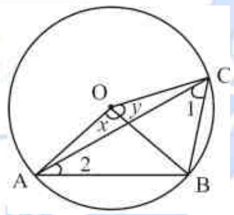


Fig 15.50

(xi) In quadrilateral ABCO in fig (15.50), OA = OB = OC.Show that (∠x + ∠y) = 2(∠1 + ∠2)

(xii) In fig (15.51), O is centre of the circle and AD bisects ∠BAC. Find ∠BCD.

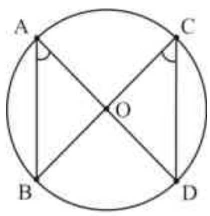


Fig 15.51

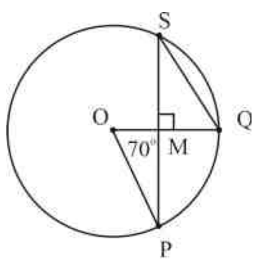


Fig 15.52

(xiii) In fig (15.52), O is centre of the circle. ∠POQ = $70^o$and PS⊥OQ. Find ∠MQS.

(xiv) ΔABCisinscribedinacirclewithcentreO,if∠AOB= 140^o and ∠BOC =100^o. Find ∠ABC.

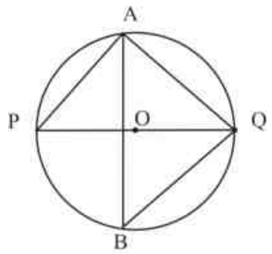


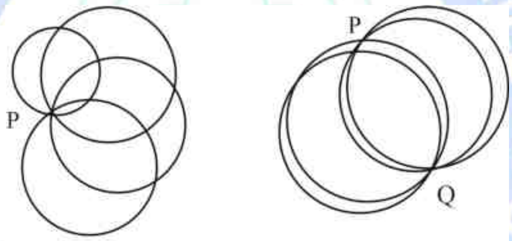
Fig 15.53

(xv) In fig (15.53) PQ is diameter of a circle with centre O. If $\angle AQP = 30^o$ find $\angle ABQ$.

15.5 CONCYCLICPOINTS

Definition: Points which lie on a circle are called Concyclic points.Let us now find certain conditions under which points are concyclic.

IfyoutakeapointP,youcandrawnotonlyonebutmanycirclespassingthroughit as in fig 15.54(i)



(i) (ii)

Fig 15.54

Now take two parts P and Q on a sheet of paper. You can draw as many circles as you wish passing through the points see fig 15.54(ii)

LetusnowtakethreepointsP,QandRwhichdonotlieon the same strength line. In this case you can draw one and one circle passing through three non-collinear points. See fig 15.55.

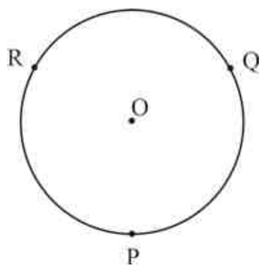
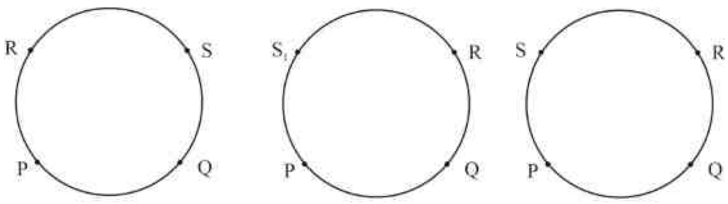


Fig 15.55

Infigure15.56as(i)and(ii) points are non-cyclic but concyclic in fig 15.56 (iii).

Note:It isnot possible todraw a circle through three collinearpointsor a pointlying on line.



(i) (ii) (iii)

Fig 15.56

Thus,weconclude–

1. Giveoneortwopointsthereareinfinitelymanycirclespassingthroughthem.
2. Threenon-collinearpointsarealwaysconcyclicandthereisonlyonecircle passing through all of them.
3. Threecollinearpointsarenotconcyclic(ornon-cyclic)
4. Fournoncollinearpointsmayormaynotbeconcyclic

15.5.1Cyclicquadrilateral

A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all its four vertices.

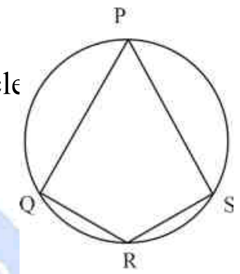


Fig 15.57

Fig(15.57)showsacyclicquadrilateralPQRS.

Theorem: Sumofoppositeanglesofacyclicquadrilateralis180^o

Given:acyclicquadrilateralABCD

Toprove:∠BAD+∠BCD

=∠ADC+∠ABC=180^o

Construction:DrawthediagonalACandDB

Proof: MinorarcABsubtends∠ACBand∠ADB

∠ACB =∠ADB(angles with same segment) and∠BAC=∠BDC

(angleswithsamesegment)

∠ACB+∠BAC=∠ADB+∠BDC=∠ADC

Adding∠ABConbothsides

∠ACB+∠BAC+∠ABC=∠ADC+∠ABC

But∠ACB+∠BAC+∠ABC=180^o(sumofanglesofatriangle)

∠ADC+∠ABC=180^o

∠BAD+∠BCD=∠ADC+∠ABC=180^o

Henceproved.

Converseofthetheoremisalsotrue.

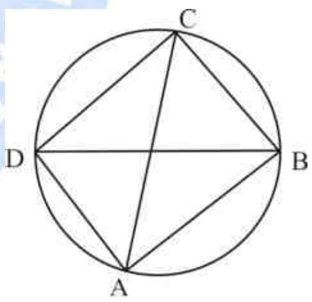


Fig 15.58

Ifapairofoppositeanglesofaquadrilateralissupplementarythenthe quadrilateral is cyclic.

Verification:

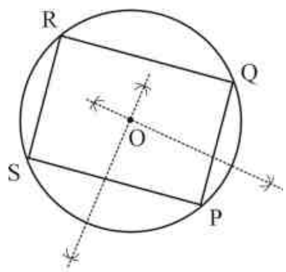


Fig 15.59

DrawaquadrilateralthroughP,Q,RandS,suchthat

$\angle P+\angle R=180^o$and$\angle Q+\angle S=180^o$

We draw right bisects of PQ and PS meeting at O. Take

OP as radius and draw a circle.

We see that the circle passes through P, Q, R and S So we conclude that PQRS is cyclic quadrilateral. We solve some examples using the above results.

Example 13: In fig 15.60,PQRS is a cyclic quadrilateral such that PQ isa diameter of the circle.If $\angle PSR = 120^o$. Find $\angle QPR$.

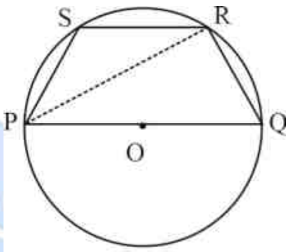


Fig 15.60

Solution:

PQRSisacyclicquadrilateral

$\angle PSR+\angle PQR=180^o$.

$\Rightarrow 120^o+\angle PQR=180^o$.

$=\angle PQR=60^o$.

$\angle QRP=90^o$(angleinasemi-circle)

Again$\angle RPQ+\angle PQR+\angle QRP=180^o$---(angleofa$\Delta$)

$\angle RPQ=180^o-60^o-90^o=30^o$

Example14: Apairofoppositesidesofacyclicquadrilateralareequal.Provethatitsdiagonals are also equal [see fig (15.61)]

Solution:

ABCD is a cyclic quadrilateral and AB = CD

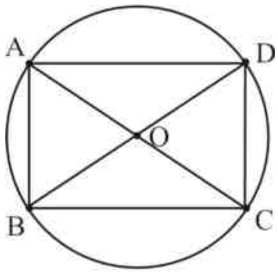


Fig 15.61

$\Rightarrow$ArcAB=ArcCD

AddingarcADtobothsides

Arc AB + arc AD = arc CD + arc AD arc BAD = arc CDA

ChordBD=ChordCA

=BD=CA

Example15: Ifdiagonalsofacyclicquadrilateralarediametersoftheci

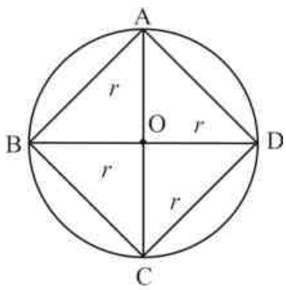


Fig 15.62

rcle,provethatitis a rectangle.

Solutions:

Letrbetheradiusofthecircle

=AB=$\sqrt{r^2+r^2}=\sqrt{2r^2}$

AD=DC=BC=AB=$\sqrt{2}$r

ACisadiameter

∴ ∠ABC=90°.

Similarly,∠BAD=∠ADC=∠DCB=90°.

In a quadrilateral ABCD, all sides are equal and each angle is 90°. ABCD is a square.

Example16:ABCDisacyclicquadrilateralif∠B=∠C=65°find∠Aand∠D.

Solutions:

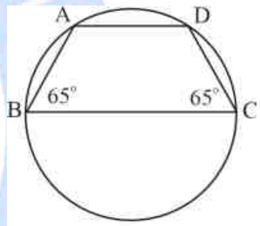


Fig 15.63

∠A+∠C=180°

∠A=180°-∠C

=180°-65°

=115°

Similarly,∠B+∠D=180°

∠D=180°-∠B=180°- 65°

=115°

Example17:Provethatacyclicparallelogramisarectangle

Solutions:Infigure(15.64)PQRSisacyclicparallelogram

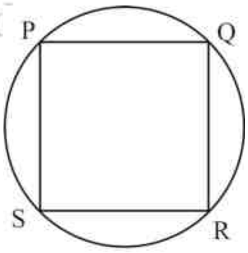


Fig 15.64

∴∠P+∠S=180°

Also∠P=∠R

But∠P+∠R=180°

∠P=∠R=90°

Similarly, we can prove that

∠Q=∠S =90° PQRS is a rectangle

Example18:Ifnonparallelsidesofatrapeziumareequal,provethattrapeziumiscyclic

Solutions: Through B draw BF || AB

It is given that AD = BC …(i)

BE || AD and AB || DE

ABEDaparallelogram

∴AD=BE …(ii)

From(i)&(ii)

BE=BC

∠1=∠2

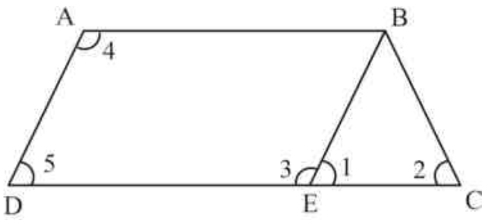


Fig 15.65

AgainAB||DCandADisatransversal

∠4+∠5=180° …………………(A)

But∠5=∠1=∠C …………………(B)

From(A)&(B)weget

∠4+∠2=180°

ABCDisacyclicquadrilateral

CHECKYOURPROGRESS15.4

(i) In fig (15.66), AB is a diameter and ABCD is a cyclic quadrilateral.If∠CDA=140° find ∠CAB.

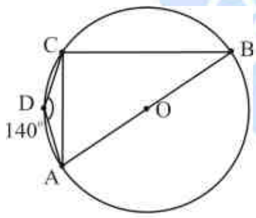


Fig 15.66

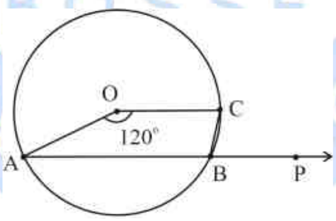


Fig 15.67

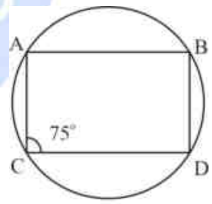


Fig 15.68

(ii) OiscentreofthecircleArcABCsubtendsanangleof120°atthecentreO.ABisextendedtoP.Find ∠PBC(fig15.67)

(iii) Provethatanyfourverticesofaregularpentagonareconcyclic.

(iv) Infig(15.68)ABCDiscyclicquadrilateralinwhichAB||DC.If∠D=75°,findtheremainingang lesofquadrilateral.

(v) Infig(15.69)ΔPQRsuchthatPQ=PRand∠PRQ=60°.Find∠QSRand∠ QTR.

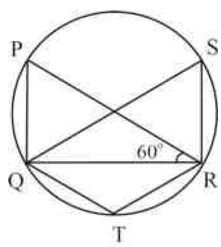


Fig 15.69

(vi) Infig(15.70),ADEisatriangleinwhichAE=DE.CirclepassingthroughA andD intersects AE and DE at B and C respectively. Prove that AD

|| BC.

(vii) CandDaretwopointsonthesemicircledescribedonadiameterABsuchthat ∠BAD=70°and∠DBC=30°.Find∠BCDand∠BDC.

(viii) D and E are points on equal sides AB and AC of an isosceles triangle ABC such thatAD = AE. Prove that the points B, C, E & D are con cyclic.

(ix) ABCDisacyclicquadrilateralwhosediagonalsintersectatE.If∠DB=70°,∠BAC =30°,find∠BCD.FurtherifAB=BC,find∠ECD.

(x) If a line is drawn parallel to the base of an isosceles triangle to intersect its equal sides shows that quadrilateral so formed is cyclic.

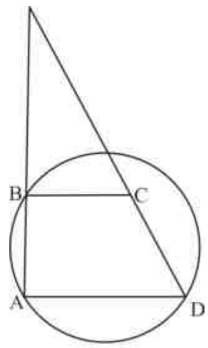


Fig 15.70

RECAPITULATIONPOINTS

- Theanglesubtendedbyanarc(orchord)atthecentreofacircleiscalledcentral angle.
- The angle subtended by an arc (or chord) on any point on the remaining part of thecircle is called inscribed angle.
- The angle subtended by arc at the centre of a circle is double the angle subtended by itatanypointontheremainingpartofthecirclei.e.,centralangleisdoublethe inscribed angle.
- Angleinasemicircleis90°(arightangle)
- Anglesinthesamesegmentofacircleareequal
- Sumofoppositeangleofacyclicquadrilateralistworightangles(180°)
- If sum of a pack of opposite angle of a quadrilateral is 180(2 right angle, then the quadrilateral is cyclic.

TERMINALEXERCISE

I. Statetrueorfalse

(i) Circlesaresaidtobeconcentriciftheyhavethesameradius.

(ii) Perpendicular from the centre to a chord divided the chord in the ratio of 1:2

(iii) Ifthediameterofacircleisincreasedby50%,theratioofcircumference

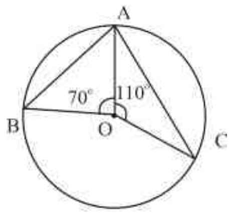


Fig 15.71

of the circle to the diameter with remain the same.

(iv) Infig(15.71),Oiscentreofthecircle∠AOB=90° and∠AOC= 110°, then ∠BAC = 80°.

(v) In fig (15.72), O is centre of the circle ∠PAQ = 40° then ∠OPQ = 50°,

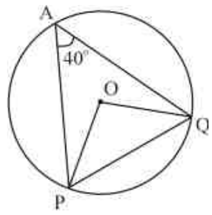


Fig 15.72

II. Choosethecorrectalternatives

(i) In fig (15.73) two congruent circles as will centre A and B angles subtraded by two are PTQ and are RVS is 70° each at centre A and B. if chord PQ = 1.6 cm, the length of chord RS is

(a) 1.7 cm

(b) 0.8 cm

(c) 3.2 cm

(d) 1.6 cm

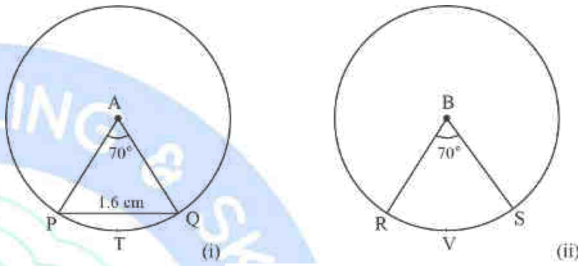


Fig 15.73

(ii) In fig (15.74) in the circle OP is perpendicular b chord Rs. If Rs=16 cm, OP = 6cm, the length of radius of circle is

(a) 4 cm

(b) 10 cm

(c) 11 cm

(d) 10.5 cm

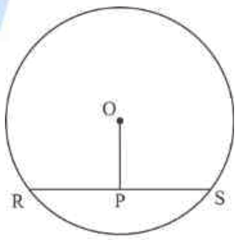


Fig 15.74

(iii) In the fig (15.75) COD is diameter of the circle. Are OHA subtends ∠DOA = 60°. If length of are D or A = 2.4 cm, the length of are ANC is

(a) 7.2 cm

(b) 3.6 cm

(c) 4.8 cm

(d) 5.2 cm

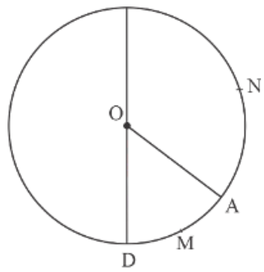


Fig 15.75

(iv) A regular pentagon is in deribad in a circle. The angle which each side of the pentagon subtend, the centre is

(a) 72°

(b) 108°

(c) 36°

(d) 54°

(v) In fig (15.76) are AB = arc BC. OB & OD and $\angle BOC = 20°$. The measurement of $\angle DOB$ is

(a) 70°

(b) 110°

(c) 80°

(d) 100°

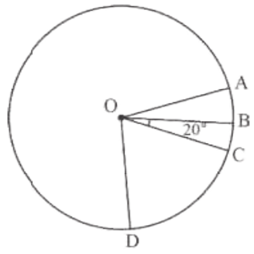


Fig 15.76

(vi) Two circles such centre O and O′ (see fig 15.77) are congruent. If PQ = 3 cm the length of RS is

(a) 2b cm

(b) $\frac{3}{2}b$ cm

(c) 3b cm

(d) $\frac{b}{2}$ cm

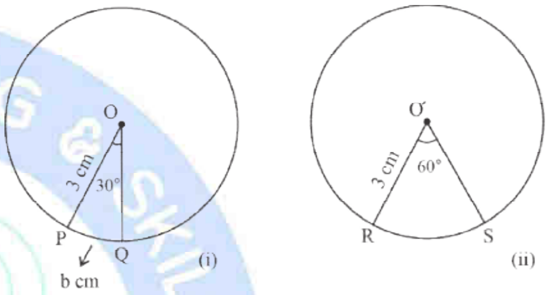


Fig 15.77

(vii) Infig(15.78),themeasureof$\angle ADB$is

(a)60°

(b)45°

(c) 30°

(d)120°

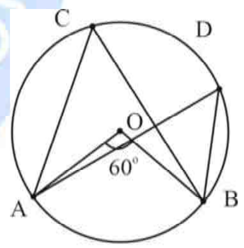


Fig 15.78

(viii) In fig (15.79), AB is a chord of the circle with centre O andOD ⊥ AB. The length of AB is

(a) 10cm

(b) 12cm

(c) 30cm

(d) 18cm

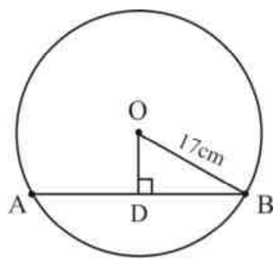


Fig 15.79

(ix) Infig(15.80),Oiscentreofcircleand$\angle ABO=60°$,$\angle ABC=$

(a)60°

(b)45°

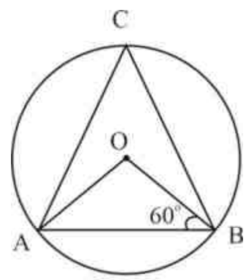


Fig 15.80

(c)30^o

(d)90^o

(x) Infig(15.81),ABCDisacyclicquadrilateral,thevalueof $(x + y)$ is

(a)80^o

(b) 90^o

(c)120^o

(d)180^o

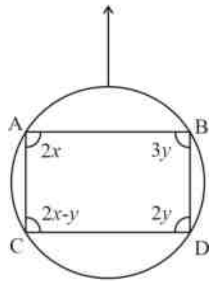


Fig 15.81

(xi) Infig(15.82),Oiscentreofcircleand$\angle$AOB=100^o,then

$\angle$ABC=

(a) 55^o

(b)110^o

(c)125^o

(d)130^o

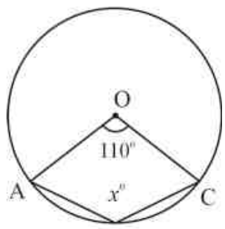


Fig 15.82

(xii) Infig(15.83),ABCDisacyclicquadrilateralandAB=AC.If

$\angle$ACB=70^o,$\angle$BDC=

(a)40^o

(b) 70^o

(c)140^o

(d)160^o

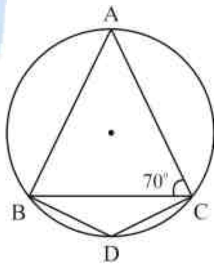


Fig 15.83

(xiii) Infig(15.84),ABCDisacyclicquadrilateralwith$\angle$ADC= 110^o,

Diagonal CA is joined and $\angle$CAB = 60^o, if AD || BC,then

$\angle$DAC=

(a)50^o

(b)80^o

(c)70^o

(d)80^o

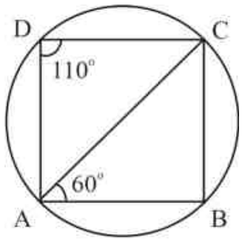


Fig 15.84

(xiv) Infig(15.85),ABCDisacyclicquadrilateraland$\angle$BAC=45^o

and$\angle$BCA = 40^o,then $\angle$ADC=

(a)80^o

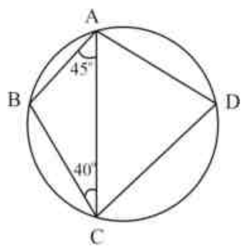


Fig 15.85

(b)85^{o}

(c) 90^{o}

(d)120^{o}

(xv) Infig(15.86),ABCDisacyclicquadrilateral,if∠DCB=120^{o} and ∠DBA = 50^{o}, ∠ADB=

(a)30^{o}

(b)40^{o}

(c)70^{o}

(d)60^{o}

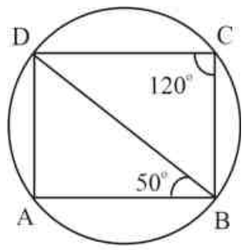


Fig 15.86

(xvi) Infig(15.87),ABisadiameterand∠ADC=110^{o},∠CABequals

(a)20^{o}

(b)40^{o}

(c)60^{o}

(d)70^{o}

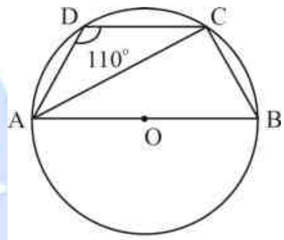


Fig 15.87

(xi) In fig (15.88), ABCD is a quadrilateral whose diagonals intersect atP, if ∠DBC =60^{o} and∠CAB = 30^{o}, then ∠BCD=

(a)60^{o}

(b) 90^{o}

(c)120^{o}

(d)130^{o}

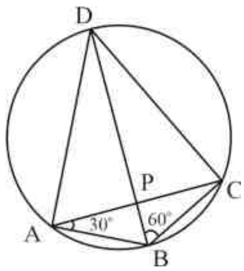


Fig 15.88

III. Statetrueorfalseforthefollowings

(i) Theperpendicularfromcentreuponachordbisectsachord.

(ii) The line joining mid-point of a chord to the centre of the circle makes an angle of90^{o} with the chord.

(iii) Thesumofadjacentanglesofcyclicquadrilateralis180^{o}.

(iv) Chordsequidistantfromthecentreofacircleareequal.

Fig 15.79

Fig 15.79

(v) Angleissemicircleisgreaterthan90^{o}.

IV. Answer the following:

(i) Two parallel chords AB and CD are of length 6 cm and 8 cm. if the distance

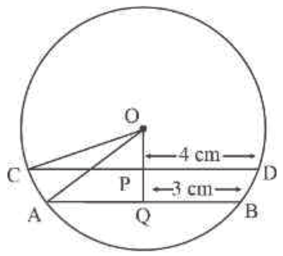


Fig 15.89

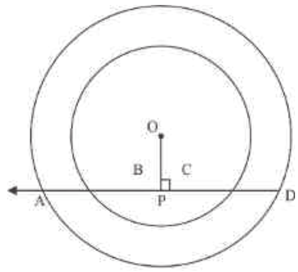


Fig 15.90

between the chords is 1 cm, solve that radius the circle is 5cm. (Fig 15.89)

(ii) If a line intersects two concentric circle, with common centre at A, B, C & D. prove that AB = CD. (Fig 15.90)

(iii) Prove that the line joining the centres of two intersecting circles subtends equal angles the two points of intersection of circles.

(iv) AB is the chord of a circle with centre O. AB is produced to C such that BC = OB. CO is joined to meet the circle at D. if ∠ACD = y and ∠AOD = x, prove that $x = 3y$

(v) If the vertices of a quadrilateral PQRS lie on a circle such that PQ = RS. Show that PR = QS

(vi) If the diameter of a circle bisects each of the two chords prove that chords are parallel.

(vii) Show that any of the two chords of a circle, the larger one is nearer to the centre than the smaller chord.

(viii) In fig (15.91), AB and CD are two parallel chords of a circle which are on opposite sides to the centre. She length of AB and CD are 6 cm and 8 cm. respectively. If the distance between AB and CD is 7 cm. if the that radius of

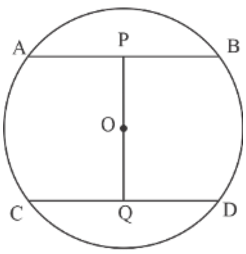


Fig 15.91

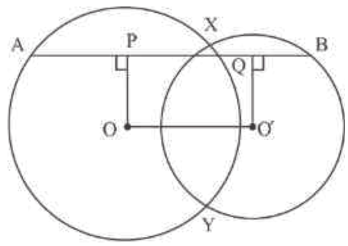


Fig 15.92

circle is 5 cm.

(ix) In fig (15.92) O and O′ are the centre of two intersecting circles at the points X and Y. AB is a line segment through X as shown in fig.(15.92) OP⊥AX and OQ⊥XB. Show that AB = 2OO′

CASESTUDYI

Five students of class X are standing at five points on a circledrawn on playground. The points are A, B, C, D and P. The points are joined by threads in such away that segmentsAB, BC, CD,DA form a quadrilateral ABCD. The diagonals AC and BD are joined. The diagonal meet at point P in such a way that ∠CAB =40°and∠DBC=50°.Thefollowingquestionsareaskedtotesttheir knowledgeregarding angles in a circle and cyclic quadrilateral.The reply given by eachof them is given by the point at which they are standing i.e., – A, B, C, D. The student at positionPisahigherclasswhowillchecktheanswer.Correctanswercarries5marksand for wrong 1 mark in deducted.

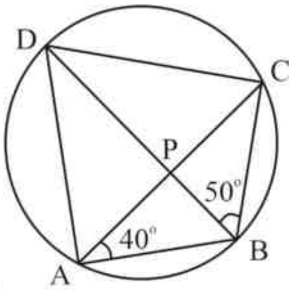


Fig 15.93

(i) Themeasureof∠BDCis

(A) 50°

(B) 40°

(C) 30°

(D) 60°

(ii) Themeasureof∠DACis

(A) 60°

(B) 40°

(C) 50°

(D) 30°

(iii) Themeasureof∠BCDis

(A) 100°

(B) 90°

(C) 80°

(D) 70°

(iv) NametheChordBDas

(A) Minor

(B) Diameter

(C) Major

(D) Can't say

(v) $BD^2 - AD^2 =$

(A) AC^2

(B) AB^2

(C) CD^2

(D) BC^2

The best scorer is …………

CASESTUDYII

Out of 4 students, one student is to be selected for mathsolympiad from class X. The teacher asked them the following questions. The names of students are Neha, Gurpreet, Sofia, Lawrence.Thebestscoreramong fouristorepresenttheschool Correct answercarries 3 marksand forwrong answer onemarkistobededucted.Theanswerby 4students aregiven by first letter of their namesi.e.N,G,S and L.the questions askedand answer given by themis as follows:

Firstletterofnameismentionedagainstanswer.

(i) Infigure(15.94),Oiscentreofthecircle∠BOA=120°and∠COA=80°, then∠BAC=

(N) 160°

(G) 120°

(S) 100°

(L) 80°

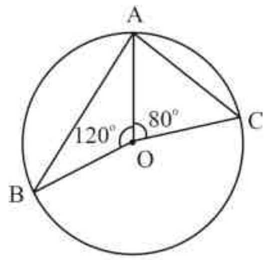


Fig 15.94

(ii) Infigure(15.95),Oiscentreofthecircle,if∠PQR=115°, then ∠POR =

(N) 230°

(G) 245°

(S) 65°

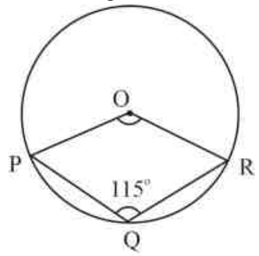


Fig 15.95

(L) 130°

(iii) Infigure(15.96),ABisachordofthecirclewithcentreO–. If $\angle ACB = 40^o$, then $\angle OAB$ =

(N) 55°

(G) 60°

(S) 80°

(L) 50°

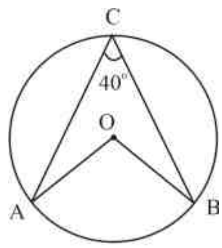


Fig 15.96

(iv) Infigure(15.97),ABCDisacyclicquadrilateralwhich

$\angle BAD=100^o$,$\angle AOC=140^o$,$\angle ABC=x^o$and$\angle BCD=y^o$.thevalueofx +yis

(N) 190°

(G) 80°

(S) 100°

(L) 110°

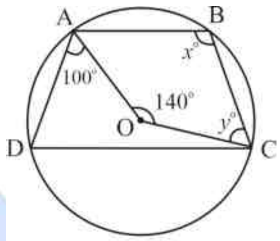


Fig 15.97

(v) In figure (15.98), O is centre of the circle if $\angle ODB$ = 60°, then $\angle BCD$ =

(N) 90°

(G) 150°

(S) 120°

(L) 80°

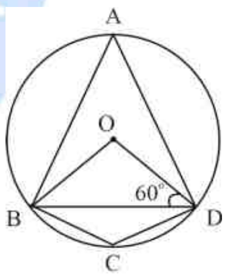


Fig 15.98

The best scorer is ………….

ANSWERSTO‘CHECKYOURPROGRESS’

CHECKYOURPROGRESS15.1

I. (i) False

(ii) True

(iii) False

(iv) False

(v) False

(vi) True

(vii) False

(viii) True

II. (i) Semi circle

(ii) Semi circle

(iii)Diameter

(iv)Interior

(v) Radii

(vi)Diameter

(vii) Radius

(viii) Interior

CHECKYOURPROGRESS15.2

(i) Minor

(ii) Perpendicular

(iii) Chord

(iv) π

(v) Two

(vi) (a) 66 cm (b) 22 cm

(vii) 12 cm

(viii) Draw Chord AP and PB. Draw right bisectors of AP and PB. The right bisector meet at a point. This point is centre of the circle.

CHECKYOURPROGRESS15.3

(i) Diameter

(ii) $\frac{\theta}{360°}, \pi r^2$

(iii) Infinite

(iv) 8 cm

(v) ∠ACB = ∠ADB

(vi) 50º

(vii) PQ

(viii) 30º

(ix) 120º, 50º

(x) 20º

(xi) Do it yourself

(xii) 45º

(xiii) 20º

(xiv) 60º

(xv) 60º

CHECKYOURPROGRESS15.4

(i) 50º

(ii) 60º

(iii) Do it yourself

(iv) ∠A = 105º, ∠B = 105º, ∠C = 75º

(v) ∠QSR = 60º, ∠QTR = 120º

(vi) Do it yourself

(vii) ∠BCD = 110º, ∠BDC = 40º

(viii) Do it yourself

(ix) ∠BCD = 80º, ∠ECD = 50º

CASE STUDY I

(i) 40º

(ii) 50º

(iii) 90º

(iv) Diameter

(v) AB^2

B is the best scorer.

CASE STUDY II

(i) (L)80º

(ii) (L)130º

(iii) (L)50º

(iv) (N)190º

(v) (G)150º

Lalita is the best scorer.

16 SECANTS AND TANGENTS

INTRODUCTION

A wheel is being moved by a boy on the road. Look at the touching point of wheel to the road. The wheel is touching the road at one point at the time of looking at it.

Similarly, if we you roll a rounded coin on a book or note book, you will find that at any instant of time, the coin touches the surface at one point.

The above example shows that a line and a circle may touch at one point.

Let us find other situations. In fig (1) we find a line and a circle in different ways.

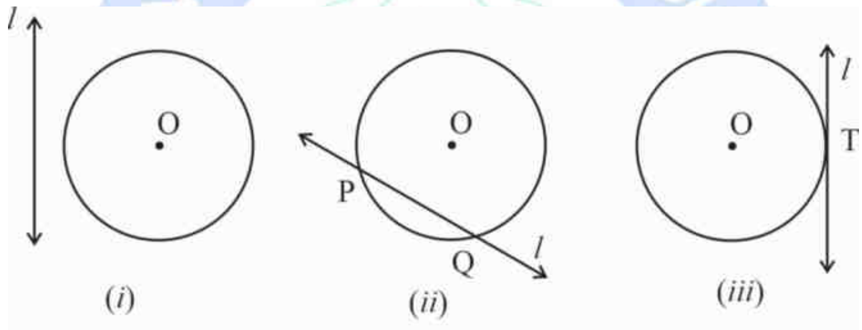


Fig 16.1

There are three situations in fig 16.1 (i), line and circle with centre have no point in common as the line *l* does not intersect the circle.

In fig. 16.1 (ii), line l intersects the circle at two points i.e., P and Q. Such a line which intersects the circle at two points in called a Secant line.

In fig. 16.1 (iii), one point (T) is common on line and circle with centre O. We can say that line l touches the circle with centre O at point T. Such a line which touches the circle at one point is called tangent. The point T is called point of tangency.

16.1 LEARNING OBJECTIVES

After completing the lesson, you will be able to:

- Define a secant and a tangent to the circle.
- Differentiate between a secant and tangent
- The angle between the tangent and radius through point of contact is 90.

- Prove that the tangents drawn from an external point to a circle are of equal length.
- Verify the results related to tangent and secant to circle experimentally.

EXPECTED BACKGROUND

(a) Measurement of angles and line segments

(b) Drawing circle of given radii and indicating centre of the circle.

(c) Drawing line perpendicular and parallel to given line

(d) Knowledge of previous results about line and angles, congruence and circles.

(e) Knowledge of Pythagorus theorem.

16.2 TANGENT

A line which touches a circle exactly at one point is called a tangent lien and point at which is touches the circle is called point of contact. In fig. 16.1 (iii) line l is tangent of the circle and the point at which it touches the circle is called point of contact. In fig. 16.1(iii) the point of contact is T.

Secant: A line which intersects the circle in two distinct points is called a secant line (generally called a secant).

In fig. 16.1 (ii) line l is a secant line (a secant) of the circle and P and Q are called the points of intersection of the line l with circle centre O.

16.2.1 Tangent as a limiting case

In fig. 16.2, secant PA of the circle with centre O. Point A is brought close to F and onward by drawing secants PB, PC PD, PE & PE. In all the cases we find that major arc PAD is becoming shorter and shorter. It attains the position of QPR.

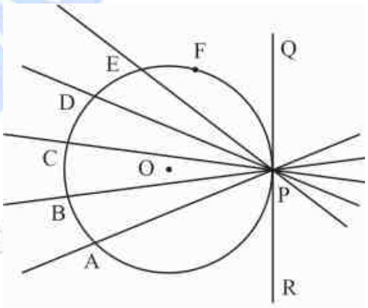


Fig 16.2

At this stage the secant becomes a tangent QPR, then we can say that:

A tangent is the limiting position of a secant when the two point of intersection coincide

Tangent and radius through the point of contact

Draw a circle with centre O. draw a line PQ touching the circle at R. Thus, PRQ is tangent to the circle. (fig. 16.3)

Draw segments OA, OB, OR, OC and OD and measurethem.

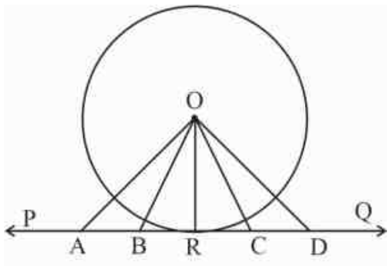


Fig 16.3

Of all the segments OA, OB, OR, OC and OD you will find OR is shortest of all.

Now measure ∠ORP or ∠ORQ. You will find that ∠ORP = ∠ORQ = 90°

Then OR ⊥ PQ.

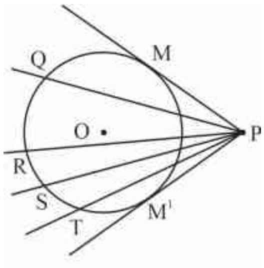


Fig 16.4

Thus, we can say that:

A radius through the point of contact of tangent to a circle, is perpendicular to the tangent at that point.

16.2.2 Tangents from a point outside the circle

Draw a circle with centre O. take any point P in the exterior of the circle. Draw some lines through P. These are shown as PM, PQ, PR, PS, PT and PM´ in fig. (16.4). We find that only two PM and PM´ are tangents to the circle or two lines touch the circle at M and M′.

Draw another circle with centre and repeat the above steps. You will get that only two lines out of all secants are tangent to the circle. Measures the length of PM and PM′ you will see that PM = PM´.

We conclude that:

From an external point, two tangents can be drawn to a circle.

We know that from a point P on the circle only one tangent can be drawn.

Think of the situation if point P lies in the interior of the circle. At line through P will always intersect the circle at two points. No tangents can be drawn from an interior point to the circle. [fig. (16.5)]

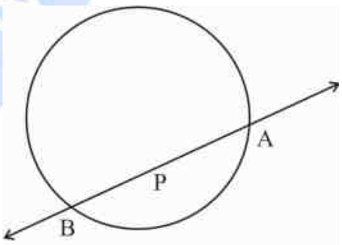


Fig 16.5

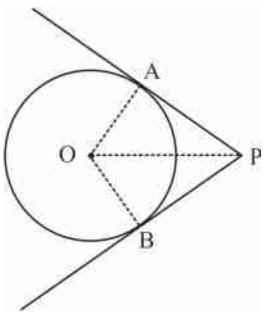
Theorem: Prove that the lengths of two tangents drawn from an external point of a circle an equal.

Given: A circle with centre O. PA and PB are two tangents from a point P outside a circle.

To prove: PA = PB

Construction: Join OP, OA and OB (see fig. (16.6))

Proof: In triangles OPA and OPB ......



∴ ∠OAP =∠OBP (90° each)

OA = OB (radii)

OP = OP (common)

∴ ΔOPA and ΔOPB are congruent

⇒ ΔOPA ≅ΔOPB (RHS)

Fig 16.6

∴ PA = PB

The length of two tangents from an external point are qual.

Also ∠OPA =∠OPB (As ΔOPA ≅ΔOPB)

The tangents drawn from an external point to a circle are equally inclined to the line joining the point to the centre of the circle.

Below are a few examples to illustrate.

Example 1: In fig. 16.7, the length of tangent drawn from T to the circle with centre O is 12 cm. The distance of T from O is 12.5 cm. Calculate the length of radius OP.

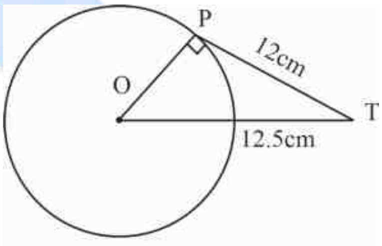


Fig 16.7

Solution: Let radius OP $=x$

∴ $(12.5)^2 = (12)^2 + x^2$ (ΔOPT is re angled)

⇒ $156.25 = 144 + x^2$

⇒ $156.25 – 144 = x^2$

⇒ $12.25 = x^2$

⇒ $(3.5)^2 = x^2$

∴ $x = 3.5$

Radius OP = 3.5 cm

Example 2: In fig. 16.8, TP and TQ are tangents from an external points T to a circle with centre O. If ∠POQ = 120° find ∠PTQ.

Solution: TP and TQ are tangents to the circle

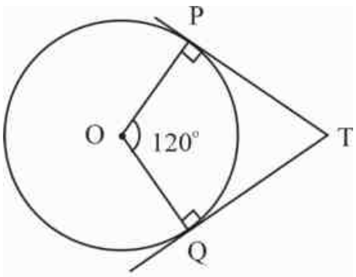


Fig 16.8

∴ ∠TPO + ∠TQO = 180°

Let ∠PTQ $=x^o$

∴ ∠OPT+∠OQT + ∠POQ + ∠PTQ = 360

∴ 90° + 90° + 120° + x^o = 360°

x = 360° – 300 = 60°

∴ ∠PTQ = 60°

Example 3: Prove that the tangents drawn from the extremities of a diameter of a circle areparallel.

Solution: In fig. 16.9, PQ is diameter of circle with centre O. SPT and XQY are tangents to the circle at point P and Q.

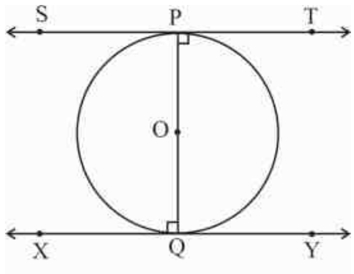


Fig 16.9

∴ $\angle QPT = 90°$ (angle between radius and tangent)

and $\angle PQX = 90°$ (___do ___)

∴ $\angle QPT = \angle PQX$

But these are alternate angles

∴ ST || XY

Example 4: The radius of two concentric circles are 5 cm and 3 cm. Calculate the length of chord of bigger circle which touches the smaller circle.

Solution: AB is chord of big circle C_2. Chord AB touches smaller circle C_1 at P

∴ $OP \perp AB$

or $\angle OPB = 90°$

In rt ΔOPB, ...

$OB^2 = OP^2 + PB^2$

$PB^2 = OB^2 - OP^2$

$PB^2 = 5^2 - 3^2 = 16 = 4^2$

∴ PB = 4cm

Again OP.. AB

⇒ AP = PB = 4cm

∴ AB = 8cm

∴ Chord AB of circle C_1 = 8cm

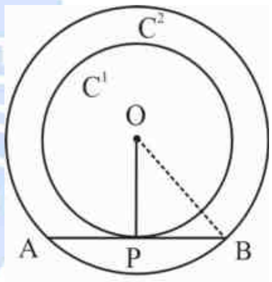


Fig 16.10

Example 5: A quadrilateral PQRS circumscribes a circle (see fig. 16.11). Prove that

PS + QR = PQ + RS.

Solution: We know that the length of tangents drawn from an external point of circle are equal:

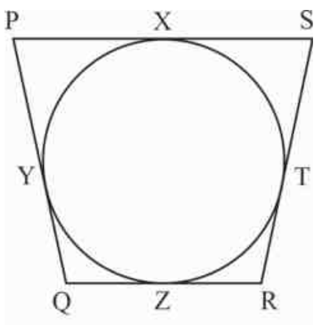


Fig 16.11

∴ PX = PY …(i)

XS = ST …(ii)

QZ = QY …(iii)

RZ = RT …(iv)

Adding (i), (ii), (iii) & (iv) we get

(PX + XS) + (QZ + ZR) = (PY + YQ) + (ST + RT)

or PS + QR = PQ + RS

This was to be proved.

Example 6: Prove that a parallelogram circumscribing a circle is a rhombus.

Solution 1: We have to prove that PQRS is a rhombus.

Solution 2: In example we proved that sum of opposite sides of a quad circumscribing a circle are equal:

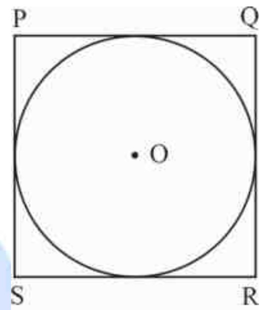


Fig 16.12

∴ PQ + SR = PS + QR …(i)

But PQRS is a || gm.

∴ PQ = SR and PS = QR

Applying the above result in (i) we get

PQ + PQ = PS + PS

⇒ 2PQ = 2PS

⇒ PQ = PS

∴ PQRS is a ||gm whose adjacent sides are equal.

∴ PQRS is a rhombus

Example 7: In the given figure 16.13, the in circle of triangle ABC touches the sides BC,CA and AB AT P, Q and R respectively. Prove that (AR + BP + CQ) = (AQ + BR + CP)

$$= \frac{1}{2} \text{ (Perimeter of } \Delta ABC)$$

Solution: We know that lengths of tangents from an exterior point to a circle are equal.

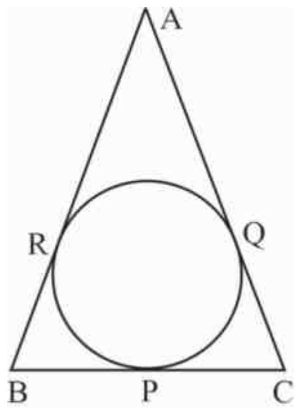


Fig 16.13

AR = AQ …(i) (tangents from A)

BP = BR …(ii) (tangents from B)

CQ = CP …(iii) (____do____)

Perimeter of ΔABC = AB + BC + CA

= (AR + BR) + (BP + CP) + (CQ + AQ)

= (AR + BP + CQ) + (AQ + BR + CP)

= $k + k = 2k$

∴ $k = \frac{1}{2}$ (Perimeter of ΔABC)

Q = (AR + BP + CQ) + (BQ + BR + CP)

Example 8: A circle is touching the sides BC of ABC at P and touching AB and AC produced at Q and R respectively. Prove that

$AQ = \frac{1}{2}$ (Perimeter of ΔABC) (see fig. 16.14)

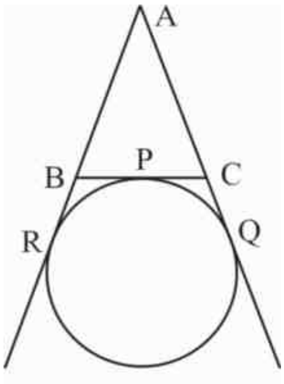


Fig 16.14

Solution: We know that length of tangents drawn from an external point to a circle are equal:

∴ AQ = AR …(i) (tangents from A)

BP = BQ …(ii) (____do____)

CP = CR …(iii) (____do____)

(Perimeter of ΔABC)

= AB + BC + CA

= AB + BP + PC + CA

= AB + BQ + CR + CA (using (ii) and (iii))

= AQ + AR

= 2AQ …….. using (i)

∴ $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

Example 9: Two tangents PA and PB are drawn to the circle with centre O from an external point P. Prove that ∠APB = 2∠OAB.

Solution: Let ∠APB $=x^o$

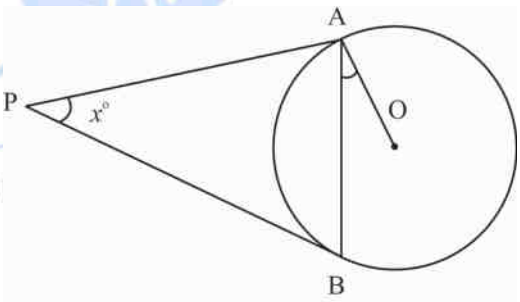


Fig 16.15

PA = PB (Tangents from an external point)

⇒ ∠PBA =∠PAB

∠APB +∠PAB + ∠PBA = 180º (sum of angles ΔAPB)

⇒ x^o + 2∠PAB = 180°

$\angle PAB = \frac{1}{2}(180 - x)$

$= 90 - \frac{x^o}{2}$

But PA is a tangent and OA is radius of given circle

∴ ∠OAB + ∠PAB = 90

∠OAB = 90 – ∠PAB

$= 90° - \left(90 - \frac{x}{2}\right)$ (proved earlier)

$= \frac{x°}{2} = \frac{1}{2}\angle APB$

∴ $\angle APB = 2\angle OAB$

Example 10: Prove that the extremities of any chord of a circle make equal angles with the chord.

Solution: To prove ∠MAT=∠MBT

In ΔMAT and MBT we have

MT= MT (common)

PA = PB (tangents from external point P)

∠MTA =∠MTB (Tangents are equally inclined to linejoining the centre to external point)

∴ ΔMAT≅ΔMBT (SAS)

∴ ∠MAT=∠MBT

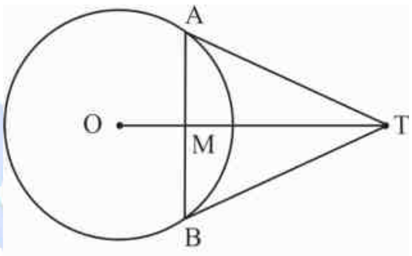


Fig 16.16

Example 11: In the given figure 16.17, two circles touch each other at the point C. Prove that the common tangent to the circles at C bisects the common tangent PQ.

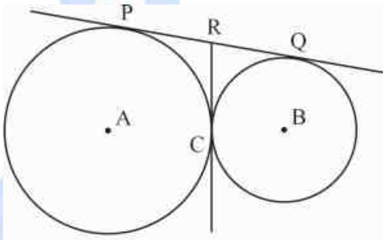


Fig 16.17

Solution: PR = RC ...... (i) (Tangent from external point Rto circle with centre A)

RQ = RC ……(ii) (____do____)

From (i) (ii)

∴ PR = RQ

∴ R is midpoint of PQ

CHECK YOUR PROGRESS 16.1

1. Fill in the blanks to make the statement true:
   (i) Tangent is a line which touches the circle at __________ point/points.
   (ii) The secant line to a circle intesects circle at_________ _.
   (iii) A _________ touches the circle at one point.
   (iv) The common point of the circle and tangent is called ___________ point.
   (v) The angle between radius and tangent at the point of contact is ________ _.

(vi) A tangent is ________ position of two points intersecting the circle.

(vii) The sum of opposite angles of a cyclic quadrilateral is _______ _.

(viii) A rectangle is a _________ quadrilateral.

(ix) If the all the four vertices of a quadrilateral lie on a circle the quadrilateral is _______ _.

(x) The maximum number of tangents drawn from are external point to the circle are ________one/two/many.

2. In the Fig. 16.18, O is the centre of AB is a chord and AT is tangent at A. If ∠AOB = 100º find ∠BAT.

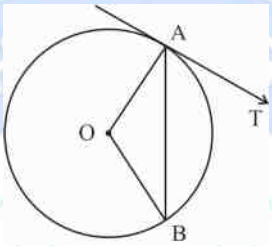
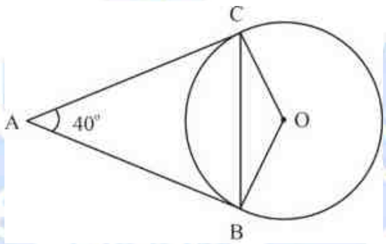


Fig 16.18

3. In fig. 16.19, AB and AC are tangents to the circle with centre O such that ∠BAC =



40º. Find ∠BOC.

4. In fig. 16.20, BC is common tangents to the given circles which touch externally at P. tangentat P meets BC at A. If BA = 3.8 cm. Find the length of BC.

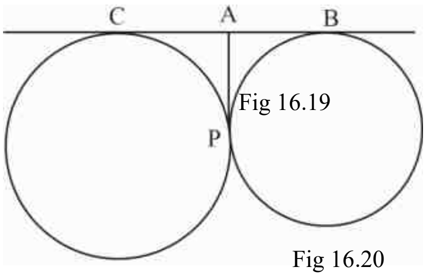


Fig 16.19

Fig 16.20

5. In fig. 16.21, a circle touches the side DF of ΔEDF at M and touches ED and EF produced at Kand N respectively. If Ek = 9 cm, fidn the perimeter of ΔDEF.

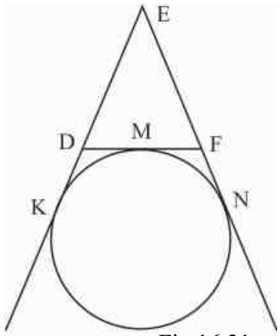


Fig 16.21

6. In a right angled ΔABC right angled at B. BC = 12 cm and AB = 5 cm, find the radius of the circle inscribed in the triangle.

7. In fig. 16.22, a circles inscribed in triangle ABC touches its sides AB, BC and AC at D, E and F respectively. If AB = 12 cm, BC = 8 cm and AC = 10 cm find the length of AD, BE and CF.

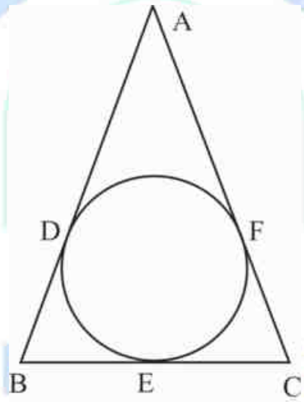


Fig 16.22

8. In fig. 16.23, from an external point P, two tangents PT and PS are drawn to the circle withcentre O and radius r. If OP = 2r show that ∠OTS =∠OST = 30°.

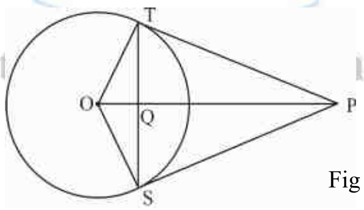


Fig 16.23

9. In fig. 16.24, XP and XQ are tangents from X to the circle with centre O. R is a point on the circle. Prove that XA + AR = XB + BR.

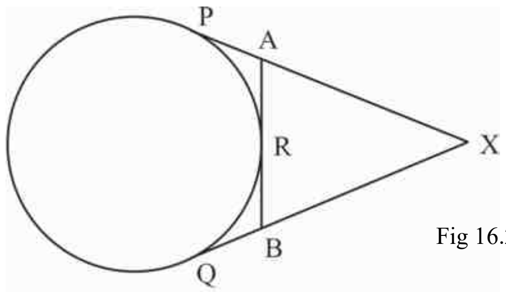


Fig 16.24

10. Prove that the angle between two tangents to a circle drawn from an external point, is supplementary to the angle subtended by the line segments joining the points of contact to the centre.
11. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angle at the centre of a circle.
12. Prove that the tangent drawn at the midpoint of an arc of a circle is parallel to the chord join the end point of the arc.
13. ABC is a isosceles triangle in which AB = AC, circumscribed about a circle as shown in fig 16.25. Prove that the base is bisected at the point of contact.

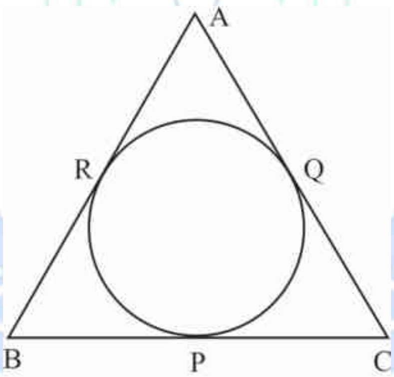


Fig 16.25

ActivityI

Intersecting chords inside and outside a circle.

Draw a circle with centre O and radius *r*. Draw two chords PQ and XY intersecting at point A inside a circle.

Measure the length of line segments PA, AQ, XA andAY.

Calculate the product of PA × AQ and XA × AY

You find that the product of (PA) (AQ) and product of (XA) (AY) issame.

∴ (PA) (AQ) = (XA) (AY).................. (i)

You can try this activity again on some other circles.

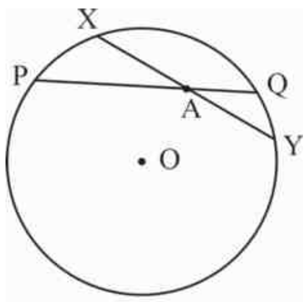


Fig 16.26

Activity II

Try the activity when the chords are intersecting outsidethe circle.

The chords AB and CD are intersecting at point M outside the circle.

Measure the length of line segments MD, MC, MB andMA.

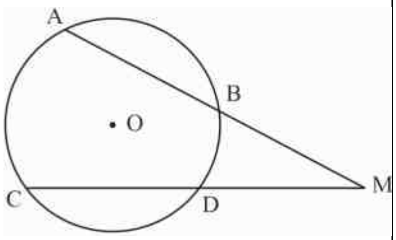


Fig 16.27

Now calculate MB × MA and MD × MC.

You will find MB × MA = MD × MC .... (ii)

Repeat the above activity on two or three other circles. You will find that result (i) and result (ii) are always true.

Thus, we can say that:

If two chords of a circle intersect at a point (inside and outside the circle) the product of distances of end points of chords from the point of intersection are equal.

Let the two chords be CD and EF intersecting at point P outside/inside a circle then PC × PD = PE × PF

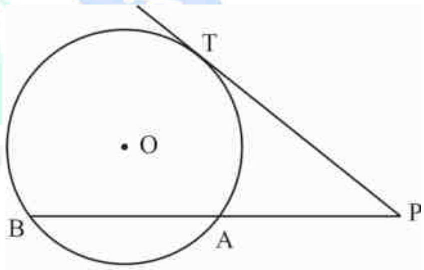


Fig 16.28

Interaction of a secant and tangent of a circle

Draw a circle of any radius (r) with centre O. take a point P outside the circles. Draw a tangent PT and a secantPAB from P to the circle. measure the length of segments PA, PB and PT.

Calculate PA × PB and PT^2= PT × PT

You will find that

PA × PB = PT^2

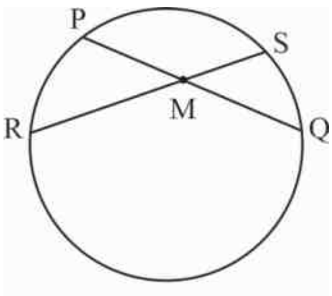
Repeat the activity on two or three more circle. Every time you will get the same results.

Thus, we can say that:

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent to the circle at T, then PA × PB = PT^2.

Let us take up some examples to illustrate the point.

Example 13: In fig. 16.29, two chords PQ and RS of a circle intersect at M inside a circle. If PM = 2cm, MQ = 3 cm, RM = 4 cm find the length of RS.



Solution: RS = RM + MS

we know the length of RM , we find MS

Show $PM \times MQ = RM \times MS$

$2 \times 3 = 4 \times MS$

∴ $MS = \dfrac{2 \times 3}{4} = 1.5$ cm

Fig 16.29

∴ Length RS = RM + MS

= 4 + 1.5 = 5.5 cm

Example 14: In fig. 16.30, A circle with centre O and radius 3.5 cm. Point P lies outside the circle at a distance of 12.5 cm from the centre. PT is tangent to the circle. Find PT.

Solution: OP = 12.5 cm

Radius OB = OC = 3.5 cm

PA = 12.5 – 3.5 = 9 cm

∴ PB = 12.5 + 3.5 = 16 cm

We know that PA. PB = PT^2

∴ $PT^2 = 9 \times 16 = 144 \text{ cm}^2$

∴ $PT = \sqrt{144} = 12$ cm

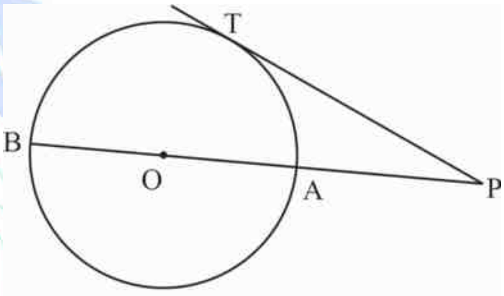


Fig 16.30

Example 15: In fig. 16.31, AR is a tangent from point A outside a circle. ABC is a secant of the circle passing through the centre O of the circle. If OA =15 cm and AR=12 cm, find the radius of the circle using $AB \times AC = AR^2$.

Solution: Let x be radius of the circle it is given that OA =15 cm

.. AB = AO – OB = $(15 - x)$cm

and AC = OA + OC = $(15 + x)$

AR = 12 cm

We know that $AB \times AC = AR^2$

$(15 - x)\,(15 - x) = 12^2$

$225 - x^2 = 144$

⇒ $x^2 = 81$ or $x = 9$ cm

∴ Radius of the circle = 9 cm

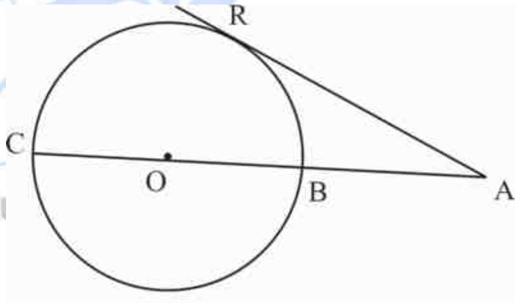


Fig 16.31

Example 16: In fig 16.32, two chords ED and NM intersect outside the circle at point C. If CD = 6 cm, CE = 15 cm, NM = 4.5 cm find MC.

Solution: We are given that CD = 6 cm, CE =15 cm, NM = 4.5 cm

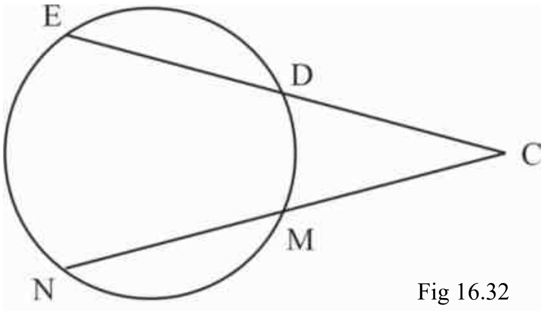


Fig 16.32

Let MC = x

We know that CD × CE = CM × CN

Or $6 \times 15 = x(x + 4.5)$

$\Rightarrow$ $x^2 + 4.5x = 90$

$\Rightarrow$ $2x^2 + 9x = 180$

$(2x - 15)(x + 12) = 0$

$\Rightarrow$ $x = -12$ (not possible) and $x = 7.5 \Rightarrow$ MC = 7.5 cm

Example 17: Two secants PA and PC intersecant P outside the circle at P. If PB = 4 cm BA = 8 cm and DP = 3 cm find chord CD.

Solution: It is given that

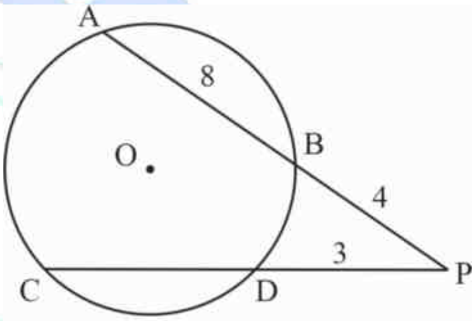


Fig 16.33

PB = 4 cm, AB = 8 cm, DP =3 cm.

Let chord CD = x cm

PB = 4 cm, PA = 8 + 4 = 12 cm,

PC = $(3 + x)$ cm

We know that PB × PA = PD × PC

$4 \times 12 = 3 (3 + x)$

$48 = 9 + 3x$

$\Rightarrow$ $x = 13$

Chord CD = 13 cm

CHECK YOUR PROGRESS 16.2

1. In fig. 16.34 if PT = 4 cm, TQ = 6 cm RT = 8 cm find TS.

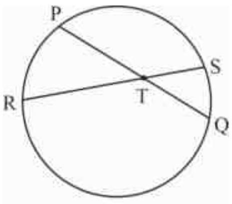
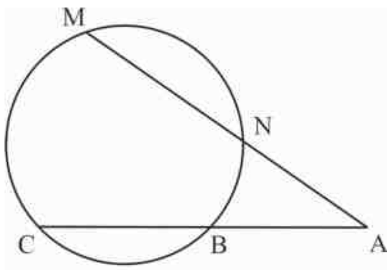


Fig 16.34

2. In fig. 16.34 PT = 6 cm, TQ = $(x + 3)$ cm RT = 9 cm, TS = $(x + 1)$ cm find x.
3. In fig. 16.35 if AN = 5 cm, MA = 8 cm, AB = 4 cm, find BC.



4. In fig. 16.36, QE = 4 cm, EF = 5 cm, find QD.
5. In fig. 16.36, if QD = 8 cm and OQ = 10 cm find the radius of the circle.
6. In fig. 16.36, if diameter FE = 12 cm and QD = $3\sqrt{5}$ cm find EQ.

Fig 16.35

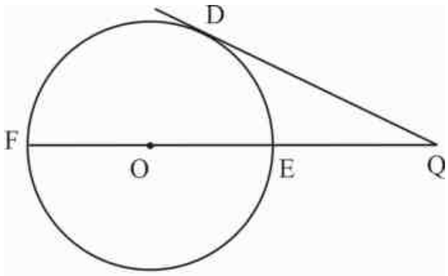


Fig 16.36

16.3 ANGLES MADE BY A TANGENT AND A CHORD.

Draw a circle with centre P. Let RQ be tangent to the circle at A. Draw a chord AB of the circle through A as shown in the figure (16.37). Take a point C on the major arc AB. Let D be a point on minor arc AB. Let D be a point on minor arc AB. Join AC, CB, AD and DB to form ∠ACB and ∠ADB. ∠ACB is in major arc ACB and∠ADB is in minor arc ADB.

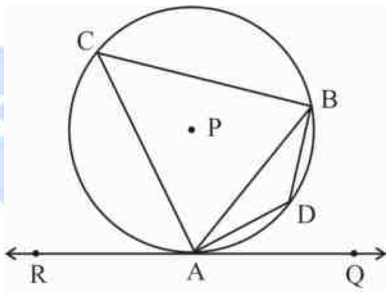


Fig 16.37

∠ACB is called angle in alternate segment for ∠DAQ and ∠ADB is angle in alternate segment for ∠BAR.

Measure ∠DAQ and ∠ACB you will find ∠DAQ = ∠ACB.

Similarly measure ∠BAR and ∠BDA. You will see that ∠BAR = ∠BDA.

Repeat this activity on two more circles of different radii.

You will find angles in alternate segments are equal.

The angles formed in the alternate segments by a chord through the point of contact of a tangent to a circle is equal to the angle between the chord and tangent.

There angles are called Angles in alternate segments.

Let us check the converse of the above fact. draw a circle with centre O. Draw a chord PR. The PR subtends ∠PTR in alternate segment is shown in fig. (16.38)

At P draw ∠RPB =∠RTP

Extent the line PB to A. Thus, line AB is formed. Join OP. Measure∠OPB. You will find that ∠OPB = 90º. This shown that line AB istangent to the circle at P.

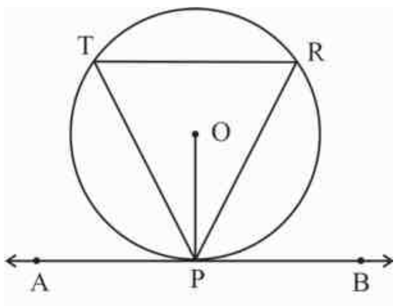


Fig 16.38

Repeat the above activity is two more circles.

You will find the same result. Thus, we can state that if a line makes with the chord angles which are equal respectively to the angles formed by the chord in alternate segments, then the line is tangent to the circle.

Let us take up a few examples to illustrate.

Example 18: In fig. 16.39, O is centre of the circle, AB is tangent to the circle at point P. If∠APQ = 58°, find measure of ∠PQR.

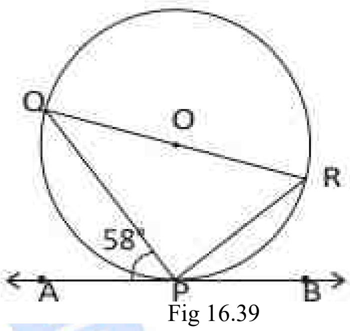


Fig 16.39

Solution: QR is diameter of the circle.

∴ ∠QPR = 58°

∠QPA = ∠QRP ... (angles in alternate segment)

∴ ∠QRP = 58° ...... (∠QPA = 58° given)

In Δ PQR

∠PQR +∠QRP +∠QPR = 180º (angle of Δ)

∴ ∠PQR + 58º + 90º = 180º

∴ ∠PQR = 180° – (90+58°) = 32°

Example 19: In the given figure (16.40) O is centre of the circle. AB is tangent to the circle at the point P. If ∠PAO = 30° then find ∠CPB +∠ACP.

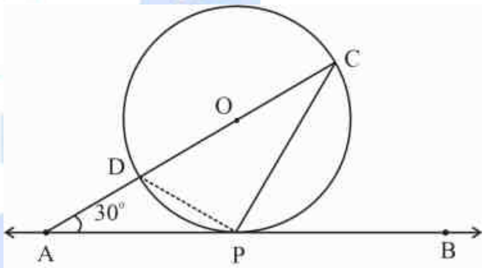


Fig 16.40

Solution:∠DPC = 90° 9angle in a semi circle)

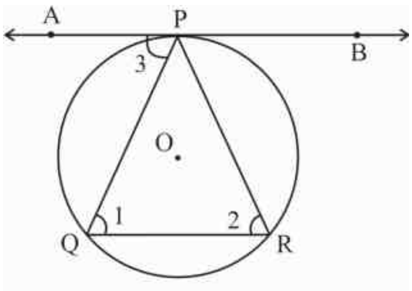
∠DPA =∠DCP __ (angles in alternate segment)

How ∠CPB +∠DPA +∠DPC = 180° (angles on a line)

∠CPB +∠DPA = 180° – 90° = 90°

∴ ∠CPB +∠DCP = 90° (∠DPA =∠DCP)

Example 20: In fig. 16.41, PQR is an isosceles triangle with PQ = PR and AB is tangent to circum circles of ΔABC. Show that AB is parallel to QR.



Solution: In PQR, PQ = PR

Fig 16.41

∴ $\angle 1 = \angle 2$

Again AB is tangent to the circle at P

∴ $\angle 3 = \angle 2$ (Angle in alternate segment)

∴ $\angle 1 = \angle 3$

But these are alternate angles

∴ AB||QR

CHECK YOUR PROGRESS 16.3

1. In figure 16.42, XY is tangent to the circle with centre O. If AOB is a diameter and $\angle PAB = 50°$find $\angle APX$ and $\angle BPY$.

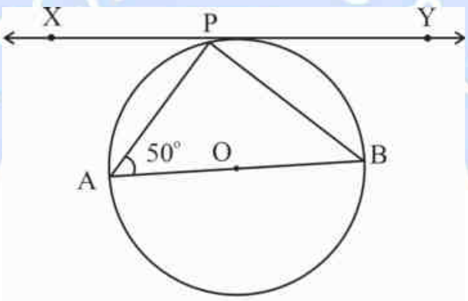


Fig 16.42

2. In fig. 16.43, PQ is tangent to the circle with centre O. A is a point of contact. If $\angle PAB = 67°$ findmeasure of $\angle AQB$.

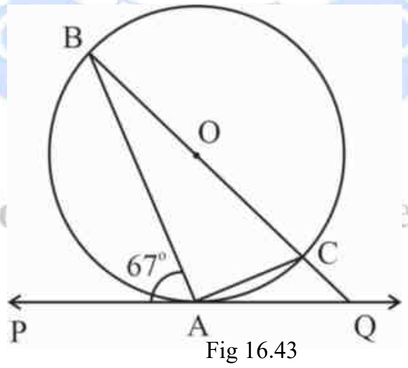


Fig 16.43

3. In fig. 16.44, PQR is tangent to the circle at Q whose centre is O. AB is a chord parallel to PRsuch that $\angle BQR = 70°$ find $\angle AQB$.

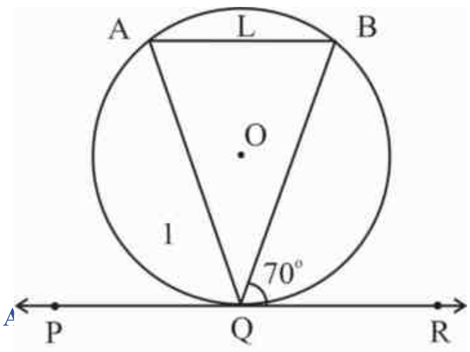


Fig 16.44

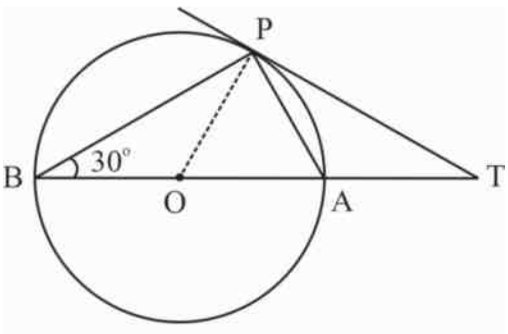


Fig 16.45

4. In fig. 16.45, O is centre of the circle. BOA is its diameter and tangent at P meets BA extendedat T. If ∠PBO = 30° then find ∠PTA.
5. In the given figure 16.46, PQ is a chord of a circle with centre O and PT is tangent. If ∠QPT = 60°find ∠PRQ.

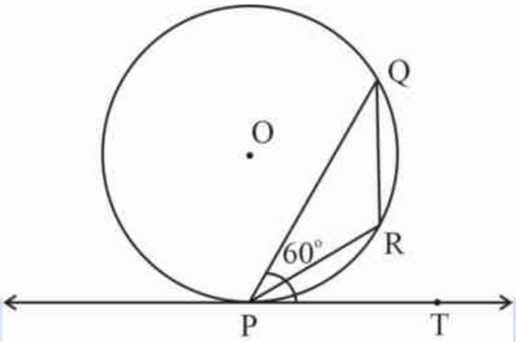


Fig 16.46

RECAPITULATION POINTS

- A line which intersects the circle at two points is called a secant of the circle.
- A line segment which meets the circle at two points is called a chord of the circle.
- A line which touches the circle at one point is called tangent the circle.
- A tangent to the limiting position of a secant when two points of intersection coincide.
- A radius of the circle through the point of contact of a tangent is perpendicular to the tangent.
- From an external point, two tangents can be drawn to a circle. The two tangents are equal in length.
- If two chords AB and XY at a circle intersect at point T inside or outside the circle the TA ×TB = (TX)(TY).
- If PAB is a secant of a circle intersecting line circle at A and B, and PT is tangent to the circle at T , then PA × PB = PT^2.
- The angles formed in alternate segments by a chord through the point of contract of a tangent to a circle are equal to the angles between the chord and the tangent.
- If a line makes with the chord angles which are respectively equal to the angles formed by the chord in alternate segments, then the line is tangent to the circle.

TERMINAL EXERCISE

1. Fill in the blanks:

(i) The largest chord of the circle is ________ _.

(ii) The area of the sector of a circle of radius $\frac{r}{2}$, which subtends angle $\frac{\theta}{2}$ at the centre is _________.

(iii) The number of diameters that can be drawn to a circle are __________.

(iv) The radius of a circle is 5 cm. If a point P is at a distance of 4.5 cm from the centre then P lies in the _________ of the circle.

(v) In the fig. 16.47 ABCD is a cyclic quadrilateral whose diagonals meet at P. If $\angle DBC = 80°$ and $\angle BAC = 40°$. $\angle BCD$ is ________ _.

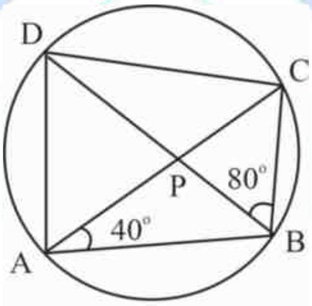


Fig 16.47

2. Choose the correct answers for a each of the following questions :

(i) In the fig. 16.48, $\angle ADB$ =

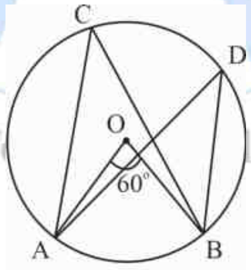


Fig 16.48

(a) 60º (b) 45º (c) 30º (d) 120º

(ii) In fig. 16.49, if $\angle OBA = 60, \angle ACB$ =

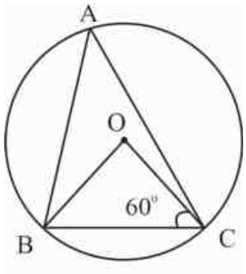


Fig 16.49

(a) 60º (b) 45º (c) 90 º (d) 30 º

(iii) In fig. 16.50, ABCD a cyclic quadrilateral with angles shown. The value of x&y is:

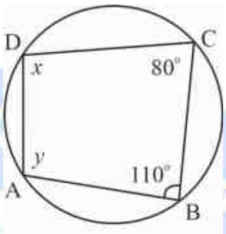


Fig 16.50

(a) 170 º (b) 150 º (c) 210º (d) 190 º

(iv) In fig. 16.51, AB = AC and ∠ACB = 50°, ∠BDC =

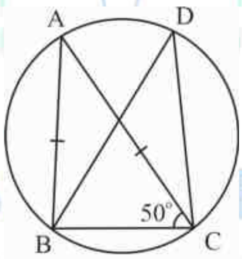


Fig 16.51

(a) 150º (b) 80º (c) 60º (d) 130º

(v) In fig. 16.52, ABCD is a cyclic quadrilateral it which ∠DBA = 50° and ∠DCB = 120°, then∠ADB =

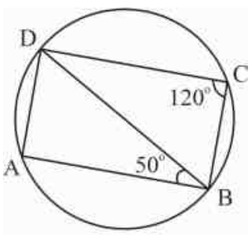
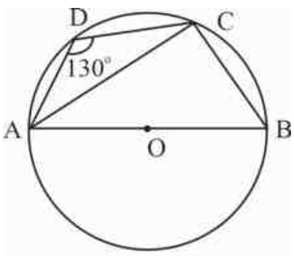


Fig 16.52

(a) 70 º (b) 30 º (c) 40 º (d) 60 º

(vi) In fig. 16.53, ABCD is a cyclic quadrilateral whose side AB is a diameter of circle



through A, B, C and D. If ∠ADC = 130°, ∠BAC =

(a) 60º (b) 30º (c) 50º (d) 40º

(vii) In fig. 16.54 O is centre of the circle. The angle subtended by arc BCD at the centre is 140°. BCis produced to P. ∠DCP =

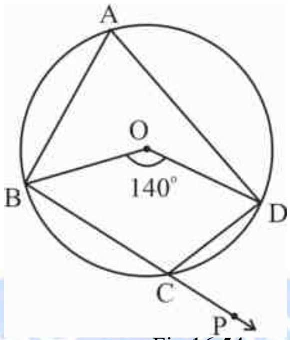


Fig 16.54

(a) 140º (b) 50º (c) 70º (d) 60º

(viii) In fig. 16.55, AB is a diameter of the circle such that ∠PAB = 45°, ∠PCA =

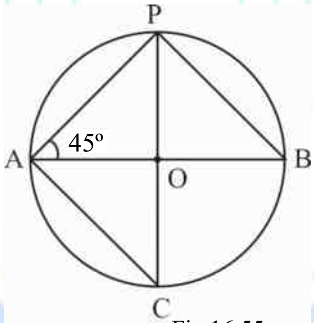


Fig 16.55

(a) 90º (b) 60 º (c) 40 º (d) 45º

3. If one side of cyclic quadrilateral is produced show that the exterior angle is equal to interior opposite angle.

4. In fig. 16.56 arc AB = arc CD prove that ∠A =∠B.

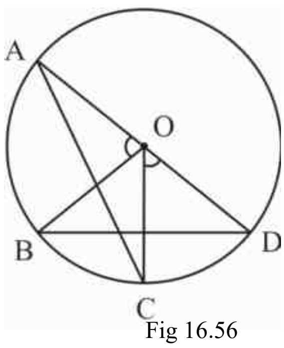


Fig 16.56

5. In fig. 16.57sides AB and DC of a cyclic quadrilateral ABCD are produced to that at E. sidesAD and BC are produced to meet at F. If ∠ADC = 80° and ∠BEC = 50° find ∠BAD and ∠CFD.

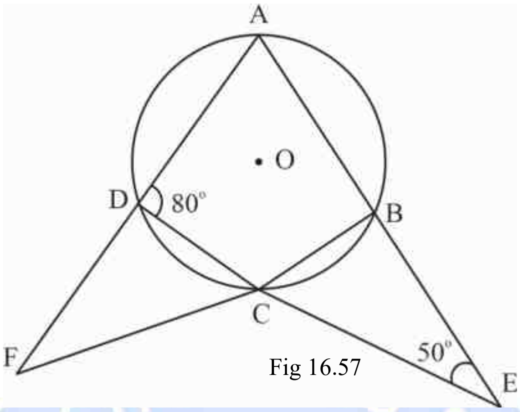


Fig 16.57

6. In fig. 16.58, PQR is a right-angled triangle with PQ = 12 cm and QR = 5 cm. a circle with centreO and radius r is inscribed in ΔPQR. Find the value of r.

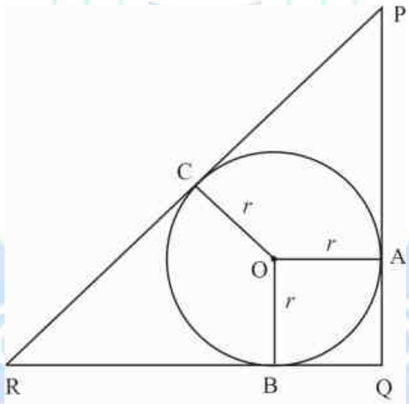


Fig 16.58

7. If two sides of a cyclic quadrilateral arc parallel prove that remaining two sides are equal and diagonals are also equal.
8. A pair of opposite sides of a cyclic quadrilateral are equal prove that its diagonals are also equals.
9. If two non-parallel sides of a trapezium are equal prove that it is cyclic.
10. Two circles intersect each other at points A and B. if AP and AQ are diameter of the two circles, prove that PBQ is a straight line.
11. ABC and ADC are two right triangles will common hypotenuse AC. Prove that ∠CAD =∠CBD.

12. Prove that the circle drawn with any side of a rhombus, passes through the point of interaction of its diagonals.

13. In fig. 16.59, PQ and RQ are two chords of the circle with centre O. Which are equidistant from the centre. prove that diameter QS bisects ∠PQR and ∠PSR.

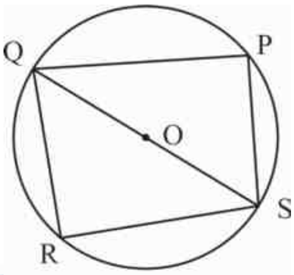


Fig 16.59

14. Two congruent circle intersect each other at the points A and B. through A, any linesegment PAQ is drawn so that P, Q lie on two circles. Prove that BP = BQ.

CASE STUDY–I

Four friends studying in class X, Asha (A), Balbir Singh (B), Mohamad (M) and Joseph (J) are playing in a circular park as shown in the figure 16.60 XY is a line touching the park at with centre O. They took their position on the boundary of the as shown in figure. Correct answer carries three clapping.

In figure 16.60, AM = MB and AB is a diameter of the circle, find ∠*x*, ∠*y* and ∠*z*.

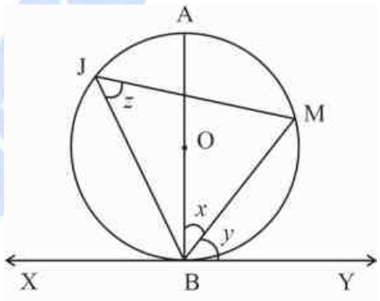


Fig 16.60

Choose the correct option:

(i) For ∠*x*:

(A) = 60º, Balbir Singh (B) = 50º

Mohammad (M) = 45º, Joseph (J) = 40º

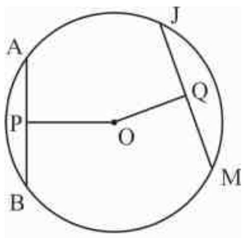
(ii) For ∠*y*:

Asha (A) = 30º, (B) = 40º (M) = 45º (J) = 50 º

(iii) For ∠*z*:

(A) = 50º, (B) = 45º, (M) = 40º, (J) = 30º

(iv) Four friends sat in the following position where O is centre.OP ⊥AB , OQ ⊥ JM and OQ < OP without measuring



the length of AB and JM , state the relation between AB and JM and reason there of:

(A) AB= JM (B) JM > AB

Fig 16.61

(M) JM < AB, (J) Can't say

(v) Four friend A, B, M and J took the position as shown in fig. 16.62. AB is a diameter and ∠AJM = 110°, ∠MAB =

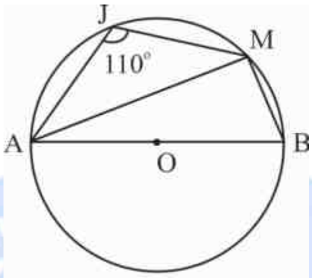


Fig 16.62

(M) 20º (B) 40º (J) 70º (A) 60º

CASE STUDY–II

Four girls Nisha (A), Balbir Kaur (B), Mona (M) and Sofia (S) took part in a quiz competition on circle in the school. Correct answer carries three marks and 1 mark is deducted for wrong answer. The following questions were asked and one girl is to be select or the basis of their scores. The question asked and their answers are given against the first letter of their names. Corret answer carries 3 marks and 1 mark is deducted for wrong answer.

(i) In fig. 16.63, PQ is a diameter of circle with centre. If ∠DCP = 20°, PD = DC, then ∠CQP is:

(A) 120º (B) 40º (M) 100º (S) 90º

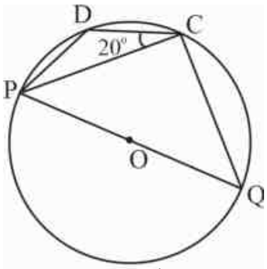


Fig 16.63

(ii) In fig. 16.64, CDEF is a cyclic quadrilateral and ∠FEC = 45°, ∠ECF = 40°, ∠EDC is:

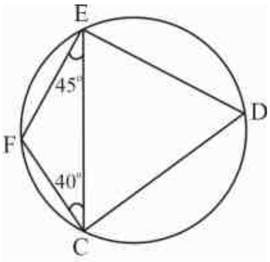


Fig 16.64

(A) 80º (B) 90º (M) 85º (S) 120º

(iii) In fig. 16.65, is ΔQPR PQ = PR and ∠PRQ = 60°. ∠QTR is:

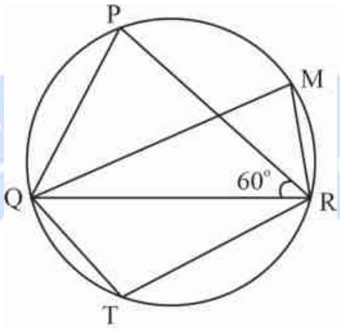


Fig 16.65

(A) 140º (B) 90 º (M) 120º (S) 110º

(iv) In fig. 16.66, C and D are points on the semi circle described on PQ as diameter. ∠QPD = 70°and ∠DQC = 30°, ∠QDC is:

(A) 40º (B) 20º (M) 50 º (S) 30º

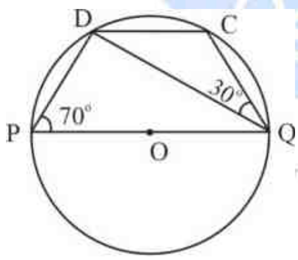


Fig 16.66

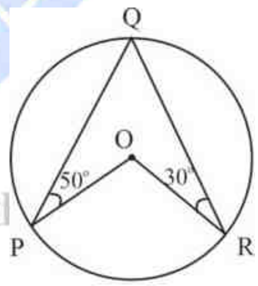


Fig 16.67

(v) In fig.16.67, m (arc PR):

(A) 110º (B) 120º (M) 100 º (S) 160º

Who got maximum marks ___________

ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 16.1

1. (i) one

(ii) two points

(iii) tangent

(iv) contact

(v) 90°

(vi) limiting

(vii) 180°

(viii) cyclic

(ix) cyclic

(x) two

2.	40	3.	140º
4.	1.9 cm	5.	18 cm
6.	2 cm	7.	AD = 7 cm, BE = 5 cm, CF = 3 cm

CHECK YOUR PROGRESS 16.2

1.	3 cm	2.	$x = 3$ cm
3.	10 cm	4.	6 cm
5.	6 cm	6.	3 cm

CHECK YOUR PROGRESS 16.3

1. $\angle APx = 40^o$, $\angle BPy = 50^o$

2.	44º	3.	40º
4.	30 º	5.	120º

CASE STUDY I

(i) M = 45º

(ii) M = 45º

(iii) B = 45º

(iv) B (DE >CF) [The chord nearest the centre is greater]

(v) M = 20º

CASE STUDY II

(i) B = 40º

(ii) M = 85 º

(iii)M = 120º

(iv)A = 40º

(v) S = 160º

The highest score is Moora.

17 CONSTRUCTIONS

INTRODUCTION

Geometricalconstructionhelpsusa lotinconstructionengineering.Youmayalsoverifycertain hypothesis andaxiomsofgeometrybyuseofconstructions.You all know how to construct an angle equal to a given angle, bisecting a given line segment and making angle of 30°, 45° and 60° etc. In this chapter we will further move to learn some more constructions related to the theoretical geometrical conceptswehave learned so far.

17.1 LEARNING OBJECTIVES

After completingthischapter,youwillbeableto:

- Divideagivenlinesegmentinternallyinagivenratio
- Constructatrianglefromthegivendatabasedon
  - (a) SSS
  - (b) SAS
  - (c) ASA
  - (d) RHS
  - (e) Perimeterandbaseangles.
  - (f) Base,sum/differenceofothertwosidesandonebaseangle.
  - (g) Twosidesandamediancorrespondingtooneofthesesides
- Constructatriangle,similartoagiventriangle
- Constructtangentstoacirclefromapoint.
  - (i) Onitusingthecentreofthecircle
  - (ii) Outside it

EXPECTED BACKGROUND

Studentsknowuseofcompassandrulertoconstruct

- Angleequaltothegivenangle
- Alinesegmentofgivenlength
- Angleof30°,45°,60°,105°,120°

- Bisectorofagivenangle

17.2 TO DIVIDE A LINE SEGMENT

Construction 1

To divide a linesegment in a given ratio internally. Let PQ be a given segment to which you want to divide in the ratio $m:n$ where both m and n are positive integers. For understanding more accurately let us take $m = 3$ and $n = 2$.

StepsofConstruction

1. DrawPQsegmentofthegivenlength.
2. DrawanyrayPX,makinganacuteanglewithPQ.
3. Draw5$(m+n)$pointsX_1,X_2,X_3,X_4andX_5onPXsothat $Px_1=X_1X_2=X_2X_3=X_3X_4=X_4X_5$
4. Join$Q$withX_5.
5. From the point X_3, draw a line parallel to QX_5, intersecting PQ at R(By making on angle equal to$\angle PX_5Q$).

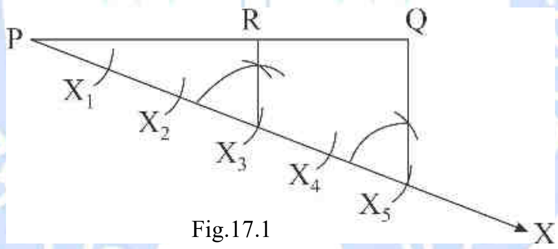


Fig.17.1

Justification: Frombasicsimilarlytheoremas $RX_3 \| QX_3$.

$$\frac{PX_3}{X_3X_5} = \frac{PR}{RQ}$$

$$\frac{3}{2} = \frac{PR}{RQ}$$

∴Point$R$dividesPQintheratio3:2.

CHECK YOUR PROGRESS 17.1

(i) Drawalinesegmentof7.6cmlong,divideitinternallyintheratio5:8.Measureeachpartand write the steps of construction.

(ii) Drawa line segment of 9 cm, divide it internallyin theratio 4:3. Justifyyouranswerand write the steps of construction.

(iii) TakenalinesegmentAB=12cmfindpointConitsuchthat $\frac{AC}{AB}=\frac{2}{5}$

[**Hint:**dividethelinesegment*AB*internallyintheratioz:3.]

17.3 CONSTRUCTION OF TRIANGLES

Construction2

Toconstructatrianglewhosethreesidearegiven(SSS).

Example: ConstructaΔABCwhosesidesareAB=3cm,BC=4cmandAC=6 cm.

StepsofConstruction

i. TakealinesegmentAB=3cm.

ii. WithAasacentreandradius6cmdrawonarc.

iii. WithBasacentreandradius4cmdrawonarcwhichisintersectingfirstarc.

iv. Takethepointofintersectionofarcas*C*.

v. JoinAandBwithC.

ΔABCistherequiredΔtriangle.

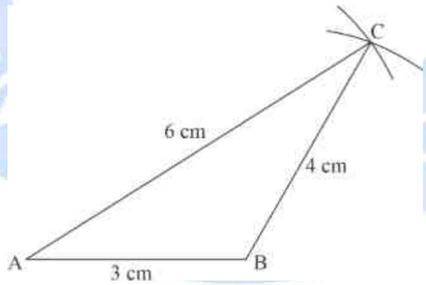


Fig.17.2

Construction3

Tocontractatriangle,whentwosidesandincludedangleisgiven(SAS).

Example: ConstructaΔABCinwhich*AB*=3.3cm*BC*=5.5cm and∠B=60°.

StepsofConstruction

1. Take*AB*=3.3cm.

2. Atpoint*B*drawanangleof60°,Let∠ABZ=60°.

3. WithBasacentreandradius5.5cmdrawanarccuttingBZatC.

4. JoinAC.

NowΔABCistherequiredtriangle.YoumayconstructitbytakingbaseBCalsointhesameway.

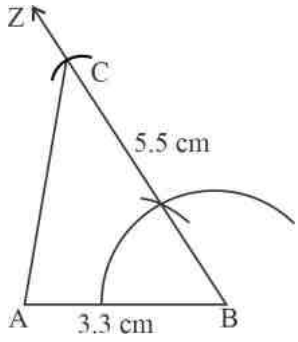


Fig.17.3

Construction4

Toconstructatriangle,whenitstwoangleandtheincludedsidearegiven(ASA).

Example: ConstructatrianglePQRinwhich$\angle P=60°$,$Q=45°$andPQ=5 cm.

StepsofConstruction:

1. DrawPQ=5cm.
2. AtPdrawanangleof60°.Let∠QPX=60°.
3. AtQdrawanangleof45°,Let$\angle PQY=45°$.
4. Letthe Ray $\overrightarrow{PX}$ and $\overrightarrow{QY}$ cuteachotherat$R$thenΔ$PQR$is therequiredtriangle.

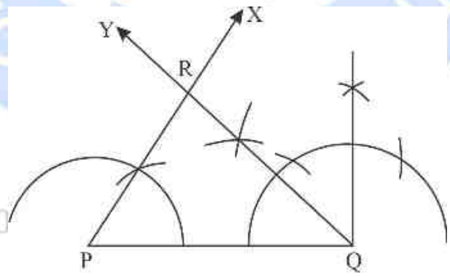


Fig.17.4

Construction5

Toconstructarighttriangle,whenitshypotenuseandasidearegiven.

Example: ConstructΔLMNinwhichMN=3cm,LN=5cmwhichisitshypotenuse.

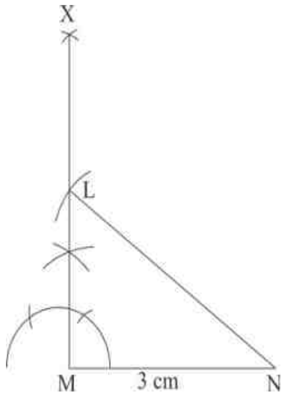


Fig.17.5

StepsofConstruction

1. TakeMN=3cm

2. AtMconstruct$\angle NMX$= 90°

3. WithNascentreandradius5cmdrawanarccutting $\overrightarrow{MX}$ at$L$.Join NL.

4. TriangleLMNistherequiredtriangle.

Construction6

Toconstructatrianglewhenitsperimeterandtwobaseanglesaregiven.

Example: ConstructatriangleABCinwhich∠B=60°,$\angle C$=45°and$AB+BC+CA$=11cm

StepsofConstruction

1. DrawPQ=11cm($AB+BC+CA$=11cm)

2. AtPDraw∠QPX=30°$\left(\frac{1}{2}\times 60^\circ\right)$ andAtQdraw∠PQY=$22\frac{1}{2}\left(\frac{1}{2}\times 45^\circ\right)$

3. LetXPandQYmeetat$A$.

4. DrawrightbisectorofPXintersectingPQat$B$

5. DrawrightbisectorofQYintersectingPQat$C$.

6. JoinABandAC.

7. ABCistherequiredtriangle.

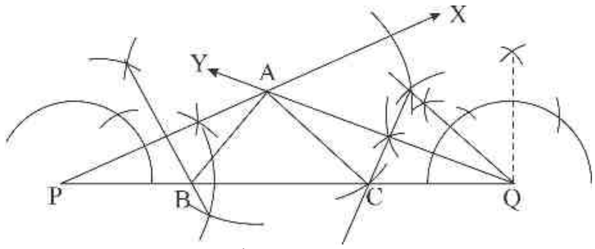


Fig.17.6

Justification:length$PB=AB$and$AC=QC$

$\therefore AB+BC+AC=PB+BC+CQ$=11cm

In Δ PBA ∠BPA = ∠BAP = 30°

∴∠ABP=120°∴∠ABC=60°

Similarly,inΔ $ACQ \angle CQA=\angle CAQ = 22\frac{1}{2}$

$\therefore \angle ACQ$=135∴$\angle ACB$= 45°

Construction 7

To construct a triangle when sum of two sides, third side and one of the angles on the third side are given.

Example: ConstructatriangleABCinwhichAB+AC=8.2cmBC=3.6cmand∠B=45°

StepsofConstruction

1. DrawBC=3.6cm
2. AtBconstruct$\angle CBK$=45°
3. FromBK,cutoffBP=8.2cm
4. DrawrightbisectorofCPintersectingBPat$A$.
5. JoinAC.

Δ ABCistherequiredtriangle.

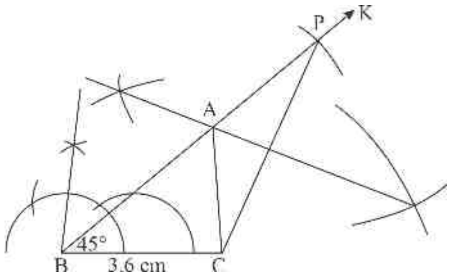


Fig.17.7

Construction 8

To construct a triangle when difference oftwo sides, the third side and one of theangle on third side are given.

Example: ConstructaΔABCinwhichBC=4.8cm$\angle B$=60°,AB–AC=1.5 cm

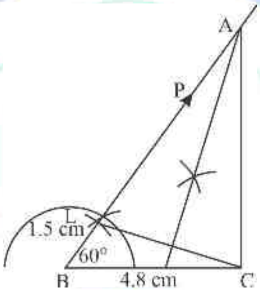


Fig.17.8

StepsofConstruction

1. DrawBC=4.8 cm
2. Atpoint$B$draw$\angle CBP$= 60°
3. FromBP,cutoffBL=1.5cmwiththehelpofcompass
4. Join $C$and$L$
5. DrawrightbisectorofCLmeetingBPat$A$
6. JoinAC

TriangleABCistherequiredtriangle

Construction 9

To construct a triangle when its two sides and a median corresponding to one of these sides are given.

Example: ConstructaΔABCinwhichAB=4.5cmBC=6cmandmedianCD=4 cm

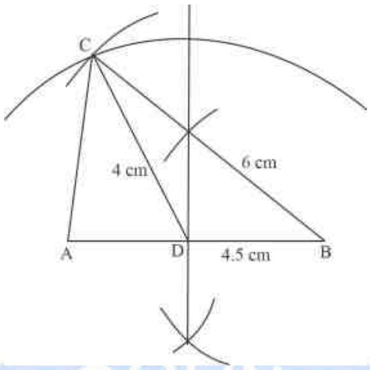


Fig.17.9

StepsofConstruction

1. DrawAB=4.5cm
2. DrawperpendicularbisectorofABmeetingitat$D$.
3. Taking D as a centre and radius 4 cm draw an arc.
4. With B as a centre and radius 6 cm draw an arc.
5. Letthetwoarcscutatpoint$C$,Join$C$with$A$and$B$.

Δ ABCistherequiredtriangle

CHECK YOUR PROGRESS 17.2

(i) Constructatriangle$\angle MN$,where$\angle M$=$\angle N$=MN=5cm.

(ii) Constructatrianglewhosesidesare3cm,4cmand5cmrespectively.

(iii) ConstructatriangleABC,givenAB=8.4cm, $\angle A$ =65°and∠B=30°.

(iv) ConstructatrianglePQY,givenPQ=4.2cmQY=5.3cmand∠Q=60°

(v) Constructarighttrianglewhosebaseis4.5cmandhypotenuseis5cm

(vi) ConstructarightangletriangleABCiswhich$\angle B$ =90°,AC=5cmandBC=4cm

(vii) Construct a triangleABCin whichBC=3cm,$\angle B$ = 30°and AB +AC =5.2 cm. Writesteps of construction.

(viii) Constructatriangle$\angle MN$whereMN=8cm,$\angle M$=45°andLM–LN =3.5cm.Writestepsof construction.

(ix) Constructatriangle*PQR*iswhich$\angle Q$=30°,$\angle R$=90°and*PQ*+*QR*+*RP*=11cm.Writethe steps of construction.

(x) Constructa triangle whosebase is 3.2 cm, median corresponding to this base is 3 cm and the other side is 4 cm. Join the median with vertex. Write steps of construction.

17.4 CONSTRUCTION OF A TRIANGLE SIMILAR TO A GIVEN TRIANGLE, AS PER GIVEN SCALE FACTOR

17.4.1Scale factor

The scale factor is the ratio of the length of a side of one figure to the length of correspondingside of other figure, here in this particular case it will be ratio of the side of the triangle to be constructed, to the corresponding sides of the given triangle.

Construction 10

construct a triangle similar to a given triangle ABC with its sides equal to $\frac{3}{4}$ of the corresponding sidesofthetriangleABC(i.e.,scale factor$\frac{3}{4}$).Thesidesoftriangle *ABC*are*AB*=6 cm, *BC* = 4 cm and *AC* = 5 cm.

StepsofConstruction

1. Construct the required triangle *ABC*, take *AB* = 6 cm, *BC* = 4 cm and *AC* = 5 cm (you have learnt how to construct a triangle when its three side are given)

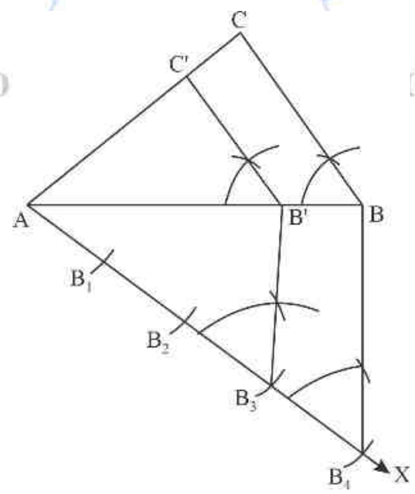


Fig.17.10

2. Drawanyray*AX*makinganacuteanglewith*AB*onthesideoppositetovertex*C*.

3. Locate4pointsB $_1$,B $_2$,B $_3$,B $_4$ onAXsuchthatAB $_1$=B $_1$B $_2$=B $_2$B $_3$=B $_3$B $_4$

4. JointB $_4$withBanddrawalineparalleltoB $_4$BfrompointB $_3$tointersectABatB′

5. DrawalinethroughB′paralleltoBCtointersectACatC′

Then AB′C′ is the required triangle

CHECK YOUR PROGRESS17.3

(i) Constructatriangleofsides5cm,6cmand7cmandthenatrianglesimilartoitwhosesidesare $\frac{2}{3}$ ofthecorrespondingsidesoffirsttriangle

(ii) ConstructatrianglesimilartoagiventriangleABCwithitssidesequalto $\frac{5}{3}$ ofthe corresponding sidesofthetriangleABC[**Hint:**Takefiveequalpartsonthelinemakingacuteanglewithbase]

(iii) Constructanequilateraltriangleofside4.8cmandthenanothertrianglewhosesidesare $\frac{7}{4}$ ofthe correspondingsidesofthefirsttriangle.

(iv) DrawatriangleABCwithBC=6cmAB=5cmand$\angle ABC$=60°.Thenconstructatrianglewhose $\frac{3}{4}$ ofthecorrespondingsidesofthetriangleABC.

17.5 CONSTRUCTION OF TANGENTS TO A CIRCLE

Construction11

Toconstructthetangenttoacirclefromapointonit.

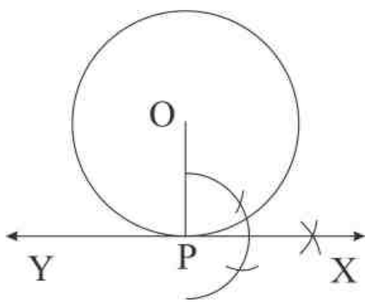


Fig.17.11

StepsofConstruction

1. Drawthecircleofgivenradius.

2. Let*P*bethepointonwhichtangentistobedrawn.

3. Join*P*withcentre*O*.

4. At*P*draw*PX*⊥*OP*.

5. Produce*XP*to*Y*

*YPX*istherequiredtangent.

Construction12

Todrawtangentstoacirclefromagivenpointoutsideit.

Suppose*C*bethegivencirclewithcentre*O*andapoint*X*outsideit.Youhavetodrawtangentstothecircle from this point.

StepsofConstruction

1. Drawthecircleofgivenradius.

2. Let*P*bethepointonwhichtangentistobedrawn.

3. Join*P*withcentre*O*.

4. At*P*draw*PX*⊥*OP*.

5. Produce*XP*to*Y*

Now*YX*isthelinetangentatpointP.

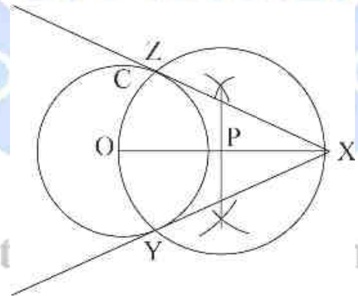


Fig. 17.12

StepsofConstruction

1. Join*OX*

2. DrawtherightbisectorofOX,LetPbethemid-pointof OX.

3. TakingPascentreandPOasradius,drawacircle.

4. ThiscircleisintersectingthegivencircleCatZandYpoints.

5. JoinXwithZandY.

XZandXYarethetworequiredtangents.Measuretheirlength,whatdoyouobserve.

Frompointoutsidethecircle,wecandrawtwotangentswhichareequalinlength.

CHECK YOUR PROGRESS17.4

(i) TakingO as acentre,draw a circleofradius 4 cm. Draw its radius OQ,draw atangent atpoint Q. Write steps of construction.

(ii) Take a circle of radius 3 cm, with centre O. Take a point P, 6 cm from centre of circle. Draw two tangents from point Pto the given circle. Measure the length of each. Write steps of construction.

TERMINAL EXERCISE

(i) Draw a line segment PQ of 8 cm. Divide it internally in the ratio 2:3. Write steps ofconstruction.

(ii) Divide a line segment of a 9 cm internally in the ratio 4:3, write the steps of construction.

(iii) Construct an equilateral triangle of side 3.2 cm. Write the steps of construction.

(iv) Construct a triangle PQR, where PQ = 4 cm, $\angle P = 60°$ $\angle R = 90°$ [**Hint:** Find angle $\angle Q$ first]. Write steps of construction.

(v) Construct a right-angle triangle whose base is 4 cm and hypotenuse is 5 cm. Write steps of construction.

(vi) Construct a right triangle in which the sides (other than hypotenuse) are of length 6 cm, 8 cm. Then construct another triangle whose sides are $\frac{3}{5}$ times the corresponding sides of the given triangle. Write 5 steps of construction.

(vii) Draw a circle of radius 3 cm. Take twoZ points P and Q on the extended diameter each at a distance of 7 cm from the centre. Draw tangents to the circle from these points P and Q. Write steps of construction.

(viii) Draw a triangle ABC in which BC = 6 cm, CA = 5 cm and AB = 4 cm. Construct a triangle similar to it and scale of factor is $\frac{5}{3}$. Write steps of construction.

(ix) Construct a right triangle whose base is 12 cm, and sum of hypotenuse and third side is 18 cm. Write the steps of construction.

(x) Construct a triangle PQR whose perimeter is 14 cm, $\angle P = 45°$ and $\angle Q = 60°$. Write the steps of construction.

(xi) Construct triangle ABC in which BC = 8 cm, $\angle B = 30°$ and AB + BC = 12 cm. Write steps of construction.

(xii) Construct triangle ABC in which BC = 7.5 cm, ∠B = 45° and AB – AC = 4 cm. Write steps of construction.

(xiii) Construct triangle ABC in which AB + BC + CA= 11.6 cm and base angle are 60° and 45° respectively. Write steps of construction.

(xiv) Construct an equilateral triangle whose altitude is 6 cm. Write steps of construction.

(xv) Draw a circle of radius 4.5 cm. Construct pair of tangents on it, which make 45° angle with each other. [**Hint:** Draw an angle of 135° at centre of the circle]. Write steps of construction.

SUPPORTIVESTUDY MATERIAL

http://w.w.w.learncbse.in

http://ncerthelp.com

http://w.w.w.learninsta.com

18 COORDINATE GEOMETRY

INTRODUCTION

You all know how to plot numbers on the number line. From origin, we plot positive integers at equal distance towards right and negative integers towards left.

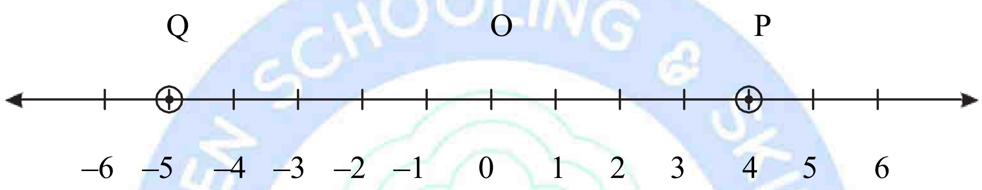


Fig.18.1

In fig.18.1, if your teacher asks the position of point *P* and *Q*. On the number line, you can easily find it with help of this number line. But look at the figure given below.

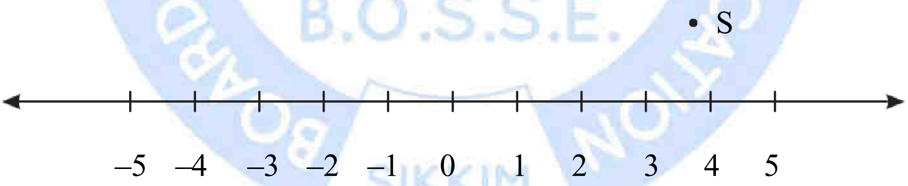


Fig.18.2

The number line is there, but can you tell position of *S* with the help of this line. Your Answer is "No" because *S* is not located at the number line. For finding the location of *S*, we need knowledge of co-ordinate geometry. Co-ordinate geometry is two-dimensional geometry with the help of which we can locate position of points in plane.

Rene Descartes (1596-1650) the French mathematician and philosopher who gave the method of finding the position of a point in plane very precisely, is known as father of Cartesian or co-ordinate geometry.

18.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Tell meaning of the terms Cartesian plane, X axis, Y axis, abscissa, ordinate.
- Locate a point in Cartesian plane whose co-ordinates are given and vice-versa.
- Find the distance between two given points.
- Find the co-ordinates of point, which divides the line segment joining two points in a given ratio.
- Find midpoint of given two points.
- Find the centroid of a triangle with given vertices.
- Find area of a triangle whose vertices are given.
- Solve problems based on above concepts.
- Relate co-ordinate geometry with other disciplines and its use in daily life.

18.2 CO-ORDINATE SYSTEM AND CARTESIAN PLANE

(a) ***X*axis:**ThehorizontalgradatednumberlineXOX'iscalled$X$axis.

(b) ***Y*axis:**TheverticalgraduatednumberlineYOY'iscalled$Y$axis

The two axis meet each other at (O, O) and are perpendicular to each other. The meeting point is called origin and is denoted by O. As is clear from the picture 18.3, the positive or right side from O is calibrated with +ve numbers and negative numbers are taken towards left of origin which is marked as OX'. Similarly, on YY' axis, OY is taken as positive distance, while OY' is taken as negative.

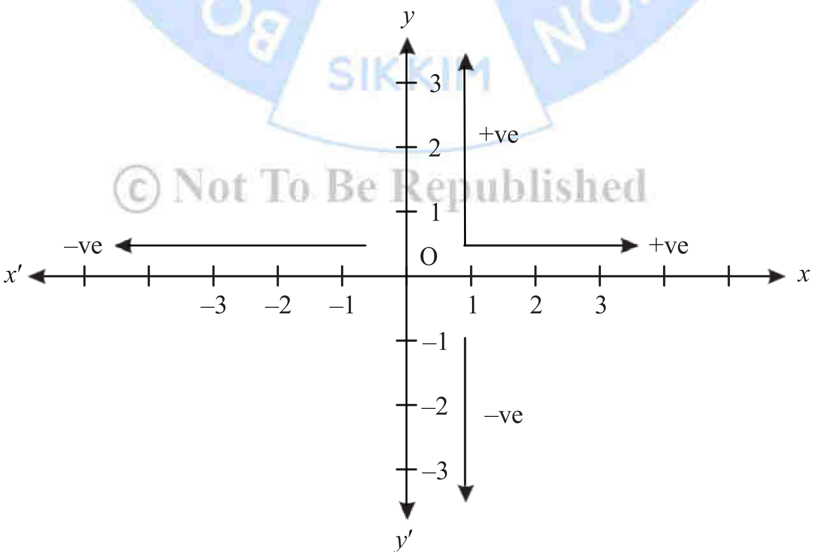


Fig 18.3

Apaper havingperpendicularaxisXX'andYY'make aCartesianplan,Aplaneisa flat,two-dimensionalsurfacethat extends infinitely far.

ACartesianplanecanbedefinedasaplaneformedbytheintersection oftwoco-ordinateaxesthatare perpendicularto each other,and intersect at (O,O) and extends infinitelyin all directions.

18.3 QUADRANTS

ThetwoaxesXX'andYY'dividetheplaneintofourpartscalledquadrants.

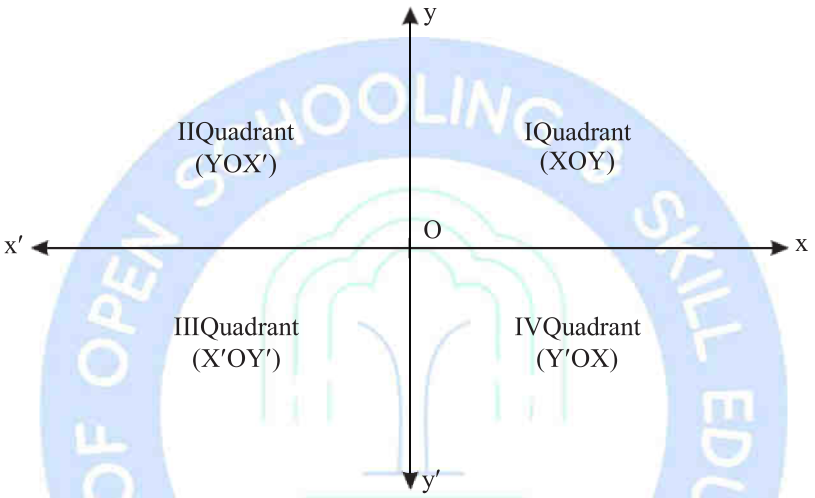


Fig. 18.4

The four quadrants are named as

***XOY′*:**	Ist Quadrant	***YOX′*:**	II Quadrant
***X′OY′*:**	IIIQuadrant	***Y′OX*:**	IV Quadrant

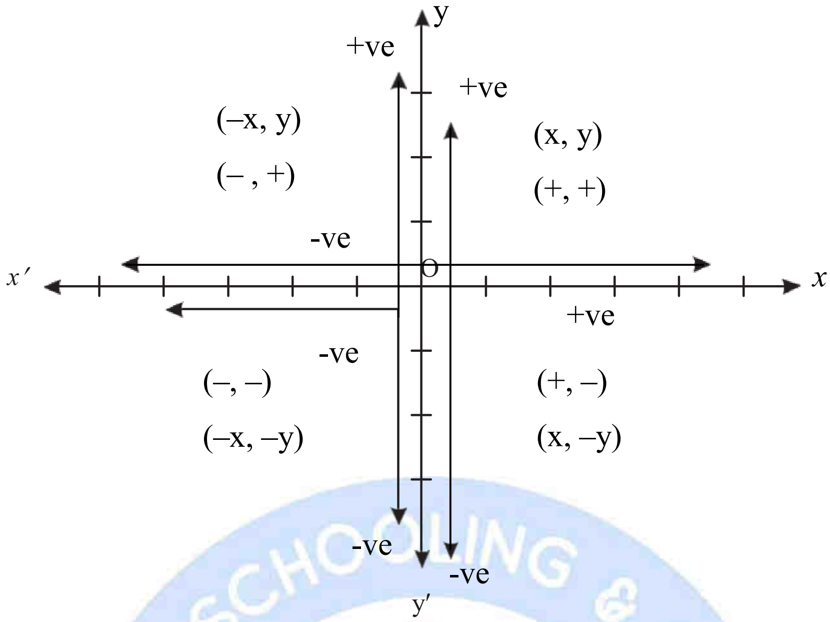


Fig 18.5

- InI[st]Quadrantweseethat*x*is+*ve*and*Y*isalsopositive∴co-ordinatesofallpointswillbe(++).
- InII[nd]Quadrant*x*isnegativean*Y*isPositive∴co-ordinatesofallpointswillbe(–,+)YoucannoweasilytelltheCo-ordinatessigninIII[rd]andIVQuadrant.

18.4 CO-ORDINATES OF A POINT

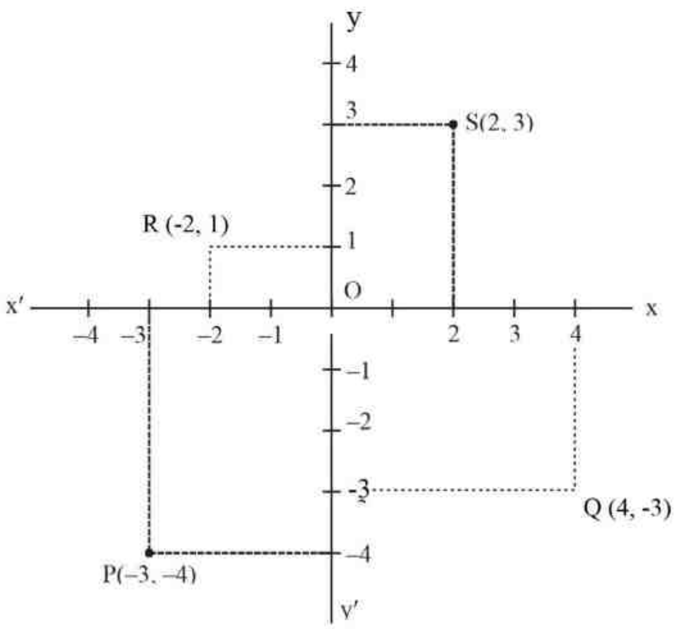


Fig. 18.6

Nowwewilllearnhow to locateposition of a pointin a co-ordinateplane look atthe fig.18.6,

Forfinding the co-ordinates of*S*

(a) DrawperpendicularfromSonXX'axis,itfallsat2,so$x$=2

(b) DrawperpendicularfromSonYY'axis,itfallsatpoint3so$y$=3.

Therefore, co-ordinates of*S* are(2, 3)

Similarly,let'sfindco-ordinatesof*P*

(a) DrawperpendicularonXX'axis,itis–3,therefore$x$=–3

(b) DrawperpendicularfrompointPonYY'axis,itis–4therefore$y$ =–4.

Henceco-ordinates of*P* are (–3, –4)

The*X* co-ordinate is called abscissa and*Y* is called ordinate

Wealwayswritetheco-ordinatesofapointasanorderedpair($x$co-ordinate,*Y*-ordinate)

> InGeneralco-ordinatesofapoint$P(x,y)$implydistanceofPfromtheYaxisin$x$unitsandit'sdistancefromX axis in*Y* units.

Example1: Writedownthe$x$ (abscissa)and$y$(ordinate)forthegivenpairs.

(a) (2, 2) (b) (–3, 2)

(c) (–4, –5) (d)(–2, 4)

Solution:

(a)(2,2)

X co-ordinate or abscissa is 2

Y co-ordinate or ordinate is 2

(b)(–3, 2)

X co-ordinate or abscissa is –3

Y co-ordinate or ordinate is 2

(c)(–4, –5)

X co-ordinate or abscissa is –4

Y co-ordinate or ordinate is –5

(d)(–2, 4)

X co-ordinate or abscissa is –2

Y co-ordinate or ordinate is -5

Example 2: In which quadrants do the following points lie

(i) (–3, 5) (ii) (–2, –5)

Solution:

(i) Inthepoint(–3,5)abscissaisnegativeandordinateispositivesothepoint(–3,5)liesinthesecondquadrant

(ii) In(–2,–5)point,abscissaisnegativeandordinateisalsonegative,sothepoint(–2,–5)liesinthirdquadrant.

Example3: Write downco-ordinates of*P*, *Q*,*R*, *S*in thefigure 18.7given below*P* (2,3)

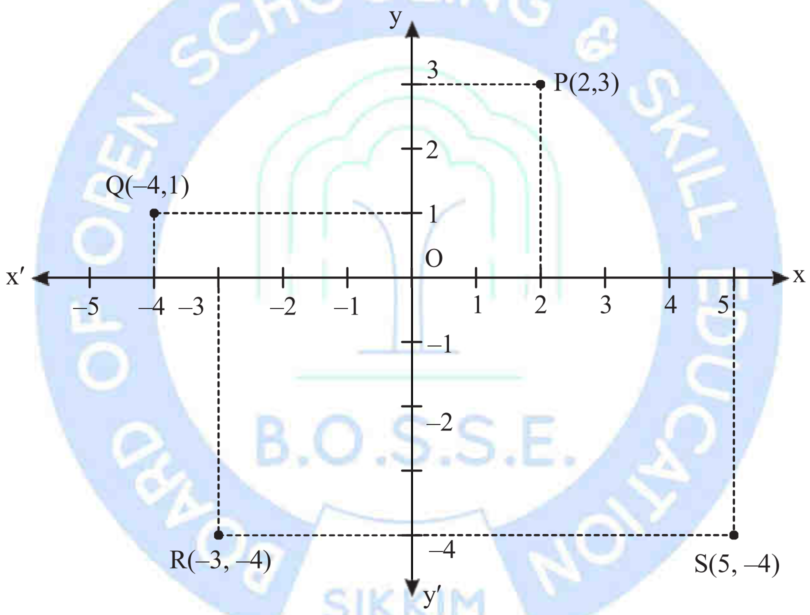


Fig. 18.7

Solution:

(i) From point*P,* draw perpendicularat*X*-axisand*Y*-axis.Nowdistance of*P* at*X*axis is2 andat*Y*axis is 3, therefore the co-ordinates of*P* are (2, 3)

(ii) From*Q*Drawperpendicular at*X*axisand*Y*axis.Nowdistanceof*Q*at*X*axisis–4andat*Y*axisis There for co-ordinates of *Q* are (–4, 1).Similarlyyoucanfindoutco-ordinatesof*R* and*S.*

CHECK YOUR PROGRESS 18.1

1. Writedownabscissaandordinateforeachofthefollowingpoints

(a) (4, 3) (b) (–2, –5)

(c) (–6, 8) (d) (–9, –2)

2. Find out the quadrant in which the below mentioned points will lie

(a) (–5, 3)	(b) (6, 4)
(c) (5, –3)	(d) (2, 2)
(e) (–6, –6)	(f) (–4, 1)
(g) (–3, 2)	(h) (8, –3)
(i) (–1, –1)	(j) (–2, 3)

18.5 PLOTTING OF A POINT WHOSE CO-ORDINATES ARE GIVEN

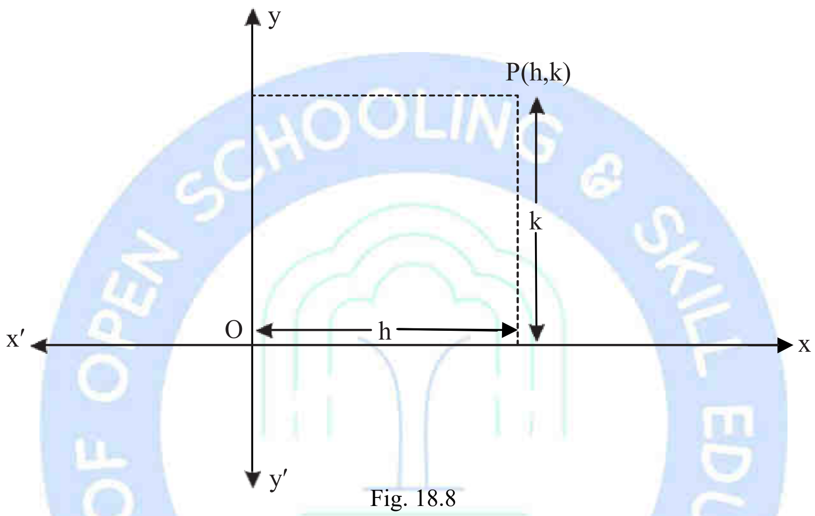


Fig. 18.8

Anypoint (h,k) canbe plottedin aCartesian plane,following thegiven belowsteps.

(i) MeasureOMequalto$h$along$X$-axis(fig. 18.8)

(ii) MeasurePMperpendiculartoOMandequalto$K$youhavetomoveaccordingtosign of coordinates on$X$axisand Y axis

Example4: Plot P(–2,4) and$Q$(2,3) ona co-ordinate plane.

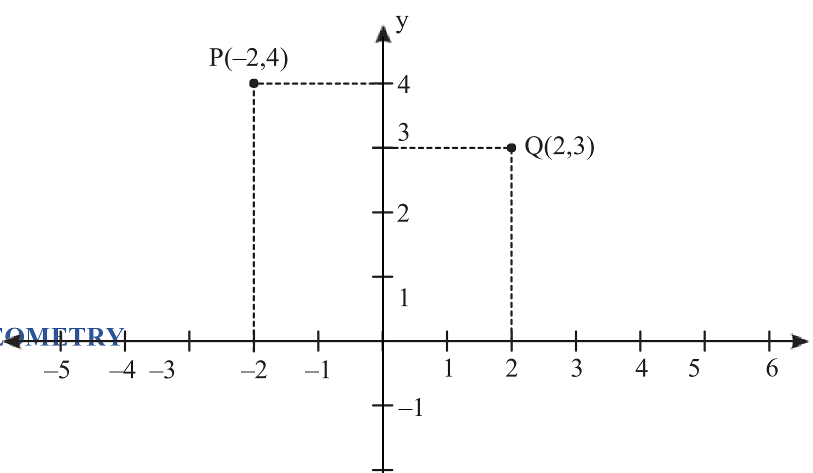


Fig. 18.9

Solution:

(i) $P(-2,4)$Locate–2at$X$axisdrawaperpendicularonit,Similarlylocate4on$Y$axisanddrawperpendicularpoint P is wherethese two perpendicular bisect.

(ii) $Q(2,3)$Locate2at$X$axis,drawa⊥onit.Locate3at$Y$axisanddraw⊥onit.$Q(2,3)$ispointofintersection of two perpendiculars.Youcaneasilyobservethat(–2,4)liesin2ndquadrantsoinfirstcaseyourperpendicularswillbein2ndquadrantwhile(2,3)lieinIstquadrant sotheperpendicularswillbeinfirstquadrant only.

18.6 DISTANCE BETWEEN TWO POINTS

Inaplane,thedistancebetweenanytwopoints$P(x_1,y_1)$and$Q(x_2,y_2)$ is the length of the line segment PQ.

Let P$(x_1,y_1)$and$Q(x_2,y_2)$beanytwopointsasshowninthefig.18.11,whosedistanceis to befind out. DrawPRandQSperpendicular to the$x$-axis.

Alsodraw a perpendicular fromthe point P onQS, let it meetthe point$T$.

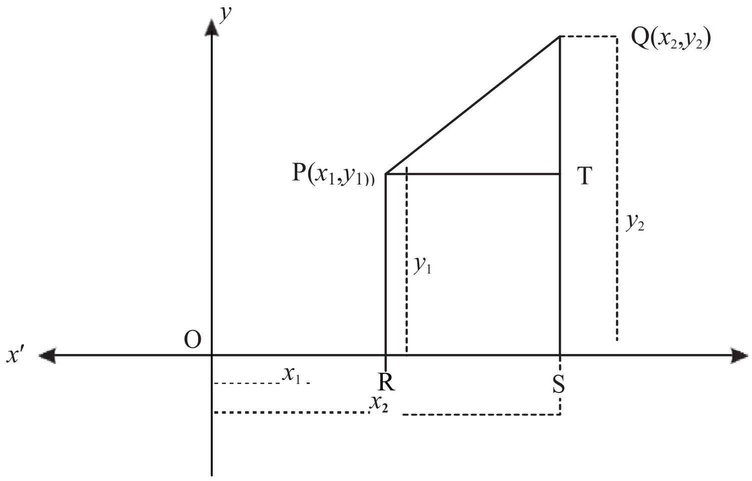


Fig.18.10

Weseethat $OR = x_1$ $OS = x_2$ So $RS = x_2 - x_1 = PT$

Also, $SQ = Y_2$ and $PR = TS = y_1$ So $QT = Y_2 - Y_1$

Now Applying the MERUPRASTHAM or Pythagoras theorem in ΔPQT

$$PQ^2 = PT^2 + QT^2$$

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$PQ^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Distanceisalwaysnon-negative,sowetakeonlypositivesquareroot.

Therefore,

Distancebetween two points $= \sqrt{(\text{difference of abscissa})^2 + (\text{difference of ordinates})^2}$

Aboveformulais calleddistanceformula.

18.7 COROLLARY

The distance of a point $P(x,y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{(x-0)^2 + (y-0)^2}$$

$$Or\ OP = \sqrt{x^2 + y^2}$$

Example 5: Find the distance between each of the following points:

(a) A (4, 7) and B (7, 3) (b) P (3, 2) and Q (–1, 0)

Solution:

(a) Herethepointare$A$(4,7)and$B$(7,3)

Byusingdistanceformula,wehave

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(7-4)^2+(3-7)^2}$$

$$=\sqrt{(3)^2+(-4)^2}$$

$$=\sqrt{9+16}$$

$$=\sqrt{25}$$

=5

HenceAB=5 unit

(b) Herethepointsare$P$(3,2)and$Q$(–1,0).

By usingdistanceformula,Wehave

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(-1,-3)^2+(0,-2)^2}$$

$$=\sqrt{(-4)^2+(-2)^2}$$

$$=\sqrt{16+4}$$

$$=\sqrt{20}\text{ units}$$

$$=\sqrt{5\times4}$$

$$=2\sqrt{5}\text{ units}$$

Example6: Findthe point on$X$ axis whichis equidistant frompoints$A$ (2,–5) and$B$(–2, 9):

Solution: Weknow,anypointon X-axisisintheform(x,0).Let X(x,0)beequidistantform$A$(2,–5)and$B$(–2,9).NowXA=XB

$$\sqrt{(x-2)^2+(0+5)^2}=\sqrt{(x+2)^2+(0-9)^2}$$

Squaringboththesides

$(x-2)^2+5^2=(x+2)^2+(-9)^2$

$x^2-4x+4+25=x^2+4x+4+81$

$-8x=56$

$x=-7$

$\therefore$ Co-ordinates of X are $(-7, 0)$.

CHECK YOUR PROGRESS 18.2

1. Findthe distancebetween thegiven points
   - (a) P (1, 0) and Q (4, 0)
   - (b) R (3, 4) and S (5, 1)
   - (c) A(c, 0) and B (0,–c)
   - (d) A(5Cosθ, 0), B(0, 5Sinθ)
2. The Centre of a circle is (3, –2), if it is passing through the point (6, –6) find its radius.
3. If A(0, –1), B(2, 1) and C(0, 3) are vertices of a triangle.
   - (a) findsidesof triangle ABC
   - (b) Are the points A (3, 1) B (0, 2) and C (1, 5) collinear?[Hint:if $AB+BC=AC$ threepointsarecalledco-linear]
   - (c) Findthe pointon Y-axis whichis equidistantfrom points P(2, –5) and Q (–2,9)

18.7 SECTION FORMULA

Whenthepointdividesthelinesegmentintheratio $m:n$ atpoint C,andthispoint C liesbetweenco-ordinatesofthe line segment, it is called internal division.

Tofindco-ordinatesofapoint,whichdividesthelinesegmentjoiningtwopoints,inagivenratiointernally. Let $A(x_1,y_1)$ and $B(x_2,y_2)$ betheendpointsofagivenlinesegment AB and $C(x,y)$ bethepointwhichdivides AB in the ratio $m:n$

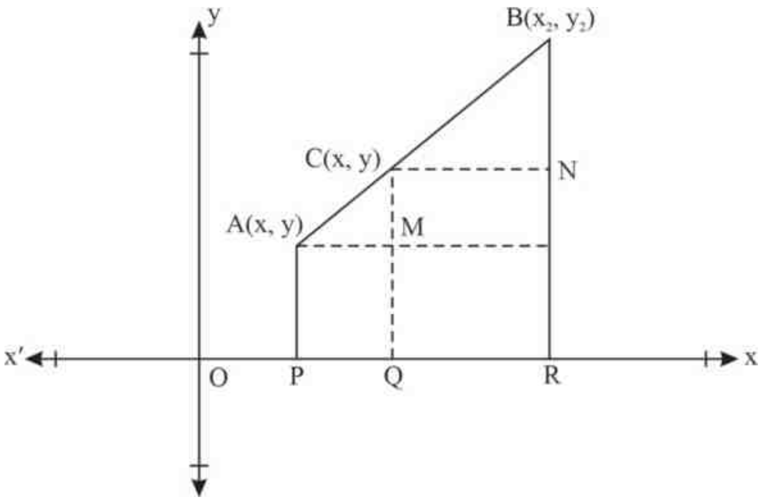


Fig. 18.11

Then $\frac{AC}{CB}=\frac{m}{n}$

Wewanttofindco-ordinatesof*C* *i.e.*(x,y)

NowDrawperpendicularfrompoint*A*,*C*,*B*parallelto*Y*axis,Letpointofjoiningoftheseperpendiculars be *P*, *Q*, *R* on *X*-axis.

Fromfigure18.11

$$AM = PQ = OQ - OP = (x - x_1)$$

$$CN = QR = OR - OQ = (x_2 - x)$$

$$CM = CQ - MQ = (y - y_1)$$

$$BN = BR - NR = (y_2 - y)$$

$\Delta AMC$and$\Delta CNB$are similartherefore,their sidesareproportional by*AA*conrule.

$$\frac{AC}{CB}=\frac{AM}{CN}=\frac{CM}{BN}$$

Substitutingthevaluesinaboverelation

$$\Rightarrow \frac{m}{n}=\frac{x-x_1}{x_2-x} \text{ and } \frac{m}{n}=\frac{y-y_1}{y_2-y}$$

Solving: $m(x_2 - x) = n(x - x_1)$ and $m(y_2 - y) = n(y - y_1)$

$$(m+n)x = mx_2 + nx_1 \text{ and } (m+n)y = my_2 + ny_1$$

$$or\ mx_2 - mn = nx - nx_1$$

$$mx_2 + nx_1 = mx - nx\ or$$

$$(m+n)x = mx_2 + nx_1$$

$$x = \frac{mx_2 + nx_1}{m+n} \text{ and } y\frac{my_2 + ny_1}{m+n}$$

Therefore, co-ordinates of$c$ (x, y) are

$$x = \frac{mx_2 + nx_1}{m+n} \text{ and } y\frac{my_2 + ny_1}{m+n}$$

18.8 MID-POINT FORMULA

Theco-ordinatesofthemid-pointofthelinesegmentjoiningtwopoints$(x_1,y_1)$and(x_2,y_2)canbeobtainedeasilyby taking$m = n$ in the sectionformula

Substituting$m$=$n$wehave

$$x = \frac{nx_2 + nx_1}{n+n} \qquad y = \frac{ny_2 + ny_1}{n+n}$$

$$= \frac{n(x_2 + x_1)}{2n} \qquad = \frac{n(y_2 + y_1)}{2n}$$

$$= \frac{x_2 + x_1}{2} \qquad = \frac{y_2 + y_1}{2}$$

Therefore,the co-ordinates ofthe mid-point joiningtwo points (x_1,y_1) and (x_2,y_2) are

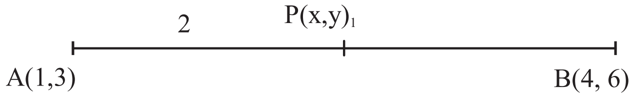
$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example7: Findtheco-ordinatesofapointwhichdividesthelinesegmentjoiningeachofthefollowing points in the given ratio.

(a) $A(1, 3)$ and$B(4, 6)$ internally in the ratio 2 : 1

(b) (3, –6) and (5,3) is divided in the ratio 3 : 2

Solution: Let $A(1, 3)$ and $B(4, 6)$ be the given points, let $P(x, y)$ divide AB in the ratio 2 :1Usingsectionformula



$$x=\frac{2\times4+1\times1}{2+1} \text{ and } y=\frac{2\times6+1\times3}{2+1}$$

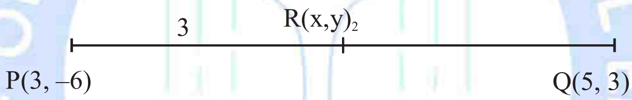
$$=\frac{8+1}{3}=\frac{12+3}{3}$$

$$=\frac{9}{3}=\frac{15}{3}$$

$$x=3\ \ y=5$$

$\therefore P\ (x, y) = (3, 5)$

(c) Let $P(3, -6)$ and $Q(5, 3)$ be the given points. Let $R(x, y)$ divides it in the ratio 3 :2.



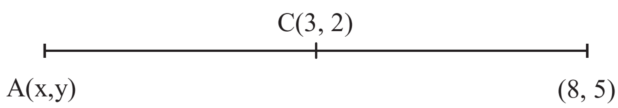
Bysectionformula

$$x=\frac{3\times5+2\times3}{3+1} \text{ and } y=\frac{3\times3+2\times(-6)}{3+2}$$

$$=\frac{15+6}{5}=\frac{9-12}{5}$$

$$=\frac{21}{5}=\frac{-3}{5}$$

Example8: Theco-ordinatesofthemidpointofasegmentare(3,2),ifco-ordinatesofoneoftheendpoints ofline segmentare (8, 5), find the co-ordinates of the other endpoint.



Solution: Letco-ordinates ofother endpoint$A$ be(x, y)By midpointformula.

$3 = \frac{x+8}{2}$ and $2 = \frac{y+5}{2}$

$6 = x+8 \qquad 4 = y+5$

Orx= –2 $\qquad y$= –1

∴ (–2, –1) are the co-ordinates of other point.

Example9: FindoutinwhichratioXaxisdividesthesegmentmadebypoints(2,–3)and(5,6).Findco-ordinates of the dividing point.

Solution: Let$X$axisdividesthesegmentformedby(2,–3)and(5,6)intheratioλ:1.Theco-ordinateofdividingpoint by sectionformula willbe

$$R\left(\frac{5\lambda+2}{\lambda+1}, \frac{6\lambda-3}{\lambda+1}\right)$$

Asthispointlieson$X$axis, weknow$Y$co-ordinateat$X$axisis0.

Therefore $\frac{6\lambda-3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{1}{2}$

∴Xaxisdividesthesegmentinthe ratio1:2co-ordinatesof point,bysubstituting $\lambda = \frac{1}{2}$

$$R\left(\frac{5\times\frac{1}{2}+2}{\frac{1}{2}+1}\right), \left(\frac{6\times\frac{1}{2}-3}{\frac{1}{2}+1}\right)$$

R(3,0) are co-ordinates of point.

18.9 CENTROID OF A TRIANGLE

Definition: The centroid of a triangle is the point of con currency of its medians and divides each median inthe ratio 2 : 1.

Tofindtheco-ordinatesofcentroidofatrianglewhoseverticesaregiven.

Let$A(x_1,y_1)$and$B(x_2,y_2)$and$C(x_3,y_3)$betheverticesoftriangleABC.LetADbeitsmedianbisecting itsbaseBC, then by mid-point formula wehave

$$D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Nowthepoint$G$onADdividesADinternallyintheratio2:1,isthecentroidofΔABC,if(x,y)beco-ordinates of G then

$$x=\frac{2\left(\frac{x_2+x_3}{2}\right)+1\times x_1}{2}=\frac{x_1+x_2+x_3}{3}$$

$$y=\frac{2\left(\frac{y_2+y_3}{2}\right)+1\times y_1}{2+1}=\frac{y_1+y_2+y_3}{3}$$

Hencethe co-ordinatesof centroidare givenby

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

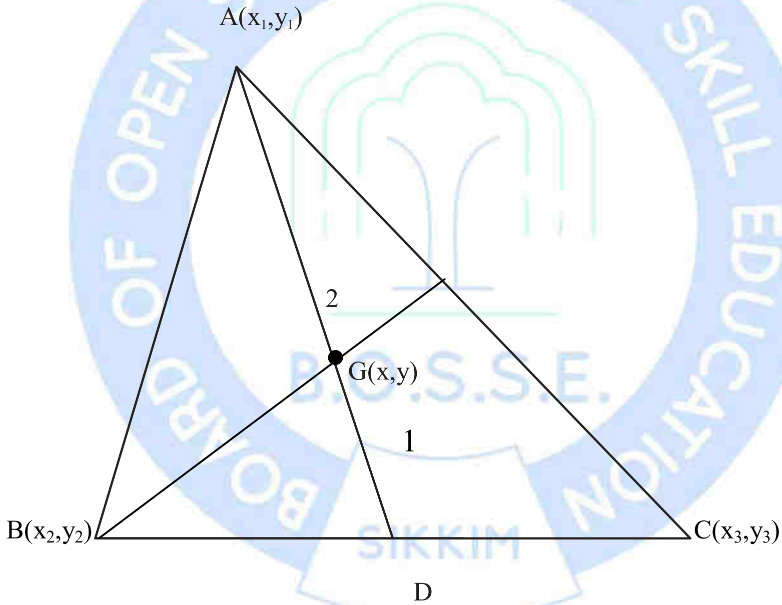


Fig. 18.12

Example 10: If vertices of a triangle are (5, 1), (3, 5) and (9, –5) find co-ordinates of its centroid.

Solution: Let $A(5, 1)$, $B(3, 5)$ and $C(9, –5)$ be the vertices of a triangle ABC.

Let$G$ (x, y)be its centroid,we know

$$x=\frac{x_1+x_2+x_3}{3},\text{ and } y=\frac{y_1+y_2+y_3}{3}$$

$$=\frac{5+3+9}{3}\qquad =\frac{1+5-5}{3}$$

$=\frac{17}{3}$ $=\frac{1}{3}$

Hence co-ordinates of centroid G are $=\left(\frac{17}{3},\frac{1}{3}\right)$

CHECK YOUR PROGRESS 18.3

1. Findtheco-ordinatesofthepointwhichdividesinternallythelinesegmentjoiningthe points

   (a)(1, 3) and (4, 6) in the ratio 2 : 1

   (b)(2, –5) and (5, 2) in the ratio 2 : 3

2. Findthe mid-pointof theline joining

   (a) (0, 2) and (6, –6)

   (b) (–4, 0) and (2, 12)

3. Findthe centroidof thetriangle whoseverticesare (5,0), (8, 0)and (8,3)

4. Ifthe mid points of the sidesof a triangle are (3, 4),(4, 6) and (5, 7) findthe vertices of the triangle.

18.10 AREAOFATRIANGLE

Youhavealreadystudied, howtocalculatetheareaoftrianglewhenitsbaseandcorrespondingheightor altitude are given.The formula used is:

$$\text{Areaofatriangle}=\frac{1}{2}\times\text{base}\times\text{altitude}$$

By using co-ordinate geometry, we can easily find out the area oftriangle,simplyusing theco-ordinates ofitsvertices.

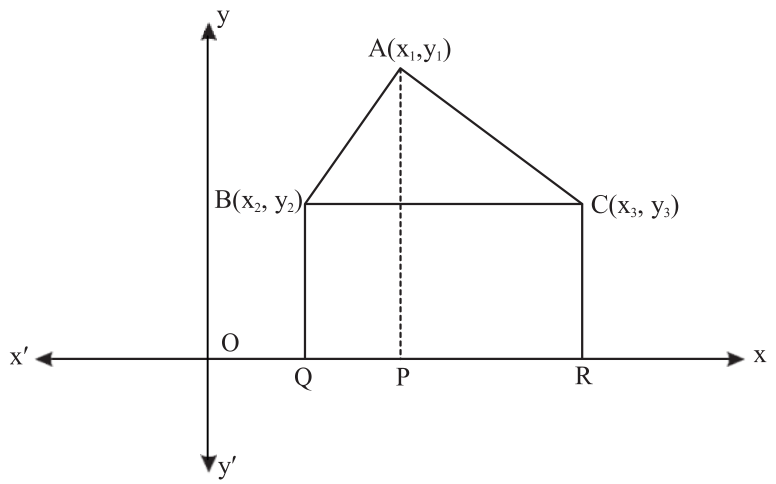


Fig. 18.13

Let ABC be any triangle whose vertices are $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$. Draw AP, BQ and CR Perpendiculars from A, B, C on X axis.

Now $ABQP$, $APRC$ and $BQRC$ are trapezium Now from fig.18.13

Area of ΔABC = Area of trapezium $ABQP$ + Area of trapezium $APRC$ – Area of trapezium $BQRC$

You know Area of trapezium $=\frac{1}{2}\times$(sum of parallel sides)(distance between them)

Therefore,

Area of $\Delta ABC=\frac{1}{2}(BQ+AP)QP+\frac{1}{2}(AP+CR)PR-\frac{1}{2}(BQ+CR)QR$

$=\frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$

$=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

Thus, the area of ΔABC is

$=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

Area is always positive, we always take numerical value of Area

Example 11: Find the area of a triangle whose vertices are $(5, 0)$, $(8,0)$ and $(8, 4)$

Solution: The area of triangle formed by the vertices $A(5, 0)$, $B(8,0)$ and $C(8, 4)$ by using the formula above.

Area of $\Delta ABC=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$

$$= \frac{1}{2}\left[5(0-4)+8(4-0)+8(0-0)\right]$$

$$= \frac{1}{2}[-20+32+0]$$

$$= \frac{1}{2}\times 12 = 6 \text{ sq. units}$$

Example 12: FindtheAreaoftriangle formed bythe points*A*(–5,7), *B*(–4, –5)and*C*(4, 5).

Solution: TheAreaofthe triangleformed bythe vertices*A*(–5, 7),*B*(–4, –5) and*C*(4, 5)is givenby:

$$\text{Area of } \Delta ABC = \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$= \frac{1}{2}\left[-5(-5-5)+(-4)(5-7)+4(7-(-5))\right]$$

$$= \frac{1}{2}\left[-5\times -10+(-4)(-2)+4\times 12\right]$$

$$= \frac{1}{2}[50+8+48]$$

$$= \frac{1}{2}\times 106 = 53 \text{ sq. units}$$

Example 13: Ifverticesof$\Delta ABC$are*A*(4,1),*B*(–3,2)and*C*(0,*k*)anditsAreais12sq.units,findthevalueof*k*.

SolutionHerex_1=4,y_1= 1,x_2=–3,y_2= 2,x_3= 0,y_3=k

NowAreaa=12sq.units

$$\Rightarrow \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right] = 12$$

$$\Rightarrow \frac{1}{2}\left[4(2-k)+(-3)(k-1)+0(1-2)\right] = 12$$

$$\Rightarrow \frac{1}{2}[8-4k-3k+3] = 12$$

$\Rightarrow \frac{1}{2}[11-7k]=12$

$\Rightarrow 11-7k=24$

$\Rightarrow -7k=13$

$k=\frac{-13}{7}$

Example 14: Findthevalueof$k$ifthe points$A(k,3)$,$B(6,–2)$and$C(–34)$arecollinear.

Solution: Since the given points are collinear, the Area of the triangle formed by them must be 0.

Herex_1= k, y_1= 3, x_2= 6,y_2= –2,x_3= –3 andy_3= 4

Sincethepointsare co-linear

$Ar(\Delta ABC)=0$

$\Rightarrow \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]=0$

$\Rightarrow \frac{1}{2}\left[k(-2-4)+6(4-1)+(-3)(3+2)\right]=0$

$\Rightarrow \frac{1}{2}\left[k(-6)+6(3)+(-3)(5)\right]=0$

$\Rightarrow -6k+18-15=0$

$\Rightarrow -6k+3=0$

$\Rightarrow k=\frac{-3}{-2}=\frac{1}{2}$

CHECK YOUR PROGRESS 18.4

1. Findtheareaoftrianglewhosevertices are

   (a) (3, 0), (7, 0) and (8, 4)

   (b) (–2, 5) , (3, –4), (7, 10)

2. Findthe area of triangle madeby the pints$A(5, 2)$,$B(4,7)$ and$C(7, –4)$

3. Showthatthepoints(2,–2),(–3,8)and(–1,4)areco-linear.

4. Forwhatvalueof$k$thepoints(k,2– 2k)(–k+1,2k)and(–4–k,6– 2k)areco-linear.

5. FindtheAreaofthequadrilateralwhosevertices,takeninorderare(–4,–2),(–3,–5),(3,–2)and(2, 3)

   [**Hint;**diagonal of a quadrilateral divides it in two triangle]

TERMINALEXERCISE

(i) Findtheordinateofthepointwhichdividesthelinesegmentjoiningthepoints(–1,7)and(4,–3)inthe ratio 2 : 3

(ii) Ifthepoints$A$(x,2), B(–3,–4) and$C$(7, –5)are collinear,thenfind valueof$x$

(iii) Findthepointon$x$axiswhichareatadistanceof $2\sqrt{5}$ fromthepoint(7,–4).Howmanysuchpoints are there

(iv) [**Hint:**Let$P$(x,0)bethepointon$x$axishavedistance $2\sqrt{5}$ frompoint$Q$(7,–4)then$PQ^2=(x-7)^2+(0+4)^2$]

(v) Findtheratioinwhich$y$-axisdividesthelinesegmentjoiningthepoints(–4,–6)and(10,12).Alsofind the co-ordinates of the point of division

(vi) Showthat thepoints$A$(7, 5),B(2, 3)and$C$(6, –7)are vertices ofa righttriangle, also finditsArea.

(vii) Can you form a triangle from points$A$(–3, 0) B(5, 0) and$C$(0, 0), if not give reason

(viii) Findthe valueof x forwhichthe distancebetween two points$P$(2, –3)and$Q$(x, 5)is 10

(ix) Provethat the points (3,0), (6, 4) and (–1,3) are vertices of aright-angled isosceles triangle.

(x) Thepoint$R$dividesthelinesegmentAB,where$A$=(–4,0)and$B$(0,6)aresuchthat $AR=\frac{3}{4}AB$, find co-ordinates or$R$.

(xi) Ifmid points ofsidesof atriangle are (4,3), (6, 0) and(7, –2)

   (a) Find its vertices

   (b) Findcentroid oftriangle soformed

RECAPITULATION POINTS

Inthischapter,youhavediscussedthefollowingpoints:

- If(4,5)aretheco-ordinatesofapoint,then Xco-ordinateorabscissais4andthe Yco-ordinateor ordinate is 5
- In (x, y), $x$is the distance from$y$axis and$Y$ is the distance from$X$ axis.
- (x, y) is always ordered pair
- Co-ordinate of origin are (O, 0)
- Y co-ordinate of every point on X-axis is O and X co-ordinate of every point on Y axis is O.
- ThetwoaxesXOX'andYOY'divide theplaneinfourquadrants.
- The distance between$P$ (x_1, y_1) and$Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- The distance of a point$P(x, y)$ from origin is $\sqrt{x^2 + y^2}$
- Theco-ordinatesofapoint,whichdividesthelinesegmentjoiningtwopoints$(x_1,y_1)$and(x_2,y_2)ina ratio$m : n$ internally are given by $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$
- Theco-ordinatesofthemid-pointofthelinesegmentjoiningtwopoints$(x_1y_1)$and(x_2y_2)aregivenby$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$
- Theco-ordinatesofthecentroidofatrianglewhoseverticesare$(x_1y_1)$and$(x_2y_2)$and$(x_3y_3)$are given by$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$
- TheAreaofthetriangleformedbypoints(x_1,y_1),$(x_2,y_2)$and(x_3,y_3)isthenumericalvalueofexpression $= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 18.1

1. (a) Abscissa–4,ordinate–3 (b) Abscissa–2,ordinate–5

   (c) Abscissa–6,ordinate8 (d) Abscissa–9,ordinate–7

2. (A)IIndquadrant, (B)Istquadrant (C)IVthquadrant (D)Istquadrant

   (E)IIIrdquadrant (F)IIndquadrant (G)IIndquadrant (H)IVthquadrant

   (I)IIIrdquadrant (J)IIndquadrant

CHECK YOUR PROGRESS 18.2

1. (a) 3 units(b) $\sqrt{13}$ units(c) $c\sqrt{2}$ units (d) 5

2. 5 units

3. $AB = 2\sqrt{2},\ BC = 2\sqrt{2},\ AC = 4$ units

4. (0, 2)

CHECK YOUR PROGRESS 18.3

1. (a) (3, 5) (b) $\left(\frac{16}{5}, \frac{-11}{5}\right)$

2. (a) (3, –2) (b) (–1, 6)

3. (7, 1)

4. (4, 5), (2, 3), (6, 9)

CHECK YOUR PROGRESS 18.4

5. (a) 8 sq.units (b) 53 sq.units

1. 2sq.units

2. $K = -1$ or $K = \frac{1}{2}$

SUPPORTIVE WEBSITES

https//www.learncbse.in

https//www.ncert.inc.in

https//www.cbsetuts.com

19 PERIMETER AND AREA OF TRIANGLES AND QUADRILATERALS

INTRODUCTION

Inthismodulewehavetolearntofindouttheperimeterofaregularpolygon,especiallytrianglesandquadrilaterals.

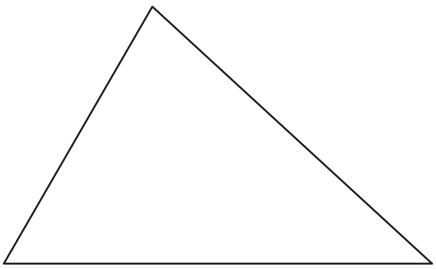
19.1 LEARNING OBJECTIVES

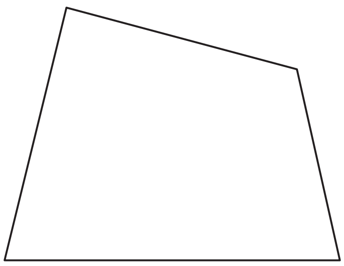
After completing this lesson, you will be able to:

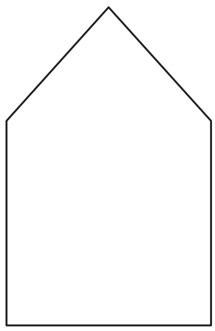
- Understand polygon and find its perimeter
- Determine the area of various rectilinear figures
- Analyse Brahmagupta Sutra, otherwise called Heron's Formula
- Find the area of Equilateral triangle

19.2 POLYGON

A Polygon is a simple closed figure formed by line segments which are in the same plane.Thus,a polygonis a closedrectilinearfigure.





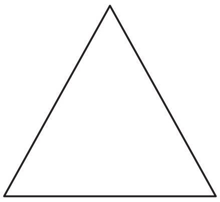


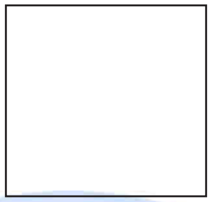
They following figures are polygons

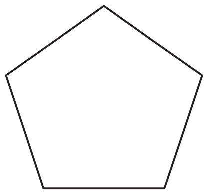
Triangle Quadrilateral Pentagon

19.2.1 Regular Polygon

A regular polygon is a polygon in which all the sides are equal.


Equilateral Triangle


Square

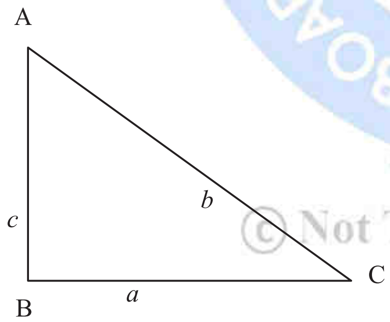
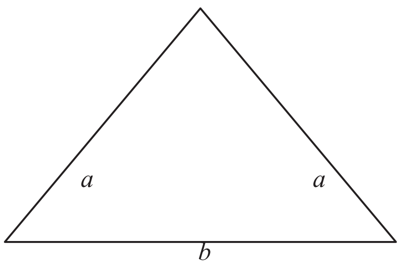

Regular pentagon

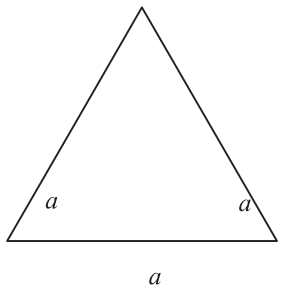
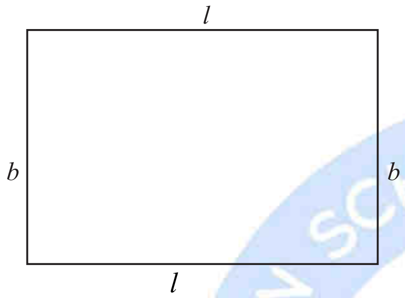
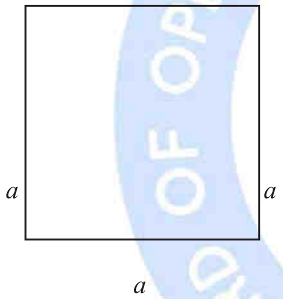
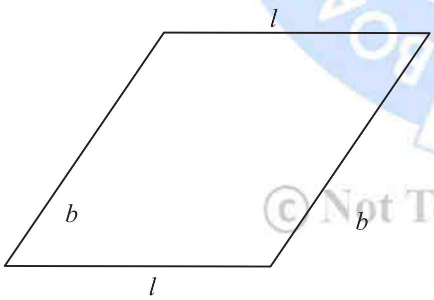
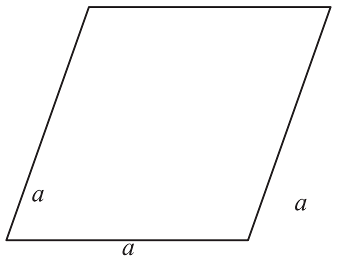
19.2.2 Perimeter of a polygon

The perimeter of a closed figure is the distance around the figure.

In other words "The sum of the length of sides of a closed figure is called the perimeter of the figure."

Watch the table below and note the perimeter of some polygons–

Figures	Name of the figure	Perimeter
	ABC is *a* scalene triangle with sides *a*, *b* and *c*	$(a + b + c)$ Sum of the length of sides
	Isosceles triangle with two equal sides '*a*' and third side is '*b*'	$a + a + b$ or $(2a + b)$ 2 × length of equal sides +length of unequal side

	Equilateral triangle with all sides of length 'a'	(a+ a + a) = 3a 3 × length of side of triangle
	Rectangle with length = l and breadth=b	l+ b + l + b =2 (l + b) 2 (sum of length and breadth)
	Square with sides 'a'	a+ a + a + a =4a= 4×length of side of square
	Parallelogram with length 'l' andbreadth 'b'	l+ b + l + b =2(l+b)
	Rhombuswithside'a'	a + a + a + a = 4a4×sidesofRhombus

Wefindfromtheabovetablethat

(i) Theperimeterof aregular polygon= numberof sides× lengthof oneside

(ii) Theperimeterof anirregular polygon= sumof lengthof allsides

Example1. Findthe perimeter offollowing figures

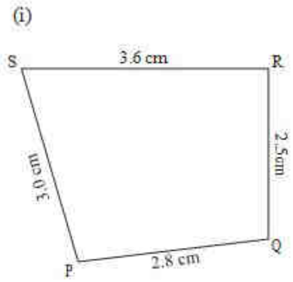
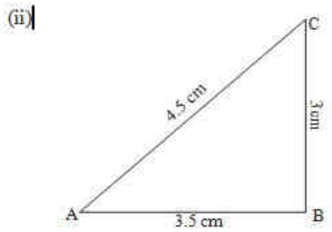


Fig 19.1



Solution: PerimeterofQuadrilateral$PQRS$

$= PQ + QR + RS + PS$

= 2.8 cm + 2.5 cm + 3.6cm + 3.0 cm

= 11.9cm

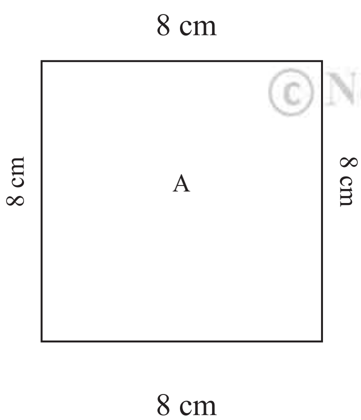
Part(ii)

PerimeterofΔABC =AB + BC + CA

= 3.5 cm + 3 cm + 4.5 cm

= 11.0cm

Example2. Findthe perimeterof followingfigures andidentity whichfigure has greaterperimeter?



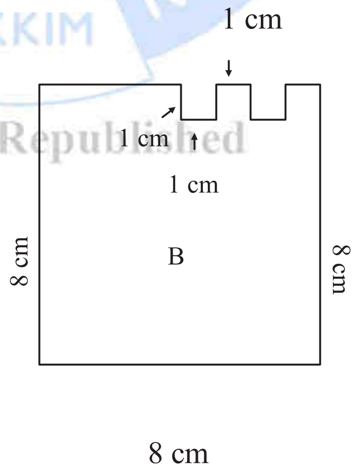


Fig 19.2

PerimeterofFigureA

= $8 \times 4 = 32$ cm

PerimeterofFigureA

=$8 + 8 + 8 + (8 + 1 + 1 + 1 + 1)$ cm = 36 cm

Here Figure 'B'has great perimeter

Weconcludethat

When a shape is cut of from given shape then the perimeter of new shape so obtained increased.

19.3AREA OF RECTILINEAR FIGURES

Thearea of a closedfigure is the spaceor region enclosed bythe closed figure.

19.3.1Rectangle

Drawarectangleoflength5cmandbreadth3cm.Drawtheverticallinesatadifferenceofonecentimetera longits length and 3 horizontal lines along its breadth. Howmany boxedare inside the entire rectangle.

5 cm

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

3 cm

There are 15unit square inside the Rectangle of 5 cm length and 3 cm breadth.Wecan writearea of rectangle= 5 cm× 3cm

Areaof rectangleis = Length× Breadth

Theunitofrectangle issquareunit.

19.3.2. Square

Lettheside ofsquarebe ‘a’uniti.e., its length is ‘a’ unit as well its breadth. Thus,Area of square =a unit × a unit=a^2squareunit

Wecanwrite

Areaofsquare=side^{2}sq.unit

19.3.3.Area ofParallelogram

Wehavealreadylearnttofindtheareaofrectangularregion.

Wealsoknowthatwhenaregionisconvertedintopieces,thenthesumoftheareasofallthepiecesmustbeequal to the area of given region. Ourobjective is to find the area of a parallelogram.

Bycuttingsomepartof theparallelogram andattaching itonthe othersidewewillfindthat

(i) Areaof theparallelogram = Base× height

(ii) Base of parallelogram=$\frac{Area}{Height}$

(iii) Height of parallelogram=$\frac{Area}{Base}$

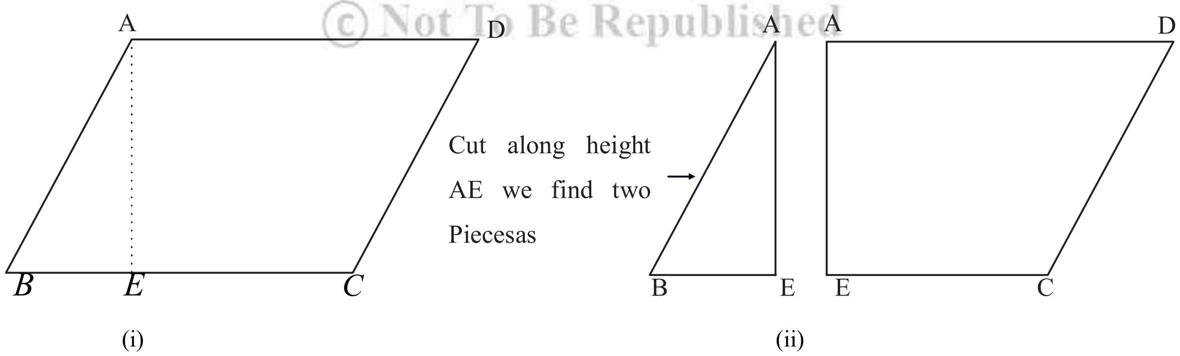


Fig 19.3

JointhesideasoversideDCwegettherectangleoflengthADand heightAE.

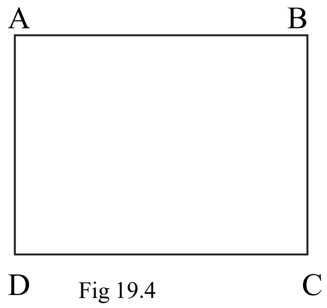


Fig 19.4

Remember: Anysideoftheparallelogramisknownasbaseofparallelogramandperpendiculardropped onthatsidefrom the opposite vertex to known as height.

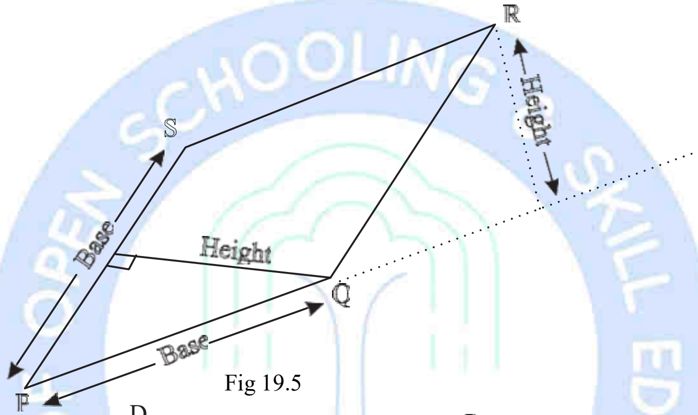


Fig 19.5

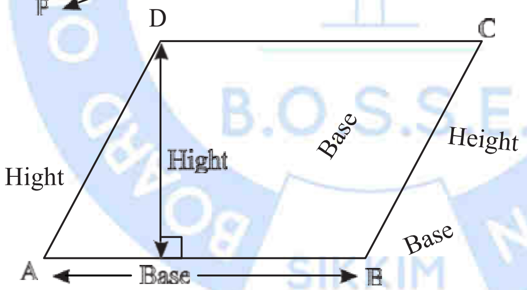


Fig 19.6

19.3.4. Area of Triangle

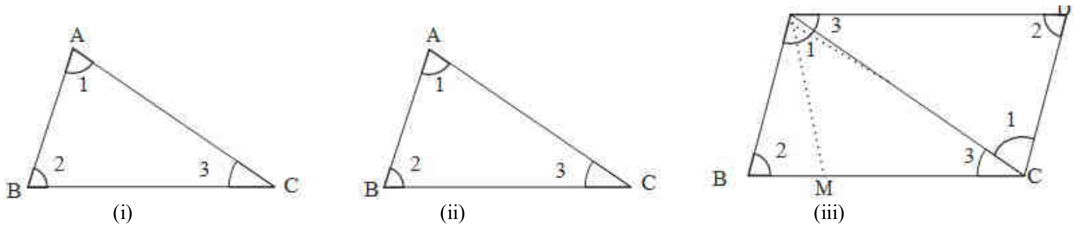


Fig 19.7

- Draw a triangle on a piece of paper

- Cutoff the triangular piece
- Place this triangle on another paper and cut out another triangular piece of same size.
- Now we have two triangular pieces of same size (same Area)
- The two triangular pieces are congruent.
- Nowplacethepiecessuchthattheir correspondingsidesarejoinedasshownin **Fig C**.

Observations

- We get a parallelogram
- The Area of each triangle is half of Area of Parallelogram.
- Also,

BaseofParallelogram=sideBCofΔABC

HeightofParallelogram= Correspondingattitude ofΔABC

AreaofParallelogram=Base×height =$BC \times AM$

Area of Traingle=$\frac{1}{2}$×Base×Height

Example19.3: Considerthesamefiguresgiveninexample2andfindtheareaineach.Alsocomparethea reaof both figures.

Solution: Fig (A) Sideofsquare=8cm

Areaofsquare=side^2

$=8^2$

=64 sq.cm

Fig(B)Areaoffig=Areaofsquare–Areaofcut regions

$=64cm^2 - 2\times 1^2$ [eachcutofarea1cm^2]

$=64cm^2 – 2cm^2$

$=62cm^2$

SofigureAhasgreaterArea.

Example19.4: FindtheAreaofgiven parallelogram

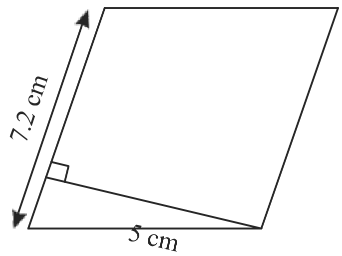


Fig 19.8

BaseofParallelogram=7.2cm

HeightofParallelogram=5cm

Area=base ×height

= 7.2 cm × 5 cm

$=36.0cm^2$

Example19.5: ThebaseofaParallelogramisthriceitsheight.Iftheareais867sq.cm.Findthebaseandheight of the Parallelogram.

Solution: AreaofParallelogram=base×height $=b \times h$

$b \times h = 867$ (given)

here$b=3h$ (given)

So $3h \times h = 867$

or $h^2 = \frac{867}{3}$

or $h^2=289$

$h^2 = 289 \therefore h = 17$cm

Given $b=3h \therefore b = 3 \times 17 = 51$

$\therefore h = 17$ cm and here$b = 51$ cm

CHECK YOUR PROGRESS 19.1

(i) Find the area of a square field whoseperimeter is 180 cm.

(ii) A door of dimension 2.5m × 1.5 m is on the wall of dimension 12 m×8.5m. Find the rate of painting the wall, if the rate of painting is Rs.3.75 per sqm.

(iii) Find the Area of shaded part.

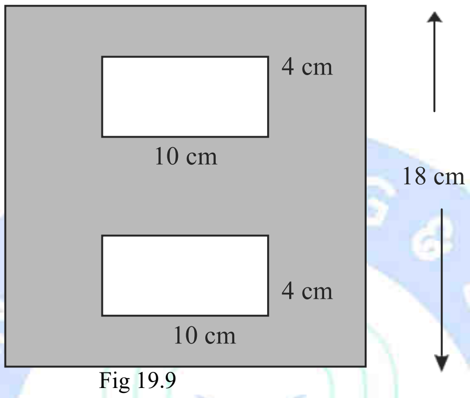


Fig 19.9

(iv) Two sides of a Parallelogram are 20 cm and 25 cm. If the attitude compounding to the sides of length 25cmis 10 cm. Find the attitude corresponding to the other pair of sides.

(v) Find the Area of Parallelogram in fig as PQ = 24 cm, ST=6 cm.

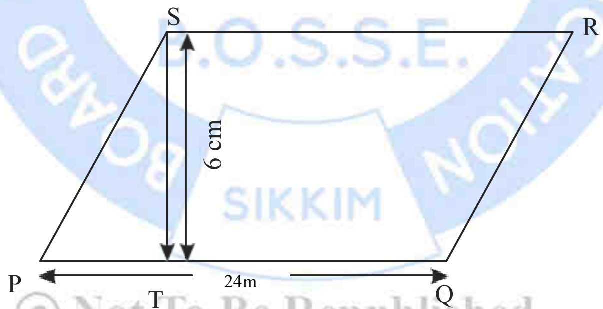


Fig 19.10

(vi) Findtheheight ofatriangle, whosebaseis 60cmand whoseareais 600sqcm.

(vii) Find unknownside 'x'ingivenfigure.

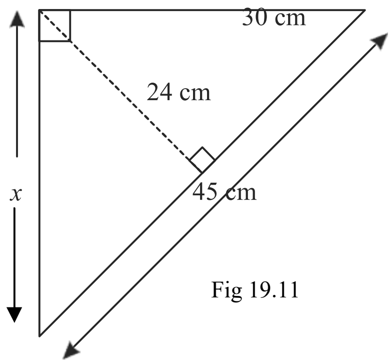


Fig 19.11

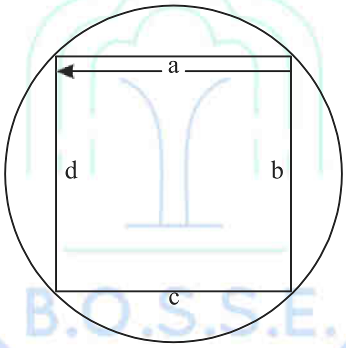
(viii) The two sides of right-angled triangle are in ratio of 3:4 and its area is 1014 sq cm. Find its hypotenuse.

19.4 BRAHMAGUPTASUTRA (HERON'S FORMULA)

Indian Mathematician Brahmagupta developed a theorem (Siddhanta) to find the area of trianglewhosethreesidesareknownbutheightisknown.Andalsoanotherresultofcyclicquadrilateralwhich is known as BrahmaguptaTheorem.

Thearea ofcyclic quadrilateralwithsides$a$, b,c, $d$isArea $=\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Where's'issemiperimeterof quadrilateral



Or $s = \dfrac{a+b+c+d}{2}$

Fig 19.12

19.4.1 Area of Triangle:

Brahmaguptasutratofindtheareaoftriangleisalsoknownas'Heron'sFormula'namedafteramathematicianofAlexandria.Accordingto thisformula—

Ifa, b,and$c$are themeasurement ofsidesof atriangleABC,wecancalculate itsArea byformula

Area$Q$= $\sqrt{s(s-a)(s-b)(s-c)}$

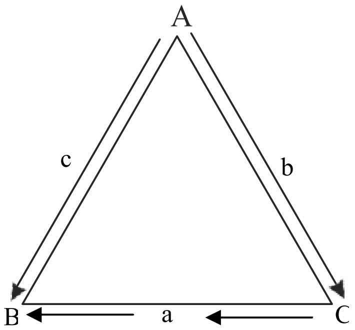


Fig 19.13

Here's' issemiperimeterof triangle(Unlikeothertriangleareaformulae,thereisnoneedtocalculateanglesorotherdistances(height) intriangle first.)

Example19.6: Findtheareaofanisoscelestriangleeachofwhoseequalsidesis13cmandwhosebaseis24 cm.

Solution: Heresidesoftriangle

a = 13 cm

b = 13 cm

c = 24 cm

Perimeter 25 = 13 + 13 + 24 = 50 cm

Semiperimeter $S = \frac{50}{2} = 25$cm

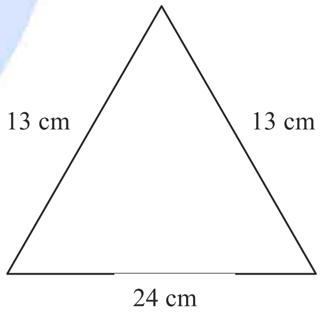


Fig 19.14

Now

$S - a = 25 - 13 = 12$ cm$S$

$- b = 25 - 13 = 12$ cm$S$

$- c = 25 - 24 = 1$ cm

Apply in gheron's formula area of$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Area= $\sqrt{25\times12\times12\times1}$

= $\sqrt{5^2\times12^2}$

$\sqrt{60^2}$

=

=60squarecm.

Example19.7: The sides of a triangle are 35cm, 54cm and 61cm. respectively. Find the length of its long estattitudes.

Solution: Here$a$=35 cm,b=54 cm,c=61 cm

$S = \frac{1}{2}(a + b + c) = \frac{1}{2}(35 + 54 + 61) = 75$ cm

$S - a = 40$ cm $s - b = 21$ cm $s - c = 14$ cm

Area ofΔ = $\sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{75 \times 40 \times 21 \times 14}$

$\Delta = \sqrt{3 \times 5 \times 5 \times 5 \times 8 \times 7 \times 3 \times 7 \times 2}$

$\Delta = \sqrt{5 \times 5 \times 5 \times 3 \times 3 \times 7 \times 7 \times 16}$

$\Delta = \sqrt{5^2 \times 3^2 \times 7^2 \times 4^2 \times 5}$

$\Delta = 5 \times 3 \times 7 \times 4 \sqrt{5} = 40 \sqrt{5}$ sq.cm

The area ofΔ =$420\sqrt{5}$ equationcm^2

Wehavetofindlongestattitude

Thelongestattitudewillbeonsmallestbase.

Hence 'h'be altitude onbase 'b'=35 cm

Area = $\frac{1}{2} \times b \times h = 420\sqrt{5}$

$= \frac{1}{2} \times 35 \times h = 420\sqrt{5}$

$h = 12 \times 2 \times \sqrt{5}$

$h = 24\sqrt{5}$ cm

19.5 AREA OF EQUILATERAL TRIANGLE

Leteachsideofequilateral trianglebe‘a’and ‘s’ issemi perimeter of equilateral triangle ,s=$\frac{3a}{2}$

then$s - a = s - b = s - c$

$= \frac{3a - a}{2}$

$= \frac{a}{2}$

ByHeron’sFormula

Areaof Δ=$\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\frac{3}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}}$

$= \sqrt{\frac{3a^4}{16}}$

$= \frac{\sqrt{3}}{4} a^2$

or

Areaofequilateraltriangleofside ‘a’=$\frac{\sqrt{3}}{4} a^2$ squnit

Example 19.8: The area of an equilateral triangle is$36\sqrt{3}$ cm^{2}find it perimeter

Solution: Letside ofequilateral triangle is‘a’its area be$\frac{\sqrt{3}}{4} a^2$

Given area=$36\sqrt{3}$ cm^2

$\therefore \frac{\sqrt{3}}{4} a^2 = 36\sqrt{3}$, $a^2 = 36 \times 4$

Ora= 6 × 2

Ora= 12 cm

Perimeterofequilateraltriangle=3a= 36 cm

CHECK YOUR PROGRESS 19.2

(i) Calculate the area of the shaded region in the given figure.

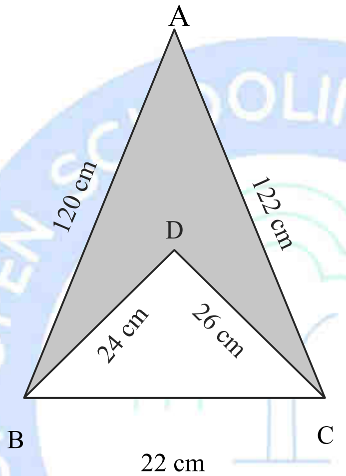


Fig 19.15

(ii) Find the area of a triangle, two sides of which are 8cm and 11cm and the perimeter is 32cm.

(iii) Sides of a triangle are in ratio 12:17:25 and its perimeter is 540cm. Find its area.

(iv) An equilateral triangle has perimeter 42cm. Find Area of triangle

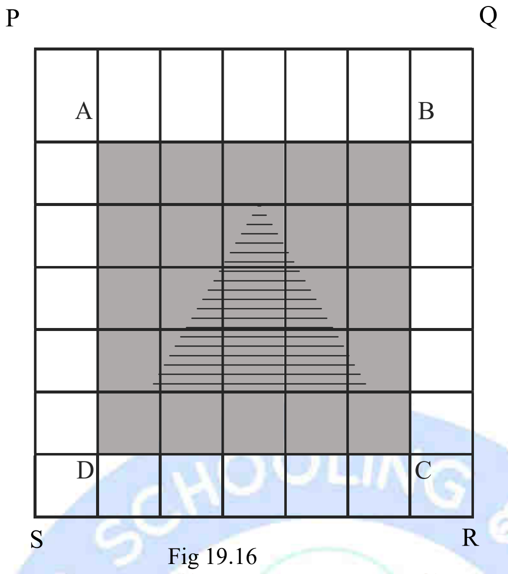


Fig 19.16

19.6AREASOFPATHSINRECTILINEARFIGURES

In this section, we shall apply the formula of area of arectangle and a square to determine the area of path around arectangleand a square.

⇒ Inthefigure below, Ifeachsquareblockis1cmlongand1 cm wide, we can find the area of the shaded regionjustbycounting thenumber ofsquaresinthe region.

Theshadedareais$5\times5=25cm^2$

Nowitiseasy tofindtheAreaofborder aroundtheshadedarea.

Letussupposetheshadedarea*ABCD*isagardenandthestriparounditisapathbetweengardensandboundarywall *PQRS*.

WehavetheArea ofPath=Areaof Rectangle*PQRS*–Areaofgarden*ABCD*

Whichisas$7\times7cm^2–5\times5cm^2$

$=49cm^2–25cm^2$

$=25cm^2$

Example19.9: Apathwayistobeconstructedaroundarectangulargrasslawnmeasuring30mby20m.Thewidthof path is 2 cm. Findthe area of the path.

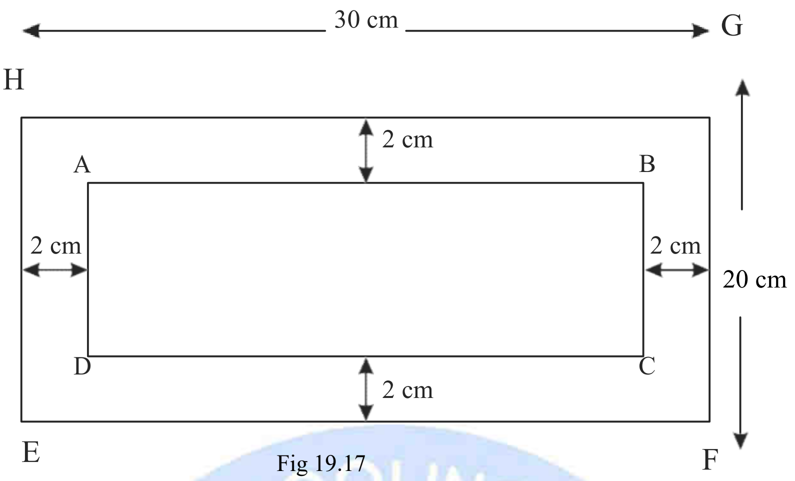


Fig 19.17

Solution: Areaofpath=Areaof*EFGH*–Areaof*ABCD*

Area of *EFGH*= 30 × 20 = 600 m^2 Areaof*ABCD*=(30–4)×(20–4)

= 26 × 16

=416m^2

Areaofpath=600m^2–416m^2

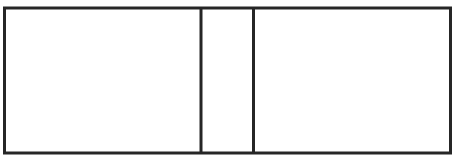
=184m^2

Example 19.10: Area of the remaining park, when two roads crosseach – otherand are parallel to the sidesofthe rectangle.

In the given figure *ABCD* is the rectangular park and *EFGH*and *PQRS* are the two roads parallel to the sides of the rectangleand crossing at the center such situation given rise two type ofproblems:

(i) Tofindareaofroad

(ii) To find area of remaining park



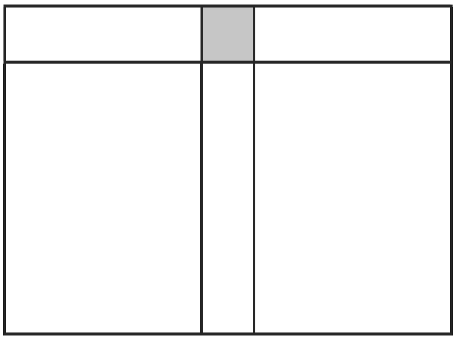


Fig 19.18

1. **Area of Roads**

=AreaofRectangle *EFGH*+AreaofrectanglePQRS–Areaofthe shadedpartofthe center

2. **Area of remaining part**

=Areaof park–Areaoftheroads

Example 19.11: The two cross roads each of 2 cm wide run at right angles through the centre of a rectangularpark 72 m by 48 m. Such that each is parallel to one of the sides of the rectangle. Find the area of the roads.Findthe area of remaining portion.

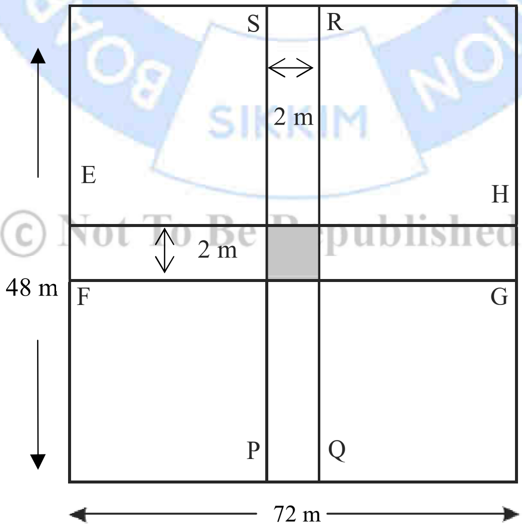


Fig 19.19

Area of park=72×48=3456m^2

Area of roads=Area of road *PQRS* + Area of Road *EFGH*–Area of shaded square

= 48 m × 2 m + 72 m × 2 m – 2 m × 2 m

=96m^2+142m^2=236m^2

Thus Area of remaining portion = Area of Park – Area of Roads

=3456m^2+236m^2

=3220m^2

TERMINAL EXERCISE

1. Apainting10 cm long and 5 cm wise, is to be pasted on a card board. Such that there is a margin of1.5 cm along each of its sides. Find the total area of margin.
2. A 3m circle path runs around outside a rectangular park of dimension 125 m by 65m. Find the area of the path.
3. A 115 m long and 64m broad lawn has two cross roads at right angles. One 2m wide running parallel to length and the other 2.5m wide, running parallel to its breadth. Find the area of road.
4. The side of a square flower bed is 1 m 8 cm. It is enlarged by digging a strip 20 cm wide all around it. Find

   (i) The area of enlarged flower bed.

   (ii) The increase in area of each flower bed.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 19.1

(i) 2025cm^2

(ii) Rs368.43

(iii) $244cm^2$

(iv) 12.5cm

(v) $144cm^2$

(vi) 20cm

(vii) 33.5cm

(viii) 65cm

CHECK YOUR PROGRESS 19.2

(i) 1074.08

(ii) $8\sqrt{30}\ cm^2$

(iii) $9000cm^2$

(iv) $84.87cm^2$

20 SURFACE AREA AND VOLUME

INTRODUCTION

You have dealt with shapes like rectangles and circles earlier. Draw these shapes in your notebook.

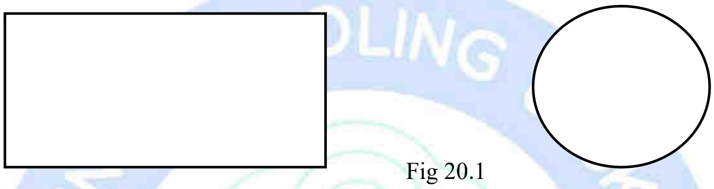


Fig 20.1

Now, look around you. Can you draw your table on a piece of paper? Try doing it.

20.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- visualize and construct three-dimensional solids
- recognize the relationship between two-dimensional shapes and three-dimensional solids and verify the technique of Reduction
- identify the concepts of Surface area and Volume for solids
- derive formulae for surface area and volume for common solids like cuboid, cylinder, cone, and sphere
- analyze the requirement and the fundamentals of integration
- apply and extrapolate the concepts learned.

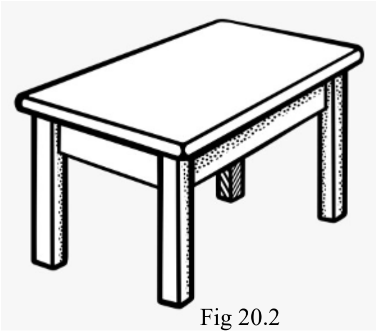


Fig 20.2

Did you notice any difference? Do you feel that the paper is inadequate to represent the table? Do you feel the need for something else, for *depth*?

As you may have noticed by now, there are different types of shapes. The ones we have dealt with till now have been in two dimensions. You only needed two perpendicular directions to draw them on paper. However, many things that you see around you are in three dimensions. You need an additional perpendicular direction – that of height or depth.

ACTIVITY 1:

Just for Fun!

Have you encountered the word 3-D anywhere? Have you heard of 3-D movies? Read up about them.

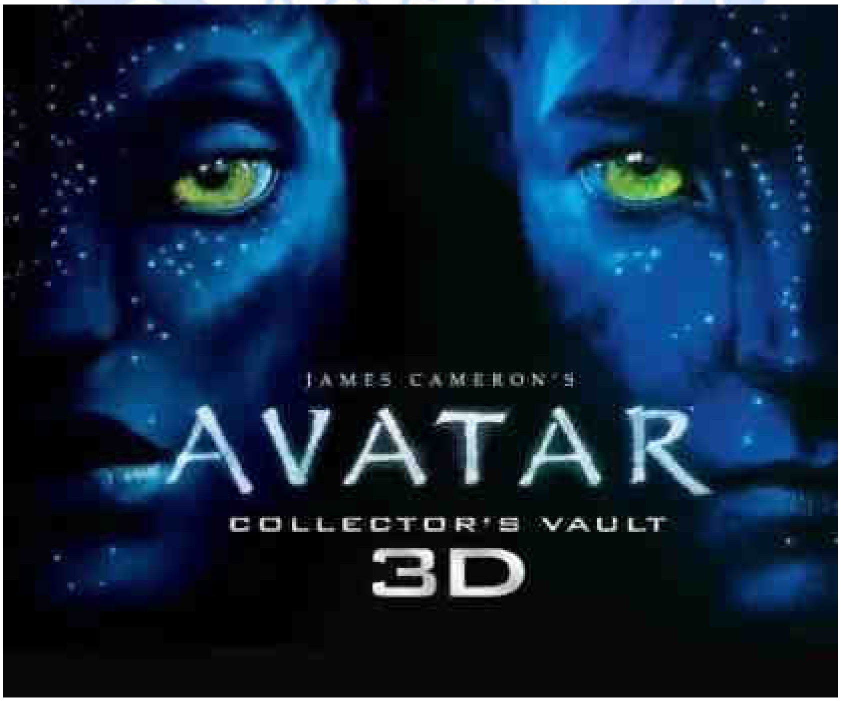


Fig 20.3

Let us now take a look at some 3-D shapes.

20.2 CUBOID – SURFACE AREA AND VOLUME

Imagine you had a rectangular piece of paper. Now, imagine that you stacked many such pieces of paper. What shape would you get?

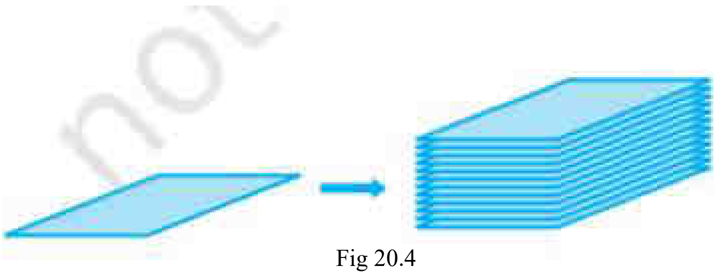


Fig 20.4

Let us construct a shape that covers this stack of papers.

Would it look something like this?

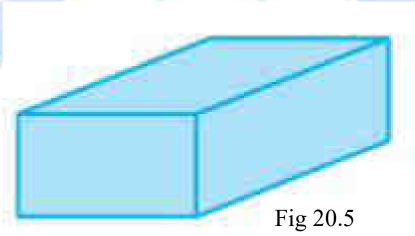


Fig 20.5

Let us try to take this shape apart to see what its constituent 2-D shapes are. This is a strategy we will follow throughout this chapter.

First, we have the rectangles at the bottom and the top. Let their dimensions be l and b, respectively. Then, we have the two rectangles at the side with lengths l and heights h, respectively. Finally, we have the two rectangles with dimensions b and h, respectively.

If we take the shape apart, we get:

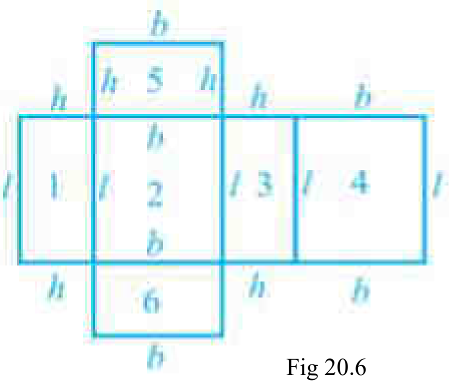


Fig 20.6

ACTIVITY 2:

Just for Fun!

Try making a cuboid box from a sheet of chart paper.

Fig 20.7

20.2.1 Surface Area of Cuboid

Now that we know what a cuboid is, can we find its surface area? How do you find the area of a rectangle?

Area of rectangle = length x breadth

In the case of a cuboid, we have already seen that it is composed of six rectangles of different dimensions. Can we say that the surface area of the cuboid is the sum of the individual surface areas of the rectangles that make the cuboid?

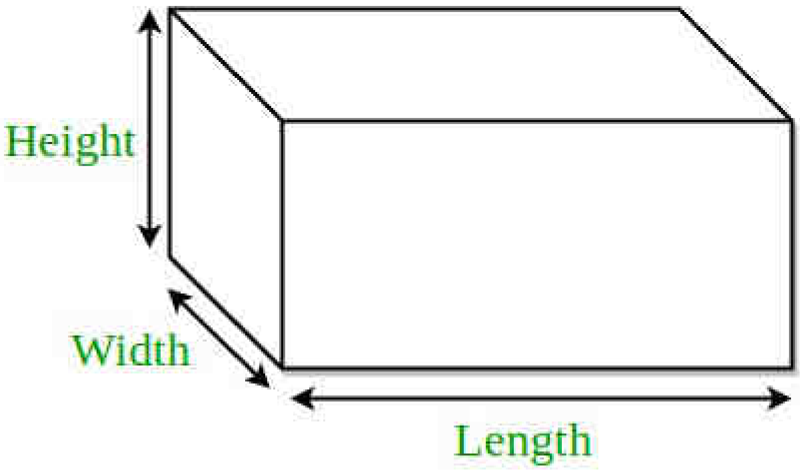


Fig 20.8

Thus, Area of cuboid = lb + lb + lh + lh + bh + bh

= 2(lb + lh + bh)

ACTIVITY 3:

Just for Fun!

Did you notice the technique we used? We reduced a three-dimensional problem to multiple two-dimensional ones to solve it. This technique is called Reduction. In further grades, you'll learn how Reduction works in areas like 3-D Geometry and Kinematics. Have you ever used it before?

Fig 20.9

20.2.2 Volume of Cuboid

In two dimensions, we learnt about perimeter and area.

Function	2-D	3-D
Covers	Perimeter	Surface Area
Fills	Area	?

If we think in these terms, the space that the 3-D shape occupies in space is called Volume. If you drop a pencil box inside a bucket of water, some of the water gets displaced to accommodate the pencil box. This amount of water is called the volume of the pencil box. In other words, if the box was hollow, this is the amount of water that would fill it completely.

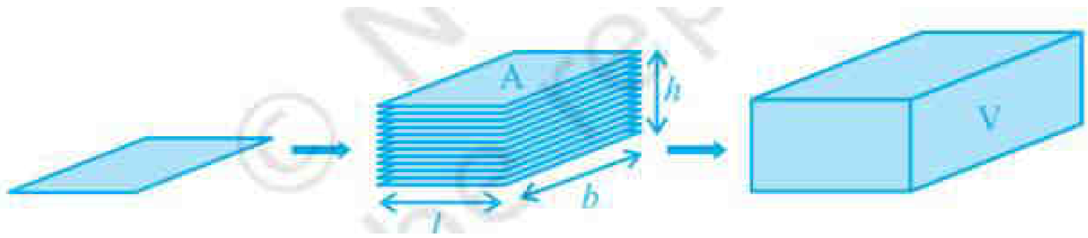


Fig 20.10

Consider that the area of each rectangle that makes up the cuboid is A. Let h be the height up to which the rectangles are stacked, and V be the volume of the rectangle.

Can we say that $V = A \times h$?

Is Volume simply adding another dimension to the surface area?

CHECK YOUR PROGRESS 20.1

1. If all three dimensions of a cuboid are equal (l = b = h = a), can you derive the surface area and volume of the resulting shape? Do you know what this shape is called? Have you seen it somewhere?

2. Fill in the blanks:

   $V = A \times h$

   Unit of Volume = Unit of area × Unit of height

   = ________ × ________

   = ________

3. Calculate the surface areas and volumes of the solids given below:

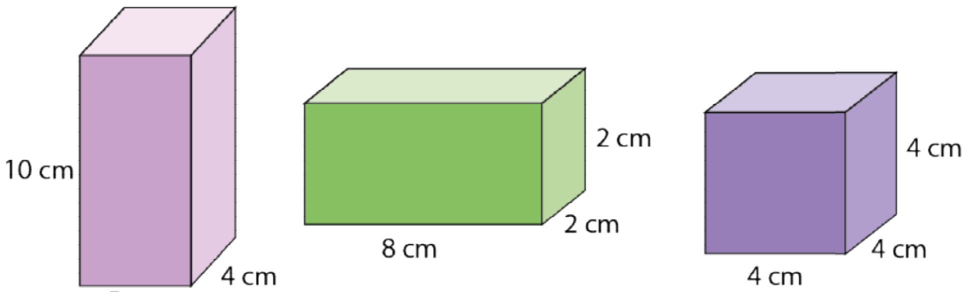


Fig 20.11

4. Say you have a cuboid, as shown below. What is the value of diagonal d? Using this example, can you generalize the value of d in terms of length, breadth and height of the cuboid? Can you do the same for a cube?

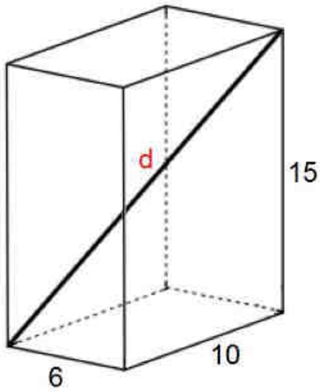


Fig 20.12

5. Can you find a cuboid (or cube) whose volume equals its surface area?

6. A painter is commissioned to paint a cuboidal building. The length, breadth and, height of the building are 55m, 20m, and 10m, respectively, and the painter is instructed to apply a coating of 1cm over the outer walls and the roof. How many tins of 13 liters does he need?(1 liter = 1000cm3)

ACTIVITY 4:

Just for Fun!

Look at the news article below. Do you think this sort of experiential learning would help you understand and appreciate the beauty of Mathematics better?

7.


This DIET Lecturer from Gangavathi makes Maths models from wedding invitations, wooden sticks for government school kids

Kavitha Diggavi, a DIET lecturer at Gangavathi has been working to make Maths concepts easy for students and teachers by making models from invitation cards, wooden sticks and broken window panes

Rashmi Patil
Edex Live

Fig 20.13

20.3 CYLINDER – SURFACE AREA AND VOLUME

Take a rectangular sheet of paper. Use adhesive tape to join the sheet breadthwise. Is the shape you get similar to the one given below?

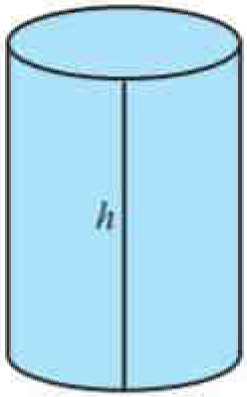


Fig 20.14

20.3.1 Surface Area of Cylinder

(Fill the blanks to complete the derivations.)

This shape is called a cylinder.

Now, is this cylinder open or closed? ________

The surface area of an open shape is called its curved surface area (CSA).

What is the curved surface area of the cylinder?

CSA of cylinder = Area of the rectangular sheet of paper (Why? Is this Reduction?)

Area of the rectangular sheet of paper = ________

For the cylinder, $b = h$, and l = Perimeter of the circular base = _______.

Thus, CSA of cylinder = $2\pi rh$

Now, to make the cylinder a closed shape, what do we need to add? The circular base and the top parts?

The surface area of a closed cylinder is called its total surface area (TSA).

TSA of cylinder = CSA of cylinder + Area of circular base + Area of circular top

= _______ + _______ + _______

$= 2\pi r(h + r)$

20.3.2 Volume of Cylinder

We are sure that you have realized what shape needs to be stacked to make up a cylinder.

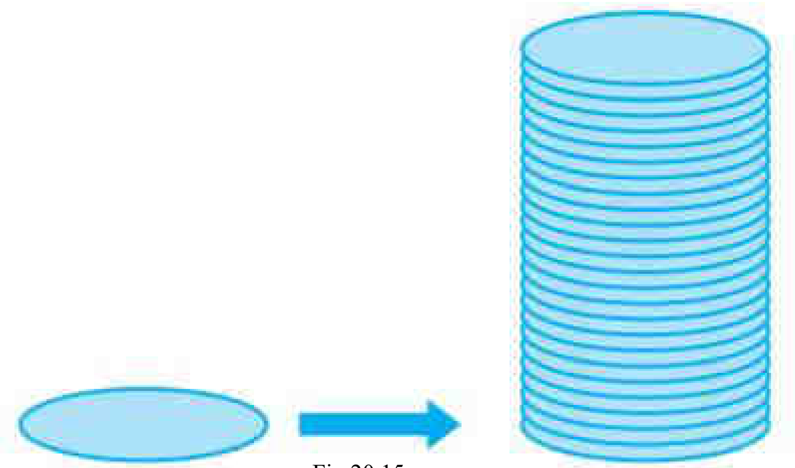


Fig 20.15

Can you figure out the Volume of this cylinder?

Is V = Area of circular base × height of the stacked circular disks?

Thus, $V = \pi r^2 h$

ACTIVITY 5:

Just for Fun!

Imagine you had a shape that looked something like this:

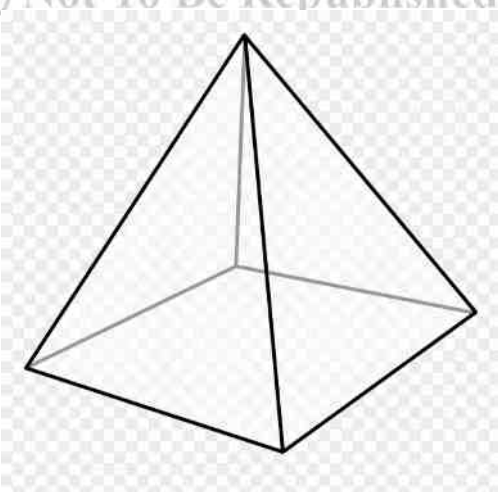


Fig 20.16

This shape is called a Pyramid. Would the Volume still be equal to Surface area times height? Why or why not?

CHECK YOUR PROGRESS 20.3

1. Look at the solids given below:

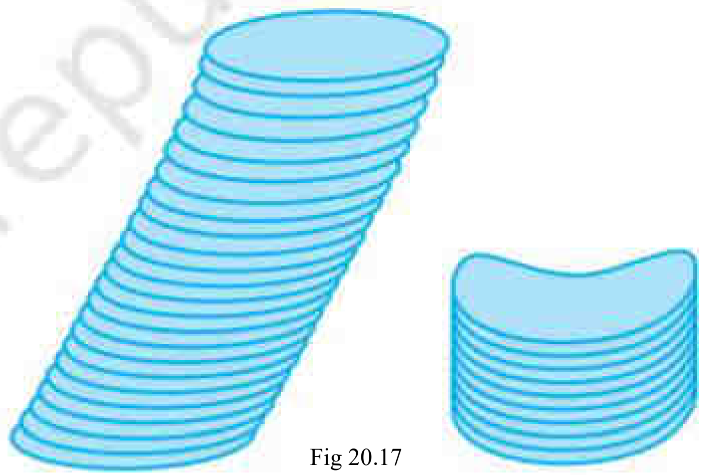


Fig 20.17

Can you calculate the surface areas and volumes of the solids above using the formulae for cylinder? Why or why not?

The solids given above are cylinders, but they are not right circular cylinders. We have been dealing with right circular cylinders up to this point in the chapter. A cylinder is a right circular cylinder if it has a circular base and is at right angles to the base.

2. An aluminum can manufacturing company is struggling to find an answer to a problem. The company has been experimenting with cylindrical-shaped cans and cuboidal-shaped cans. They want to use the most cost-effective shape out of the two. Imagine that the company created a cylindrical can with radius r and height h. They also made a cuboidal can with length r, breadth π (as nearly as possible), and height h. What is the condition for the cuboidal can to be more cost-effective than the cylindrical can?

3. What is the volume of cement required to cast the following solid?

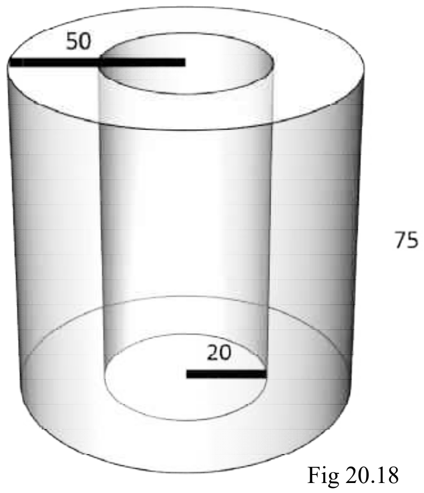


Fig 20.18

4. In a cylinder of 400cm3, how many cubes of 1cm3 can be fitted? What does this tell us about the unit cm3?

5. Look at the chart below:

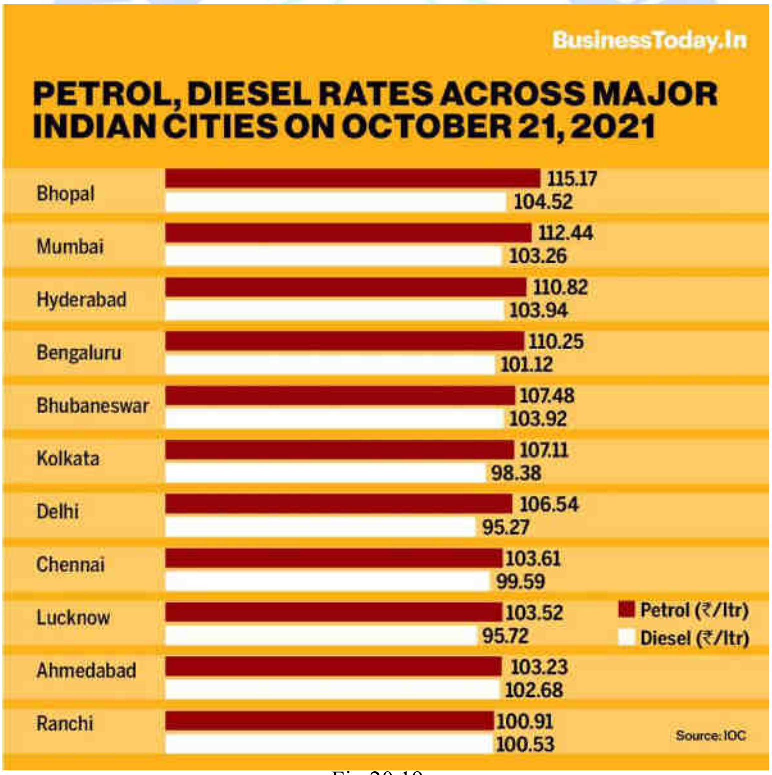


Fig 20.19

Mr. Sharma runs a bus service and wants to fill up his buses. He has approximately 10,005/- in cash. How many liters of petrol can he purchase in Bhopal and Lucknow, respectively? If he fills the petrol in a large tank of radius $r = 10m$, what is the height to which the petrol reaches in both cases?

20.4 CONE – SURFACE AREA AND VOLUME

What would happen if the areas of the circles that we stack while constructing a cylinder decrease uniformly?

Would we get the shape shown below?

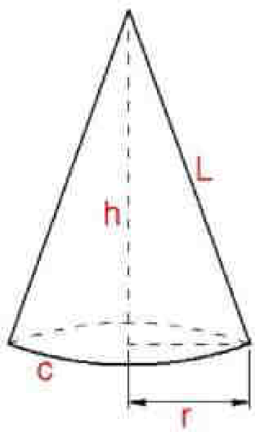


Fig 20.20

If we unfold the cone, we get a sector as shown below.

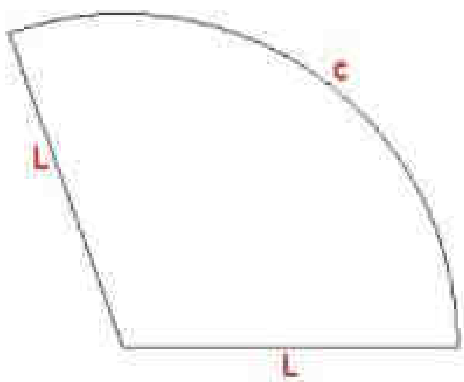


Fig 20.21

20.4.1 Surface Area of Cone

We know that Area of a sector $=\left(\dfrac{\theta}{360^{o}}\right)\times \pi r^{2}$

Also, Arc length of a sector $=\left(\dfrac{\theta}{360^{o}}\right)\times 2\pi r$

Thus, area of sector $=\dfrac{1}{2}\times\left(\text{Arc length}\times r\right)$

For the sector that we got by unfolding the cone, we get $Area = \frac{1}{2} \times (cL)$

Curved Surface Area of Cone = Area of the sector

$= \frac{1}{2}(cL)$

$= \frac{1}{2}(2\pi rL)$

$= \pi rL$

What is the total surface area (TSA) of the cone?

Is TSA = CSA + Area of the circular base?

Thus, TSA of cone $= \pi rL + \pi r^2$

$= \pi r(L + r)$

ACTIVITY 6:

Just for Fun!

Can you find the relation between *r*, *h*, and *L*? Here, r is the radius of the base of the cone, *h* is the height of the cone, and *L* is called the slant height of the cone. Write the formulae of Surface areas of the cone using r and h.

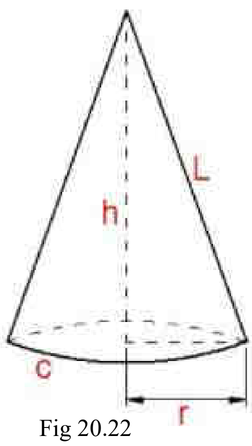


Fig 20.22

20.4.2 Volume of Cone

A cone is composed of circular disks of uniformly decreasing radius stacked up to height h. You saw earlier that the volume of such solids cannot be found by the technique we have been using till now.

We will not give the entire procedure to derive the formula here. You will learn about it in higher grades. For now, we will outline the procedure and provide links to a few experiments that you can do to verify the formula for yourself.

Say you have a cone as shown below:

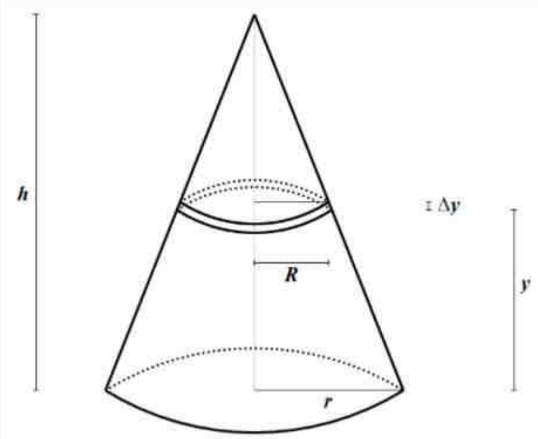


Fig 20.23

As shown in the figure, suppose you slice the cone to get a thin cross-section. The cross-section would be so thin that you essentially get a circular disk of radius R.

The Volume of this disk = $\pi R^2 \Delta y$

We have many such disks where R varies from 0 to r. Can you figure out how many?

Thus, we essentially add up the volumes of all these thin circular disks to get the volume of cone.

Volume of cone $= \frac{1}{3}\left(\pi r^2 h\right)$

ACTIVITY 7:

Just For Fun!

Suppose we give you a cylinder and a cone having the same base radius r and the same height h. Can you figure out the relation between their volumes?

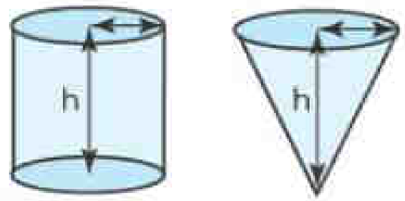


Fig 20.24

Credits: https://www.cuemath.com/measurement/volume-of-cone/

Fig 20.25

CHECK YOUR PROGRESS 20.4

1. Imagine you have a solid cone. Your sibling comes and slices the top of the cone, leaving you with the following solid:

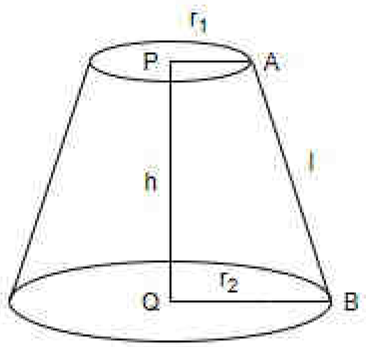


Fig 20.26

What is the CSA and TSA of this new solid if the height of the original cone was H, and its slant height was L? What is the volume of this new solid?

2. Children starting school are given a cone to celebrate their first school day. These school cones are filled with chocolates, sweets, and goodies.

   A mother wants to sew a cloth cone for her child. How much cloth would she need for a cone of height 30cm and radius 10cm?

3. A cone is made by rolling a metal sheet in the form of a sector with radius 36m and the central angle 150°. What is the CSA of the cone formed?

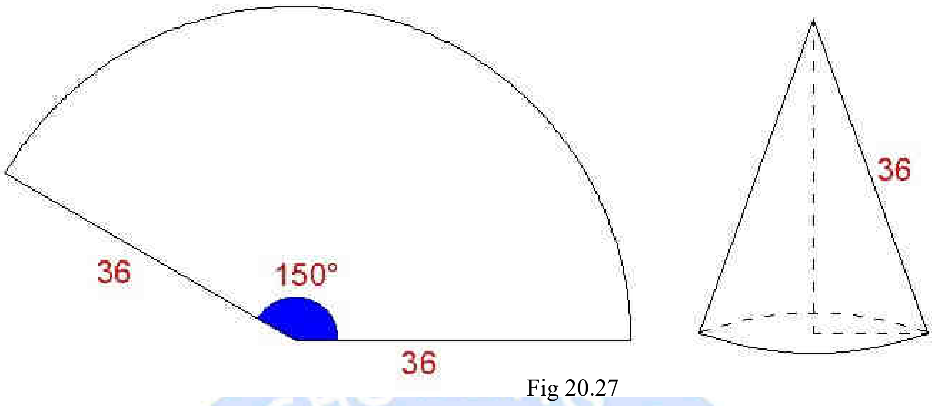


Fig 20.27

4. Amul's new ice cream contains a disc of chocolate on top of the ice cream, as shown below.

Fig 20.28

What is the volume of the chocolate if the radii of the disc are 4cm and 3cm, respectively, and its thickness is 3cm? Note that the total height of the ice cream is 15cm.

5. Look at the picture below:

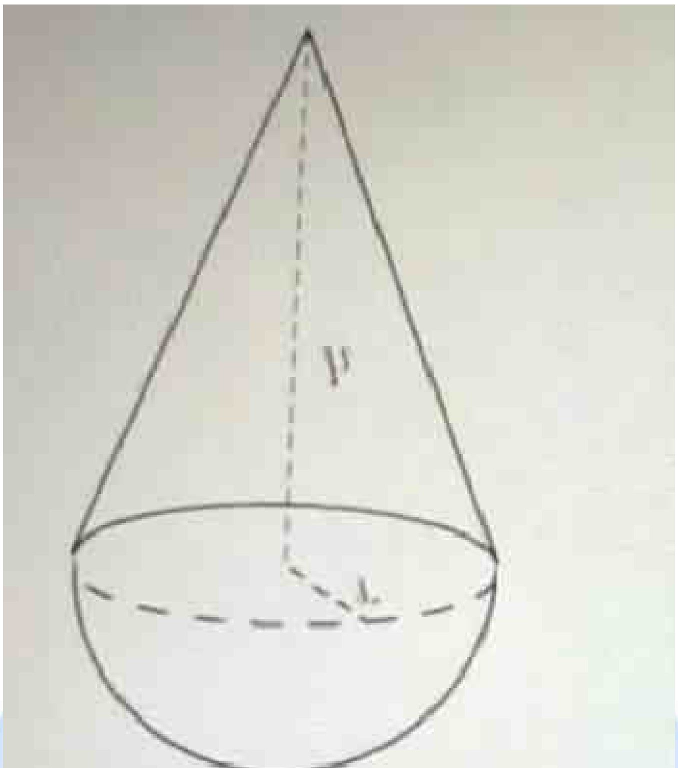


Fig 20.29

The volume of the hemisphere is $2094m^3$. What is the volume of the conical roof if its height is double its radius?

20.5 SPHERE – SURFACE AREA AND VOLUME

Draw a circle on a sheet of paper. Cut it out. Keep the circle horizontal and rotate it across a vertical axis perpendicular to it.

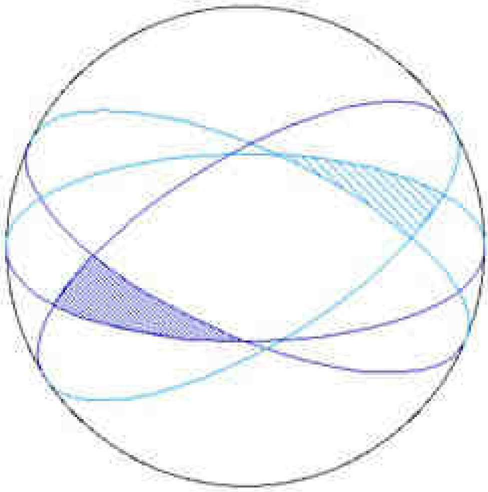


Fig 20.30

This solid that you get is called a sphere. In other words, a sphere is a collection of all those points in three dimensions which are at a fixed distance from a point, which is the center of the sphere. Does this definition seem similar to anything you have encountered before?

20.5.1 Surface Area of Sphere

Till now, you have been able to unfold all the solids you have encountered. Is that possible for a sphere?

To find the surface area of a sphere, let us go back in time to ancient Greece. Archimedes, one of the greatest Mathematicians of the age, was trying to figure out the same problem. Then, he discovered something that helped him solve the problem. He found that the surface area of a sphere of radius r was the same as the CSA of a cylinder of radius r and height 2r.

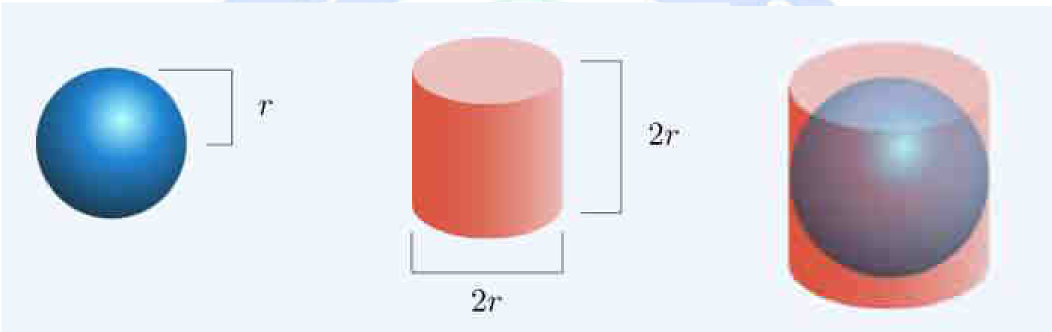


Fig 20.31

This might not seem intuitive, but we are not providing a complete proof here. You can attempt using the technique of slicing illustrated above to get closer to the proof.

Thus, Surface Area of Sphere = CSA of Cylinder

$= 2\pi rh$

$= 2\pi r\ (2r)$

$= 4\pi r^2$

ACTIVITY 8:

Just for Fun!

Suppose we have a hollow sphere, and we slice it into two halves. Would the resultant halves be open or closed shapes?

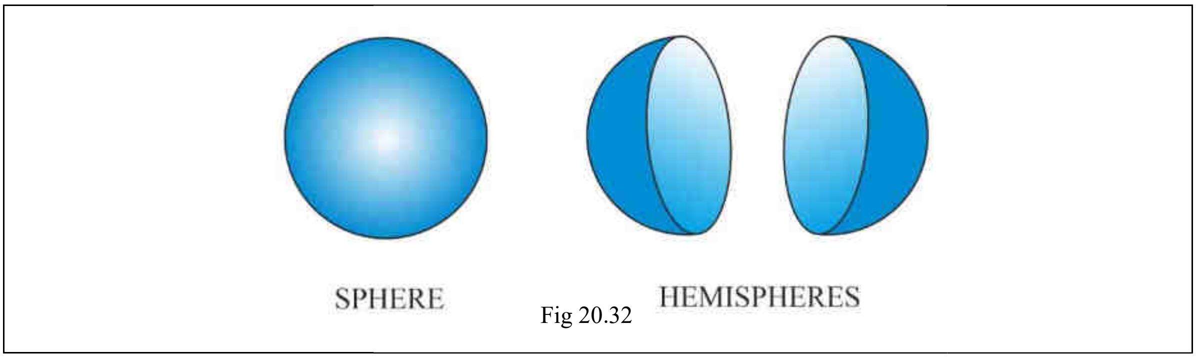


Fig 20.32

The shapes shown above are called hemispheres. What is the CSA of a hemisphere of radius r? What is its TSA?

20.5.2 Volume of Sphere

To find the volume of a sphere, we go back to Archimedes. Archimedes first imagined a hemisphere inside a cylinder.

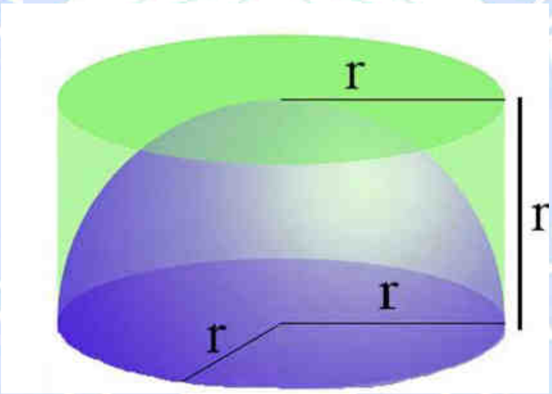


Fig 20.33

Archimedes then discovered that the volume unoccupied by the hemisphere was equal to the volume occupied by a cone of radius r and height r.

Thus, Volume of hemisphere = Volume of the cylinder – Volume of the cone

$= \pi r^3 - 1/3(\pi r^3)$

$= 2/3\ (\pi r^3)$

Hence, Volume of sphere = 2 x (Volume of hemisphere)

$= 4/3\ (\pi r^3)$

CHECK YOUR PROGRESS 20.4

1. Say you have a circular area of 64π sq m. If you install solar panels in that area, what shape will give you maximum surface area? Consider all the solids you have learned in this chapter, and take the direction of the sunlight to be perpendicular to the ground.
2. Look at the picture given below. This is Gol Gumbaz, the largest dome structure in India. If the dome's diameter is 45m, what is its CSA and its volume?

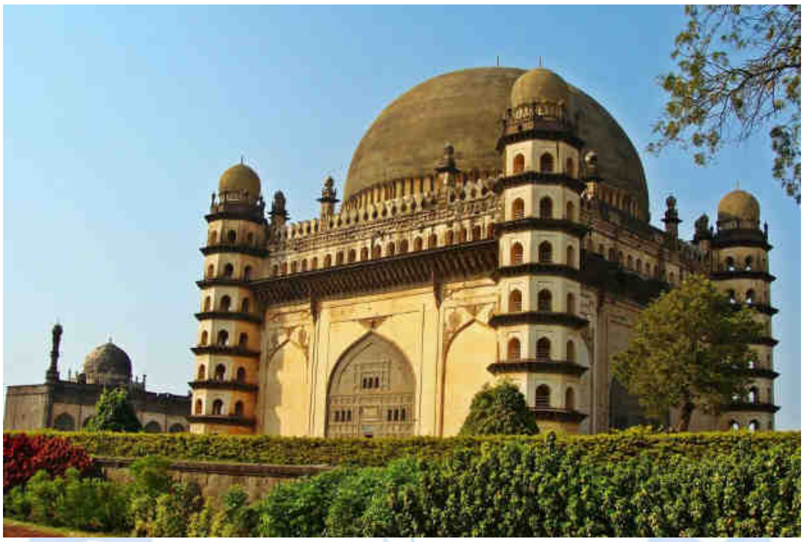


Fig 20.34

3. The average diameter of a single SARS-CoV-2 (the virus that causes COVID) is 100 nanometers. What is the Volume of a single SARS-CoV-2? If there are around 2×10^{17} virus particles globally, what is the volume that can accommodate all these particles?
4. You know that if you go on adding sides to polygons, you will approach a circular shape. Can you guess how that works for a sphere?
5. Consider a cube of volume $216 cm^3$. What is the surface area and volume of:

   (I) The sphere inscribed within the cube?

   (II) The sphere that circumscribes the cube?

RECAPITULATION POINTS

1. CUBOID

Volume = (l x b x h) cubic units.

Surface area = $2(lb + bh + lh)$ sq. units.

Diagonal = $l^2 + b^2 + h^2$ units.

2. **CUBE**

Volume = a^3 cubic units.

Surface area = $6a^2$ sq. units.

Diagonal = $3a$ units.

3. **CYLINDER**

Volume = $(\pi r^2 h)$ cubic units.

Curved surface area = $(2\pi rh)$ sq. units.

Total surface area = $2\pi r(h + r)$ sq. units.

4. **CONE**

Slant height, $l^2 = h^2 + r^2$ units.

Volume = $1/3\ (\pi r^2 h)$ cubic units.

Curved surface area = $(\pi r\mathrm{l})$ sq. units.

Total surface area = $(\pi r\mathrm{l} + \ r^2)$ sq. units.

5. **SPHERE**

Volume = $4/3\ (\pi r^3)$ cubic units.

Surface area = $(4\pi r^2)$ sq. units.

6. **HEMISPHERE**

Volume = $2/3\ (\pi r^3)$ cubic units.

Curved surface area = $(2\pi r^2)$ sq. units.

Total surface area = $(3\pi r^2)$ sq. units.

Note: 1 litre = 1000 cm^3

7. **We also learned about the techniques of Reduction and integration.**

TERMINAL EXERCISE

1. The mass of a wooden log is 30 kg. What is its volume if its density is 900 kg/m^3? What is its height if its radius is 4cm?
2. If the length, breadth and height of a cuboid are in the ratio 3:4:5, and its volume is $480cm^3$, find its dimensions and surface area.
3. Find the surface areas and volumes of all the 3-D objects we have encountered by keeping all important measures as 1cm.
4. The radius of a sphere is tripled. What is the ratio of the volume of the new sphere and the volume of the old sphere?
5. A metal sphere of $36\pi m^3$ is melted and casted into a cylinder of radius of radius $2m$. What is the height of this cylinder?
6. A cylindrical tent is covered with a conical roof. If the radius of the tent in $10m$, its height is $15m$, and the total volume is $100\ \pi\ m^3$, what is the height of the conical roof alone?

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 20.1:

1. Try inputting $l = b = h = a$ in the formulae for surface area and volume of a cuboid. The resulting shape is called a cube. You might have seen this shape as a popular toy in stores.
2. Unit of volume = cm^3. Does it now make sense to say that we are adding another dimension to the area to get the volume?
3. You can simply apply the formulae you have derived earlier.
4. Hail Pythagoras!
5. There is no simple method to find this. Try playing with the values to find an answer. How many solutions do you think there are?

CHECK YOUR PROGRESS 20.2:

1. Try finding the surface area to be painted and the resultant volume.

2. The first cylinder given is called an oblique cylinder. The formulae for a right circular cylinder cannot be applied to either of them.
3. Find the surface areas of the cans in both cases in terms of the variables provided. Then, apply the inequality to find the necessary condition.
4. Volume of the cement required = Volume of the outer cylinder – Volume of the inner cylinder.
5. The unit cm^3 is nothing but the volume of a cube having sides $1cm$. Thus, we have a physical representation of the unit of volume.

CHECK YOUR PROGRESS 20.3:

1. Find out the volumes of petrol in both cases. Then, you can try finding the height of the cylinder, which can accommodate the volumes in both cases.
2. Keep in mind the discarded portion of the cone. Use that to find the quantities for the new solid. This new solid is called a frustum.
3. You can simply apply the formula you have derived earlier.
4. Use your knowledge of the geometry of sectors to figure this one out.
5. You can simply apply the formula you have derived in Q3.

SUPPLEMENTARY STUDY MATERIAL

Activities on cones and sphere: https://tapintoteenminds.com/3act-math/cones-and-spheres/

Read more about Archimedes' work here: https://www.famousscientists.org/archimedes-makes-his-greatest-discovery/

Read about the role of air in ice-creams:
https://cosmosmagazine.com/science/chemistry/science-of-ice-cream/

What does this tell us about the ice cream question in the exercises?

For more explanation and exercises on surface area and volume, work through:
https://www.khanacademy.org/math/in-in-grade-10-ncert/x573d8ce20721c073:surface-areas-and-volumes

For more on the coronavirus, read: https://theprint.in/science/all-the-coronavirus-in-the-world-could-fit-inside-a-coke-can-with-room-to-spare/603998/

How much water does the earth actually have? https://mtstandard.com/news/science/this-wild-video-shows-all-of-earths-water-in-a-single-sphere/video_4c15cb1b-af78-590b-8b7f-8252187d705a.html

The Geometry of Social distancing: https://www.quantamagazine.org/the-math-of-social-distancing-is-a-lesson-in-geometry-20200713/

21 TRIGONOMETRY

INTRODUCTION

Knowledge of the trigonometrical ratio's sine, cosine and tangent, is vital in many fields of engineering, mathematics and physics. Trigonometry, as the name might suggest, is all about triangles.

More specifically, trigonometry is about right-angled triangles, where one of the internal angles is 90°. Trigonometry is a system that helps us to work out missing or unknown side lengths or angles in a triangle.

21.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- write the trigonometric ratios of an acute angle of right triangle;
- find the sides and angles of a right triangle when some of its sides and trigonometric ratios are known;
- write the relationships amongst trigonometric ratios;
- establish the trigonometric identities;
- solve problems based on trigonometric ratios and identities;
- find trigonometric ratios of complementary angles and solve problems based on these.

Expected Background Knowledge

- Concept of an angle
- Construction of right triangles
- Drawing parallel and perpendiculars lines
- Types of angles- acute, obtuse and right
- Types of triangles- isosceles and equilateral
- Complementary angles.

21.2 TRIGNOMETRY RATIOS

Trigonometric Ratios are applicable only for a right-angle triangle. A right-angle triangle is a special triangle in which one angle is 90^o and the other two are less than 90^o. Furthermore, each side of the right-angle triangle has a name.

- **Hypotenuse:** It is the largest side of the triangle. Also, it is opposite the right angle of the triangle.
- **Base:** The side on which the right-angle triangle stands is known as its base. Moreover, any of the two sides other than the hypotenuse can be chosen as the base for performing the calculation.
- **Perpendicular:** It is the side to the base of the right-angled triangle.

DEFINITION

Trigonometric ratios are the ratios of sides of a right-angle triangle. The most common trigonometric ratios are sine, cosine, and tangent.

Consider a right-angle triangle *ABC*, right-angled at *C*. In that case, side *AB* will be the hypotenuse. Also, if we chose *AC* as the base and BC as the perpendicular. Then, for □*BAC*, value of

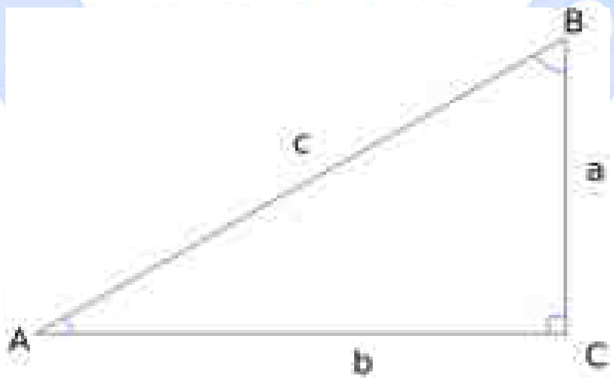


Fig 21.1

21.2.1 Concepts of Trigonometric Ratios

Fixing the base and perpendicular can be difficult sometimes. For example, in the triangle above,

For $\angle BAC, \sin\theta_1 = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{BC}{BA}$

But for $\angle ABC, \sin\theta_2 = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{AC}{AB}$

We will name different sides of the right-angled triangle as adjacent, opposite and hypotenuse.

- Adjacent: It is the side adjacent to the angle being considered.
- Opposite: It is the side opposite to angle being considered.
- Hypotenuse: It is the side opposite to the right angle of the triangle (or the largest side).

Now, formulas for ratios are as follows:θ

$$\text{Sine or Sin}\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\text{BC}}{\text{BA}} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos ine \text{ or } \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{tangent or } \tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\text{Opposite}}{\text{Adjacent}}$$

The reciprocal of sin, cos, and tan can also have names. Also, it's obvious that they are trigonometric ratios. They are as follows:

$$\text{cosecant or cosec}\theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{\text{Hypotenuse}}{\text{Opposite}}$$

$$\text{secant or sec}\theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$

$$\text{cotangent or cot}\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{\text{Adjacent}}{\text{opposite}}$$

Example21.1: Consider a right-angle triangle ABC right angled at C. If the hypotenuse $= AB = 5cm$, perpendicular $= BC = 4cm$ and base $= AC = 3cm$. Then, for $\angle BAC = \theta$, calculate the value of sinθ, cosθ and tanθ.

Solution:

For right angle triangle ABC,

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{4}{3}$$

Example 21.2: If tan $A = \frac{3}{4}$, then find the other trigonometric ratio of angle A.

Solution:

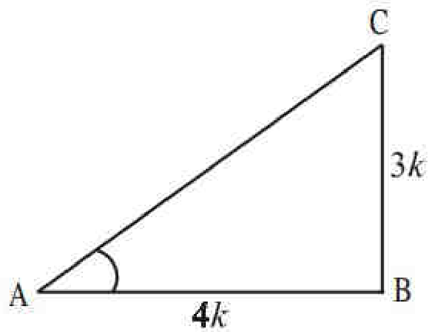


Fig 21.2

Given tan $A = \frac{3}{4}$

Hence tan $A = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{4}$

Therefore, opposite side: adjacent side = 3:4

For angle A, opposite side = $BC = 3k$

Adjacent side = AB = 4k (where k is any positive number)

Now, we have in triangle ABC (by Pythagoras theorem)

$$AC^2 = AB^2 + BC^2$$

$$= (3k)^2 + (4k)^2 = 25k^2$$

$$AC = \sqrt{25k^2}$$

$=5k$ = Hypotenuse

Now, we can easily write the other ratios of trigonometry

$$\sin A = \frac{3k}{5k} = \frac{3}{5} \text{ and } \cos A = \frac{4k}{5k} = \frac{4}{5} \text{ Sin } A$$

and also,

$$cosecA = \frac{1}{Sin\ A} = \frac{5}{3},\ secA = \frac{1}{cosA} = \frac{5}{4}, cotA = \frac{1}{tanA} = \frac{4}{3}.$$

Example 21.3: In ΔABC right angled at B, $AB = 24\ cm$, $BC = 7\ cm$. Determine

(i) sin A, cos$A$

(ii) sin$C$, cos$C$

Solution:

Applying Pythagoras theorem for ΔABC, we obtain

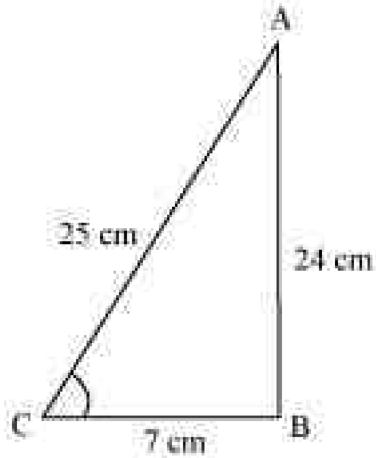


Fig 21.3

$$AC^2 = AB^2 + BC^2$$

$= (24\text{ cm})^2 + (7\text{ cm})^2$

$= (576 + 49)\text{ cm}^2$

$= 625\text{ cm}^2$

$$\therefore AC = \sqrt{625}cm = 25cm$$

(ii)

$$\sin C = \frac{\text{Side opposite}\angle C}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{24}{25}$$

$$\cos C = \frac{\text{Side adjacent to}\angle C}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{7}{25}$$

Example21.4: If sin $A = \frac{3}{4}$, calculate cos$A$ and tan$A$.

Solution:

Let ΔABC be a right-angled triangle, right-angled at point B.

Given that,

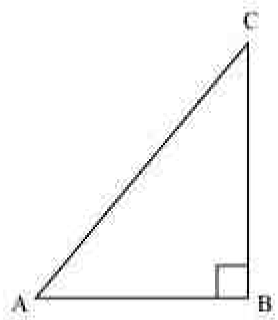


Fig 21.4

$$\sin A = \frac{3}{4}$$

$$\frac{BC}{AC} = \frac{3}{4}$$

Let BC be $3k$. Therefore, AC will be $4k$, where k is a positive integer.

Applying Pythagoras theorem in ΔABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$$(4k)^2 = AB^2 + (3k)^2$$

$$16k^2 - 9k^2 = AB^2$$

$$7k^2 = AB^2$$

$$AB = \sqrt{7}k$$

$$\cos A = \frac{\text{Side adjacent to} \angle A}{\text{Hypotenuse}}$$

$$= \frac{AB}{AC} = \frac{\sqrt{7}k}{4k} = \frac{\sqrt{7}}{4}$$

$$\tan A = \frac{\text{Side opposite to} \angle A}{\text{Side adjacent to} \angle A}$$

$$= \frac{BC}{AC} = \frac{3k}{\sqrt{7}k} = \frac{3}{\sqrt{7}}$$

Example 21.5: Given 15 cot$A$ = 8. Find sin$A$ and sec$A$

Solution:

Consider a right-angled triangle, right-angled at B.

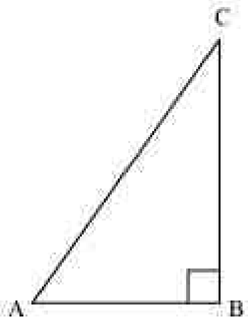


Fig 21.5

$$\text{Cot}A = \frac{\text{Side opposite to} \angle A}{\text{Side adjacent to} \angle A}$$

$$= \frac{AB}{BC}$$

It is given that,

$$\cot A = \frac{8}{15}$$
$$= \frac{AB}{BC} = \frac{8}{15}$$

Let AB be $8k$.Therefore, BC will be $15k$, where k is a positive integer.

Applying Pythagoras theorem in ΔABC, we obtain

$$AC^2 = AB^2 + BC^2$$

$$= (8k)^2 + (15k)^2$$

$$= 64k^2 + 225k^2$$

$$= 289k^2$$

$$AC = 17k$$

$$\sin A = \frac{\text{Side opposite to} \angle A}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$= \frac{15k}{17k} = \frac{15}{17}$$

$$\sec A = \frac{\text{Hypotenuse}}{\text{Side adjacent to} \angle A}$$

$$= \frac{AC}{AB} = \frac{17}{8}$$

CHECK YOUR PROGRESS 21.1

1. Given $\sec\theta = \frac{13}{5}$, calculate all other trigonometric ratios.

2. If $\cot\theta = \frac{7}{8}$, evaluate

$$\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

3. If 3 cot$A$ = 4, check whether $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2$ or not.

4. In ΔABC, right angled at B. If, $\tan A = \frac{1}{\sqrt{3}}$, find the value of

   (i) $\sin A \cos C + \cos A \sin C$

   (ii) $\cos A \cos C - \sin A \sin C$

5. In ΔPQR, right angled at Q, $PR + QR = 25\ cm$ and $PQ = 5\ cm$. Determine the values of $\sin P$, $\cos P$ and $\tan P$.

6. State whether the following are true or false. Justify your answer.

   (i) The value of tan A is always less than 1.

   (ii) $\sec A = \frac{12}{5}$ for some value of angle A.

   (iii) Cos$A$ is the abbreviation used for the cosecant of angle A.

   (iv) Cot$A$ is the product of cot and A.

   (v) $\sin\theta = \frac{4}{3}$, for some angle θ.

21.3 TRIGONOMETRIC RATIOS OF SOME SPECIFIC ANGLES

We already know about isosceles right-angle triangle and right-angle triangle with angles 30º, 60º and 90º.

Can we find sinv30º or tan60º or cos45º etc. with the help of these triangles? Does sin0º or cos0º exist?

Trigonometric ratios table helps to find the values of trigonometric standard angles such as 0°, 30°, 45°, 60° and 90°. It consists of trigonometric ratios – sine, cosine, tangent, cosecant, secant, cotangent. These ratios can be written in short as sin, cos, tan, cosec, sec and cot. The values of trigonometric ratios of standard angles are essential to solve the trigonometry problems. Therefore, it is necessary to remember the values of the trigonometric ratios of these standard angles.

$\angle$A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Example 21.6: Find the value of tan 45° + 2 cos60° $^{-}$ sec 60°.

Solution:

From the trigonometry table, $\tan 45^o = 1,\ \cos 60^o = \frac{1}{2}$ and $\sec 60^o = 2$

Therefore, $\tan 45^o + 2\cos 60^o - \sec 60^o = 1 + 2 \times \frac{1}{2} - 2 = 1 + 1 - 2 = 0$

Example 21.7: Find the value of sin 75°.

Solution:

We can write,

$\sin 75^o = \sin\left(45^o + 30^o\right) = \sin 45^o \cos 30^o + \cos 45^o \sin 30^o$

{Since, Sin(*A*+*B*)=Sin *A* Cos *B*+Cos *A* Sin *B*}

$= \frac{1}{\sqrt{2}} \times \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore$Sin 75°= $\frac{\sqrt{3}+1}{2\sqrt{2}}$

Example 21.8: Using the trigonometric table, evaluate $\sin^2 30^o + \cos^2 30^o$.

Solution:

By the trigonometric identities, we know that $\sin^2\theta + \cos^2\theta = 1$.

But let us prove this using the trigonometric table.

$$\sin^2 30^o + \cos^2 30^o = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1$$

Example21.9: Evaluate: sin 65° – cos 25°.

Solution:

We know, $\sin A = \cos(90° - A)$

So, sin 65° = cos (90° – 65°) = cos 25°

Therefore,

Sin 65° –cos 25° = cos 25° – cos 25° = 0

Example 21.10: Express cot75° + sin75° in terms of trigonometric ratios of angles between 0° and 45°.

Solution:

$$\cot 75° + \sin 75° = \cot(90° - 15°) + \sin(90° - 15°) = \tan 15° + \sin 15°$$

CHECK YOUR PROGRESS 21.2

1. Find the value of $2\sin^2 30° \tan 60° - 3\cos^2 60° \sec^2 30°$
2. Find the values of the following
   (i) $2\tan^2 45° + \cos^2 30 - \sin^2 60°$
   (ii) $4\cot^2 45° - \sec^2 60° + \text{cosec}^2 30° + \cot 90°$
3. Show that: sin 90° = 2cos 45° sin 45°
4. (i) If cos(40° + x) = sin 30°, find the value of x.
   (ii) If tan y = sin 45° cos 45° + sin 30°then find the value of y.
5. Find the values of the following.
   (i) cos 38°cos 52°−sin 38° sin 52°
   (ii) $\frac{\cos 80^o}{\sin 10^o}$ + cos 59° cosec 31°

21.4 GIVEN ONE TRIGONOMETRIC RATIO, TO FIND THE OTHERS

Example 21.11: In each of the following figures, ΔABC is a right triangle, right angled at B. Find all the trigonometric ratios of *θ*.

Solution:

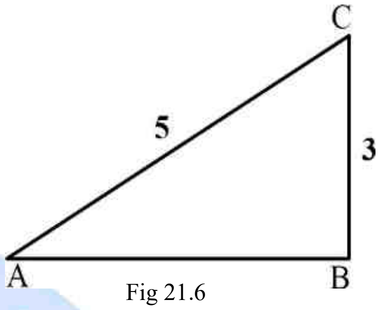


Fig 21.6

We have,

Sin $A = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{3}{5}$

So, we draw a right triangle right angled at B such that Perpendicular =BC=3 unit and, Hypotenuse =*AC*=5 units. By Pythagoras theorem, we have

$AC^2 = AB^2 + BC^2$

$\Rightarrow 5^2 = AB^2 + 32$

$\Rightarrow AB^2 = 25 - 9 = 16$

$\Rightarrow AB = 4$

When we consider the trigonometric ratios of ∠*C*, we have

Base =*BC*=3, Perpendicular=*AB*=4, and Hypotenuse =*AC*=5

Sin $C = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{4}{5}$

Cos $C = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{3}{5}$

Tan $C = \frac{\text{Perpendicular}}{\text{Base}} = \frac{4}{3}$

Cosec $C = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{5}{4}$

Sec C$= \frac{\text{Hypotenuse}}{\text{Base}} = \frac{5}{3}$

And,

Cot $C = \frac{\text{Base}}{\text{Perpendicular}} = \frac{3}{4}$

Example 21.12: Consider right triangle *ABC*, right angled at *B*. If *AC* = 17 units and *BC* = 8**.** units, determine all the trigonometric ratios of angle *C*.

Solution:

Length of the side $BC = 8$ units

Length of the side $AC = 17$ units

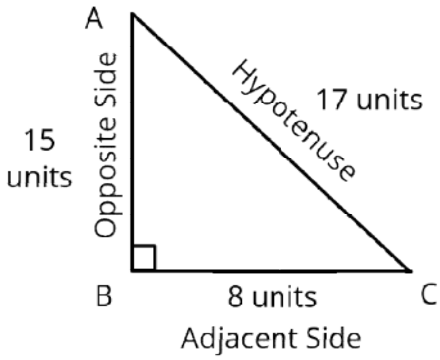


Fig 21.7

In ΔABC, using Pythagoras theorem,

$AB = \sqrt{AC^2 - BC^2} = \sqrt{17^2 - 8^2}$

$289 - 64 = 225$

$\Rightarrow AB = 15$ units

$\text{Sin } C = \frac{AB}{AC} = \frac{15}{17}$

$\text{Cos } C = \frac{BC}{AC} = \frac{8}{17}$

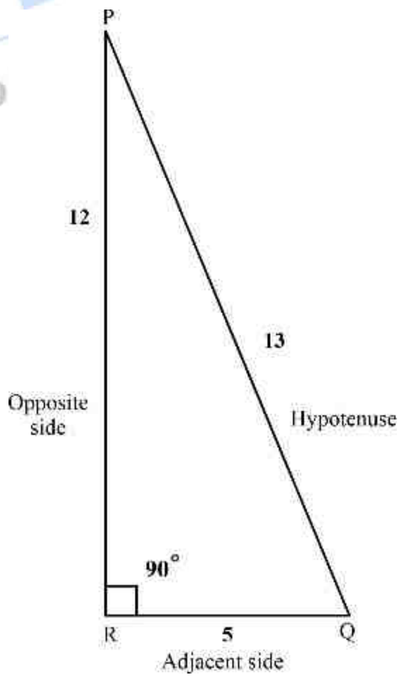
$\tan C = \frac{AB}{BC} = \frac{15}{8}$

$\cot C = \frac{1}{\tan C} = \frac{BC}{AB} = \frac{8}{15}$

$\sec C = \frac{1}{CosC} = \frac{AC}{BC} = \frac{17}{8}$

$\cos ecC = \frac{1}{SinC} = \frac{AC}{AB} = \frac{17}{15}$

Example 21.13: *PQR* is right triangle at *R*. If *PQ* = 13*cm* and *QR* = 5*cm*, which of the following is true?



(i) $\sin\theta + \cos\theta = \frac{17}{13}$

(ii) $\sin\theta - \cos\theta = \frac{17}{13}$

(iii) $\sin\theta + \sec\theta = \frac{17}{13}$

(iv) $\tan\theta + \cot\theta = \frac{17}{13}$

Solution:

Here $PR = \sqrt{PQ^2 + QR^2} = \sqrt{13^2 + 5^2} = \sqrt{144} = 12cm$

$\therefore \sin\theta = \frac{PR}{PQ} = \frac{12}{13}$ and $\cos\theta = \frac{QR}{PQ} = \frac{5}{13}$

Fig 21.8

$\therefore \sin\theta + \cos\theta = \frac{17}{13}$

Hence statement (i) i.e., $\therefore \sin\theta + \cos\theta = \frac{17}{13}$ is true.

CHECK YOUR PROGRESS 21.3

1. If sin A = 34, calculate cos$A$ and tan$A$.
2. Given 15 cot A = 8, find sin$A$ and sec$A$.
3. In Fig., find tan$P$ – cot$R$.

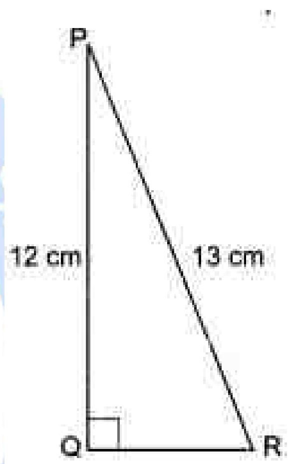


Fig 21.9

21.5 TRIGNOMETRIC IDENTITY

Definition of trigonometrical identity:

An equation which is true for all values of the variable involved is called an identity. An equation which involves trigonometric ratios of an angle and is true for all the values of the angle is called trigonometrical identities.

When the solutions of any trigonometric ratio problems represent the same expression in the L.H.S. and R.H.S. and the relation is satisfied for all the values of θ then such relation is called a trigonometrical identity.

Mutual relations among the trigonometrical ratios are generally used to establish the equality of such trigonometrical identities.

To solve different types of trigonometrical identity, follow the formula:

- $\sin\theta \cdot \csc\theta = 1 \Rightarrow \csc\theta = \frac{1}{\sin\theta}$
- $\cos\theta \cdot \sec\theta = 1 \Rightarrow \sec\theta = \frac{1}{\cos\theta}$

- $\tan\theta \cdot \cot\theta = 1 \Rightarrow \cot\theta = \frac{1}{\tan\theta}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- $\sin^2\theta$ implies $(\sin\theta)^2$ similarly, $\tan^3\theta$ means $(\tan\theta)^3$ etc.
- $\sin^2\theta + \cos^2\theta = 1$

  $\cos^2\theta = 1 - \sin^2\theta$

  $\sin^2\theta = 1 - \cos^2\theta$
- $\sec^2\theta = 1 - \tan^2\theta$

  $\sec^2\theta - \tan^2\theta = 1$

  $\tan^2\theta = \sec^2\theta - 1$
- $\csc^2\theta = 1 + \cot^2\theta$

  $\csc^2\theta - 1 = \cot^2\theta$

  $\csc^2\theta - \cot^2\theta = 1$
- The trigonometrical ratios of a positive acute angle θ are always non-negative and
  (i) Sinθ and cosθ can never be greater than 1;
  (ii) Secθ and cscθ can never be less than 1;
  (iii) Tanθ and cotθ can have any value.

Example 21.14:Proof the identity:

$$\tan^2\theta - \frac{1}{\cos^2\theta} + 1 = 0$$

Solution:

L.H.S $\tan^2\theta - \frac{1}{\cos^2\theta} + 1$

$= \tan^2\theta - \sec^2\theta + 1 \left[\text{since, } \frac{1}{\cos\theta} = \sec\theta\right]$

$= \tan^2\theta - \left(1 + \tan^2\theta\right) + 1 \ \left[\text{since, } \sec^2\theta = 1 + \tan^2\theta\right]$

$= \tan^2\theta - 1 - \tan^2\theta + 1$

= 0 = **R.H.S. Proved**

Example 21.15: Verify that:

$1/(\sin\theta + \cos\theta) + 1/(\sin\theta - \cos\theta) = 2\sin\theta/(1 - 2\cos^2\theta)$

Solution:

L.H.S $= 1/(\sin\theta + \cos\theta) + 1/(\sin\theta - \cos\theta)$

$= [(\sin\theta - \cos\theta) + (\sin\theta + \cos\theta)]/(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$

$= [\sin\theta - \cos\theta + \sin\theta + \cos\theta]/(\sin^2\theta - \cos^2\theta)$

$= 2\sin\theta/[(1 - \cos^2\theta) - \cos^2\theta]$ [since, $\sin^2\theta = 1 - \cos^2\theta$]

$= 2\sin\theta/[1 - \cos^2\theta - \cos^2\theta]$

$= 2\sin\theta/[1 - 2\cos^2\theta]$ = **R.H.S. *Proved***

Example 21.16: Prove that:

$\sec^2\theta + \csc^2\theta = \sec^2\theta \cdot \csc^2\theta$

Solution:

L.H.S. $= \sec^2\theta + \csc^2\theta$

$= 1/\cos^2\theta + 1/\sin^2\theta$ [since, $\sec\theta = 1/\cos\theta$ and $\csc\theta = 1/\sin\theta$]

$= (\sin^2\theta + \cos^2\theta)/(\cos^2\theta\sin^2\theta)$

$= 1/\cos^2\theta \cdot \sin^2\theta$ [since, $\sin^2\theta + \cos^2\theta = 1$]

$= 1/\cos^2\theta \cdot 1/\sin^2\theta$

$= \sec^2\theta \cdot \csc^2\theta$ = **R.H.S. *Proved***

Example 21.17: Prove the identity

$\cos\theta/(1 + \sin\theta) = (1 + \cos\theta - \sin\theta)/(1 + \cos\theta + \sin\theta)$

Solution:

R. H. S. $= (1 + \cos\theta - \sin\theta)/(1 + \cos\theta + \sin\theta)$

$= \{(1 + \cos\theta - \sin\theta)(1 + \cos\theta + \sin\theta)\}/\{(1 + \cos\theta + \sin\theta)(1 + \cos\theta + \sin\theta)\}$ [multiplying both numerator and denominator by $(1 + \cos\theta + \sin\theta)$]

$= \{(1 + \cos\theta)^2 - \sin^2\theta\}/(1 + \cos\theta + \sin\theta)^2$

$= (1 + \cos^2\theta + 2\cos\theta - \sin^2\theta)/\{(1 + \cos\theta)^2 + 2 \cdot (1 + \cos\theta)\sin\theta + \sin^2\theta\}$

$= (\cos^2\theta + 2\cos\theta + 1 - \sin^2\theta)/\{1 + \cos^2\theta + 2\cos\theta + 2\cdot(1 + \cos\theta)\cdot\sin\theta + \sin^2\theta\}$

$= (\cos^2\theta + 2\cos\theta + \cos^2\theta)/\{2 + 2\cos\theta + 2\cdot(1 + \cos\theta)\cdot\sin\theta\}$ [since, $\sin^2\theta + \cos^2\theta = 1$ and $1 - \sin^2\theta = \cos^2\theta$]

$= \{2\cos\theta\,(1 + \cos\theta)\}/\{2\,(1 + \cos\theta)(1 + \sin\theta)\}$

$= \cos\theta/(1 + \sin\theta)$ = **L.H.S.** ***Proved***

Example 21.18: Verify the trigonometrical identity:

$(\cot\theta + \csc\theta - 1)/(\cot\theta - \csc\theta + 1) = (1 + \cos\theta)/\sin\theta$

L.H.S. = $(\cot\theta + \csc\theta - 1)/(\cot\theta - \csc\theta + 1)$

$= \{\cot\theta + \csc\theta - (\csc^2\theta - \cot^2\theta)\}/(\cot\theta - \csc\theta + 1)$[$\csc^2\theta = 1 + \cot^2\theta \Rightarrow \csc^2\theta - \cot^2\theta = 1$]

$= \{(\cot\theta + \csc\theta) - (\csc\theta + \cot\theta)(\csc\theta - \cot\theta)\}/(\cot\theta - \csc\theta + 1)$

$= \{(\cot\theta + \csc\theta)(1 - \csc\theta + \cot\theta)\}/(1 - \csc\theta + \cot\theta)$

$= \cot\theta + \csc\theta$

$= (\cos\theta/\sin\theta) + (1/\sin\theta)$

$= (1 + \cos\theta)/\sin\theta$ = **R.H.S.** ***Proved***

Example 21.19: Find the value of $1 - \text{Sin}^2 A$.

Solution:

Using the identity, $\sin^2 A + \cos^2 A = 1$

Substituting the value of 1 in the given equation,

$1 - \sin^2 A = \left(\sin^2 A + \cos^2 A\right) - \sin^2 A \;= \cos^2 A$

Example 21.20: Prove that

$\sec^2 P - \tan^2 P - \cos ec^2 P + \cot^2 P = \sec^2 P - \tan^2 P - \cos ec^2 P + \cot^2 P = 0.$

Solution:

Using the identities,

- $\sec^2 a = 1 + \tan^2 a$
- $\cos ec^2 a = 1 + \cot^2 a$

Substituting the value of $\sec^2 a$ and $\text{cosec}^2 a$ in the given equation,

$$\sec^2 P - \tan^2 P - \text{cosec}^2 P + \cot^2 P = 1 + \tan^2 P - \tan^2 P - 1 + \cot^2 P + \cot^2 P$$
$$= 1 + 0 - 1 - \cot^2 P + \cot^2 P = 0$$

Example 21.21: Prove $(1 - \sin A)/(1 + \sin A) = (\sec A - \tan A)^2$

Solution:

L.H.S = $(1 - \sin A)/(1 + \sin A)$

$= (1 - \sin A)^2/(1 - \sin A)(1 + \sin A)$,

Multiplying both numerator and denominator by $(1 - \sin A)$

$= (1 - \sin A)^2/(1 - \sin^2 A)$

$= (1 - \sin A)^2/(\cos^2 A)$, [Since $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$]

$= \{(1 - \sin A)/\cos A\}^2$

$= (1/\cos A - \sin A/\cos A)^2$

$= (\sec A - \tan A)^2$ = **R.H.S.** ***Proved.***

Example 21.22: Prove that, $\sqrt{\{(\sec\theta - 1)/(\sec\theta + 1)\}} = \text{cosec}\,\theta - \cot\theta$.

Solution:

L.H.S.= $\sqrt{\{(\sec\theta - 1)/(\sec\theta + 1)\}}$

$= \sqrt{[\{(\sec\theta - 1)(\sec\theta - 1)\}/\{(\sec\theta + 1)(\sec\theta - 1)\}]}$;

[multiplying numerator and denominator by $(\sec\theta - 1)$ under radical sign]

$= \sqrt{\{(\sec\theta - 1)^2/(\sec^2\theta - 1)\}}$

$= \sqrt{\{(\sec\theta - 1)^2/\tan^2\theta\}}$; [since, $\sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta - 1 = \tan^2\theta$]

$= (\sec\theta - 1)/\tan\theta$

$= (\sec\theta/\tan\theta) - (1/\tan\theta)$

$= \{(1/\cos\theta)/(\sin\theta/\cos\theta)\} - \cot\theta$

$= \{(1/\cos\theta) \times (\cos\theta/\sin\theta)\} - \cot\theta$

$= (1/\sin\theta) - \cot\theta$

$= \text{cosec}\ \theta - \cot\theta =$ **R.H.S.** ***Proved.***

Example 21.23: $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

Solution:

L.H.S $= \tan^4\theta + \tan^2\theta$

$= \tan^2\theta\ (\tan^2\theta + 1)$

$= (\sec^2\theta - 1)\ (\tan^2\theta + 1)$ [since, $\tan^2\theta = \sec^2\theta - 1$]

$= (\sec^2\theta - 1)\ \sec^2\theta$ [since, $\tan^2\theta + 1 = \sec^2\theta$]

$= \sec^4\theta - \sec^2\theta =$ **R.H.S.** ***Proved.***

NOTE: It is noted that, these trigonometry identities are the basis of trigonometry and can be used to express each trigonometric ratio in terms of other trigonometric ratios. Also, if you know the value of any trigonometry ratio, then you can find the values of other trigonometric ratios also.

CHECK YOUR PROGRESS 21.4

(i) $\cos\theta/(1 - \tan\theta) + \sin\theta/(1 - \cot\theta) = \sin\theta + \cos\theta$

(ii) Show that, $1/(\csc A - \cot A) - 1/\sin A = 1/\sin A - 1/(\csc A + \cot A)$

(iii) $(\tan\theta + \sec\theta - 1)/(\tan\theta - \sec\theta + 1) = (1 + \sin\theta)/\cos\theta$

(iv) If $x = a\cos\theta$, $y = b\sin\theta$, then find the value of $b^2x^2 + a^2y^2 - a^2b^2$.

(v) If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.

(vi) Prove that $1-\sin\theta/1+\sin\theta = (\sec\theta - \tan\theta)^2$

(vii) Prove that: $(\text{cosec}\ \theta - \cot\theta)^2 = 1-\cos\theta 1+\cos\theta$

(viii) Prove that: $(\sin A + \text{cosec}\ A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$.

(ix) If $3\cot A = 4$, check whether $1-\tan^2 A/1+\tan^2 A = \cos^2 A - \sin^2 A$ or not.

21.6 TRIGONOMETRY RATIOS OF COMPLEMENTARY ANGLES

If the sum of two angles is one right angle or 90°, then one angle is said to be complementary of the other. Thus, 25° and 65°; θ° and $(90 - \theta)$° are complementary to each other.

Suppose a rotating line rotates about O in the anti-clockwise sense and starting from its initial position.

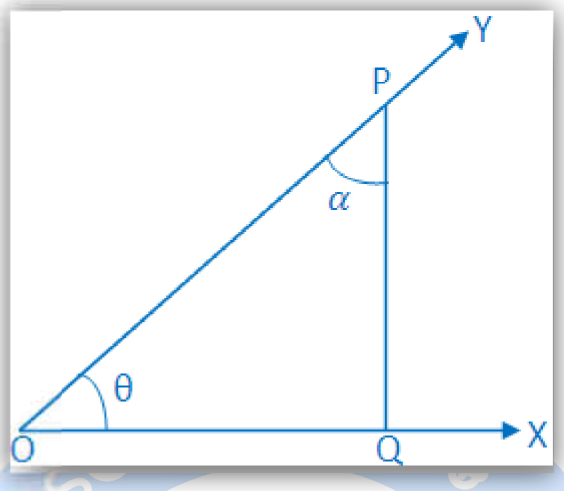


Fig 21.10

Angle $\angle XOY = \theta$, where θ is acute.

Take a point P on OY and draw PQ perpendicular to OX. Let, $\angle OPQ = \alpha$. Then, we have,

$\alpha + \theta = 90°$

or, $\alpha = 90° - \theta$.

Therefore, θ and α are complementary to each other.

Now, by the definition of trigonometric ratio,

$\sin\theta = \frac{PQ}{OP}$ ……………… (i)

$\cos\theta = \frac{OQ}{OP}$ ……………… (ii)

$\tan\theta = \frac{PQ}{OQ}$ ……………(iii)

And, $\sin\theta = \frac{OQ}{OP}$ …..……… (iv)

$\cos\theta = \frac{PQ}{OP}$ ……………..… (v)

$\tan\theta = \frac{OQ}{PQ}$ ……………….. (vi)

From (i) and (iv) we have,

sin α = cos θ

or, sin (90° –θ) = cos θ;

From (ii) and (v) we have,

cos α = sin θ

or, cos (90° – θ) = sin θ;

From (iii) and (vi) we have,

And $\tan\alpha = \dfrac{1}{\tan\theta}$

or, tan (90° –θ) = cot θ.

Similarly, csc (90° –θ) = sec θ;

sec (90° –θ) = csc θ

and cot (90° –θ) = tan θ.

Therefore,

Sine of any angle = cosine of its complementary angle;

Cosine of any angle = sine of its complementary angle;

Tangent of any angle = cotangent of its complementary angle.

Corollary:

Complementary Angles: Two angles are said to be complementary if their sum is 90°. Thus,θ and (90° –θ) are complementary angles.

(i) sin (90° – θ) = cos θ (ii) cos (90° – θ) = sin θ

(iii) tan (90° – θ) = cot θ (iv) cot (90° – θ) = tan θ

(v) sec (90° – θ) = csc θ (vi)csc (90° – θ) = sec θ

We know there are six trigonometrical ratios in trigonometry. The above explanation will help us to find the trigonometrical ratios of complementary angles.

Example 21.24: If A, B and C are the interior angles of a right–angle triangle, right–angled at B then find the value of A, given that tan 2A = cot(A – 30°) and 2A is an acute angle.

Solution:

Using the trigonometric ratio of complementary angles,

cot $(90°–A) = \tan A$

From this ratio, we can write the above expression as:

$\Rightarrow \tan 2A = \cot (90°– 2A)$ ….(1)

Given expression is $\tan 2A = \cot (A – 30°)$ …(2)

Now, equate the equation (1) and (2), we get

$\cot (90°– 2A) = \cot (A – 30°)$

$\Rightarrow 90°– 2A = A – 30°$

$\Rightarrow 3A = 90° + 30°$

$\Rightarrow 3A = 120°$

$\Rightarrow A = \frac{120}{3}$

$\Rightarrow A = 40°$

Thus, the measure of the acute angle A can be easily calculated by making use of trigonometry ratio of complementary angles.

Example 21.25: Without using trigonometric tables,

Evaluate $\frac{\tan 65°}{\cot 25°}$

Solution:

$\frac{\tan 65°}{\cot 25°} = \frac{\tan 65°}{\cot(90°-65°)}$

$= \frac{\tan 65°}{\tan 65°}$, [Since $\cot (90° - \theta) = \tan\theta$]

$= 1$

Example 21.26: Without using trigonometric tables, evaluate $\sin 35° \sin 55° – \cos 35° \cos 55°$

Solution:

$\sin 35° \sin 55° – \cos 35° \cos 55°$

$= \sin 35° \sin (90° – 35°) – \cos 35° \cos (90° – 35°)$,

$= \sin 35° \cos 35° – \cos 35° \sin 35°$,

[Since $\sin (90° – \theta) = \cos \theta$ and $\cos (90° – \theta) = \sin \theta$]

$= \sin 35° \cos 35° – \sin 35° \cos 35°$

$= 0$

Example21.27: If $\sec 5\theta = \csc (\theta – 36°)$, where 5θ is an acute angle, find the value of θ.

Solution:

$\sec 5\theta = \csc (\theta – 36°)$

$\Rightarrow \csc (90° – 5\theta) = \csc (\theta – 36°)$, [Since $\sec \theta = \csc (90° – \theta)$]

$\Rightarrow (90° – 5\theta) = (\theta – 36°)$

$\Rightarrow –5\theta – \theta = –36° – 90°$

$\Rightarrow –6\theta = –126°$

$\Rightarrow \theta = 21°$, [Dividing both sides by $–6$]

Therefore, $\theta = 21°$

Example 21.28: Using trigonometrical ratios of complementary angles prove that $\tan 1° \tan 2° \tan 3°$ .... $\tan 89° = 1$

Solution:

$\tan 1° \tan 2° \tan 3°$ ...... $\tan 89°$

$= \tan 1° \tan 2°$ ...... $\tan 44° \tan 45° \tan 46°$ ...... $\tan 88° \tan 89°$

$= (\tan 1° \cdot \tan 89°) (\tan 2° \cdot \tan 88°)$ ...... $(\tan 44° \cdot \tan 46°) \cdot \tan 45°$

$= \{\tan 1° \cdot \tan (90° – 1°)\} \cdot \{\tan 2° \cdot (\tan 90° – 2°)\}$ ...... $\{\tan 44° \cdot \tan (90° – 44°)\} \cdot \tan 45°$

$= (\tan 1° \cdot \cot 1°)(\tan 2° \cdot \cot 2°)$ ...... $(\tan 44° \cdot \cot 44°) \cdot \tan 45°$, [Since $\tan (90° – \theta) = \cot \theta$]

$= (1)(1)$ ...... $(1) \cdot 1$, [since $\tan \theta \cdot \cot \theta = 1$ and $\tan 45° = 1$]

$= 1$

Therefore, $\tan 1° \tan 2° \tan 3°$ ...... $\tan 89° = 1$

REMARKS:

1. Sin A or sin θ is one symbol and sin cannot be separated from A or θ. It is not equal to sin $\times\ \theta$. The same applies to other trigonometric ratios.
2. Every t–ratio is a real number.
3. For convenience, we use notations $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$ for $(\sin\theta)^2$, $(\cos\theta)^2$, and $(\tan\theta)^2$ respectively. We apply the similar notation for higher powers of trigonometric ratios.
4. We have restricted ourselves to t–ratios when A or θ is an acute angle.

CHECK YOUR PROGRESS 21.5

1. Evaluate without using trigonometric table:

   (i) $\sin 54° - \cos 36°$

   (ii) $\tan 10° - \cot 72°$

   (iii) $\csc 12.5° - \sec 77.5°$

2. If the csc of an angle complementary to A be $23\sqrt{23}$, find tan A.

3. Evaluate without using trigonometric table:

   (i) $\tan 63° \tan 27°$

   (ii) $\cos 13° \csc 77°$

   (iii) $\sin 40° \sec 50°$

   (iv) $(\cos 15° - \cos 75°) : (\sin 75° - \sin 15°)$

   (v) $\sin80°\cos10°\sin80°\cos10° + \sin 59° \sec 31°$

4. Evaluate without using trigonometric table:

   (i) $\sin 10° \cos 80° + \cos 10° \sin 80°$

   (ii) $\cos 65° \cos 25° - \sin 65° \sin 25°$

   (iii) $\sec 37° \csc 53° - \tan 37° \cot 53°$

   (iv) $3 \cos 80° \cdot \csc 10° + 2 \cos 59° \cdot \csc 31°$

21.7 HEIGHT AND DISTANCE

Height and Distance: One of the main applications of trigonometry is to find the distance between two or more than two places or to find the height of the object or the angle subtended by any object at a given point without actually measuring the distance or heights or angles. Trigonometry is useful to astronomers, navigators, architects and surveyors etc. in solving problems related to heights and distances.

The directions of the objects can be described by measuring:

- Angle of elevation
- Angle of depression

Angles of elevation or angles of depression of the objects are measured by an instrument called **Theodolite.** Theodolite is based on the principles of trigonometry, which is used for measuring angles with a rotating telescope. In 1856, Sir George Everest first used the giant theodolite, which is now on display in the Museum of the survey of India at Dehradun.

21.7.1 Angle of Elevation

Let P be the position of the object above the horizontal line OA and O be the eye of the observer, then angle AOP is called angle of elevation. It is called the angle of elevation, because the observer has to elevate (raise) his line of sight from the horizontal OA to see the object P. [When the eye turns upwards above the horizontal line.

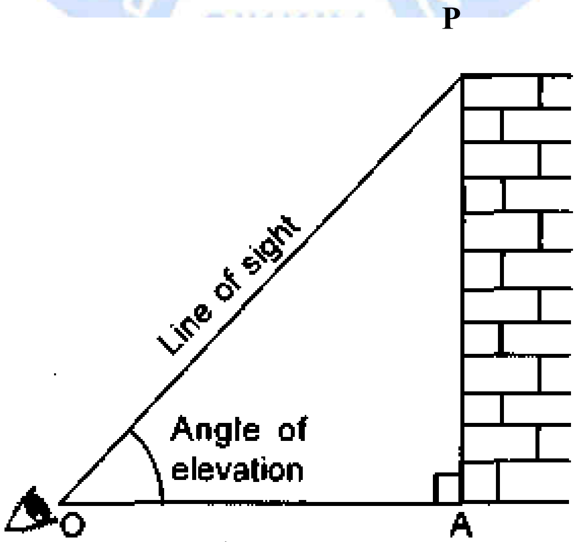


Fig 21.11

Line of sight: It is the line drawn from the eye of an observer to the object viewed.

21.7.2 Angle of Depression

Let P be the position of the object below the horizontal line OA and O be the eye of the observer, then angle AOP is called angle of depression.

It is called the angle of depression because the observer has to depress (lower) his line of sight from the horizontal OA to see the object P.

Below the eye level: If the object lies below the horizontal plane of our eyes, then we have to move our head downwards to view it. In doing so, our lines of sight moves downwards through an angle and the angle, which the line of sight now makes with the horizontal line, is called the angle of depression of the object from our eyes.

Example 21.29: Let OH be the horizontal line at the eye level. If a person at O looks at an object P lying below the eye level, then, ∠HOP is the angle of depression of P as seen from O.

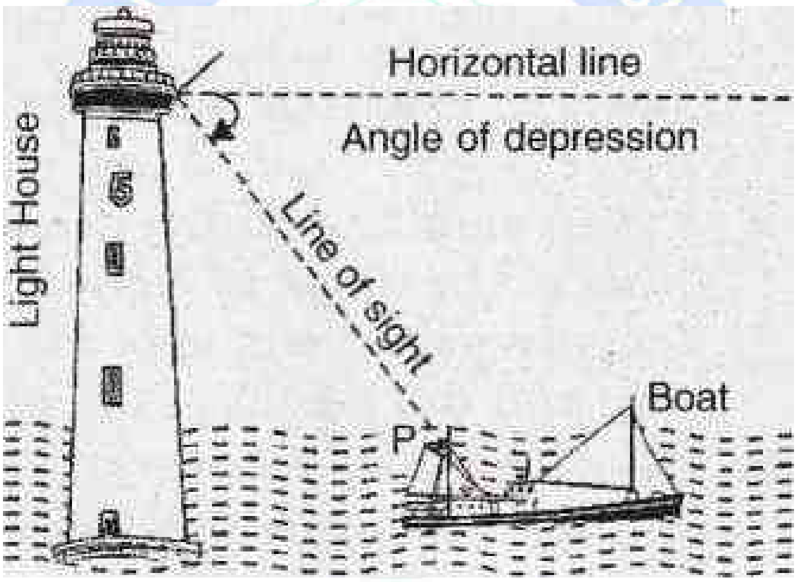


Fig 21.12

Note: Angle of depression of P as seen from O = angle of elevation of O, as seen from P.

21.7.3 Heights and Distances: Trigonometry Applications, Examples,Formulas

Height and Distance: One of the main applications of trigonometry is to find the distance between two or more than two places or to find the height of the object or the angle subtended by any object at a given point without actually measuring the distance or heights or angles. Trigonometry is useful to astronomers, navigators, architects and surveyors etc. in solving problems related to heights and distances.

Height and Distance Formulas

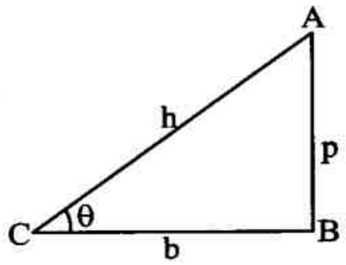


Fig 21.13

(i) In a triangle ABC,

$\sin\theta = \frac{p}{h}, \cos\theta = \frac{b}{h}$ and $\tan\theta = \frac{p}{b}$

(ii) In any triangle ABC,

If $BD : DC = m : n$ and $\angle BAD = \alpha, \angle CAD = \beta$ and $\angle ADC = \theta$,
then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

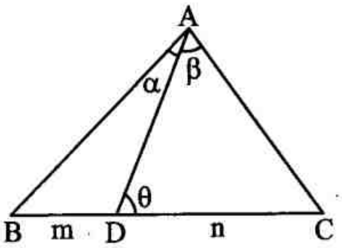


Fig 21.14

- If the observer moves towards the objects like tower, building, cliff, etc. then angle of elevation increases and if the observer moves away from the object, the angle of elevation decreases.

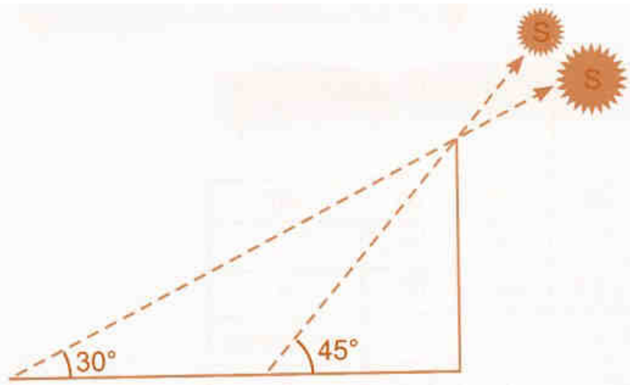


Fig 21.15

- If the angle of elevation of sun decreases, then the length of shadow of an object increases and vice–versa.
- If in problems, the angle of elevation of an object is given, then we conclude that the object is at higher altitude than observer. The angle of depression implies that observer is at higher altitude than object.

Example21.30: If a pole 6 *m* high casts a shadow $2\sqrt{3}\,m$ long on the ground, find the Sun's elevation.

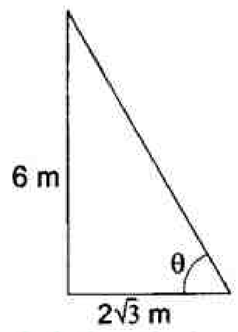


Fig 21.16

$$\tan\theta = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$
$$\Rightarrow \theta = 60^o$$

Example21.31: An observer 1.5 m tall is 20.5 *m* away from a tower 22 *m* high. Determine the angle of elevation of the top of the tower from the eye of the observer.

$$PQ = MB = 1.5m$$
$$AM = AB - MB = 22 - 1.5 = 20.5m$$

Now in triangle ΔAPM

$$\tan\theta = \frac{AM}{PM}$$

$$= \frac{20.5}{20.5} = 1$$

$$= 45^o$$

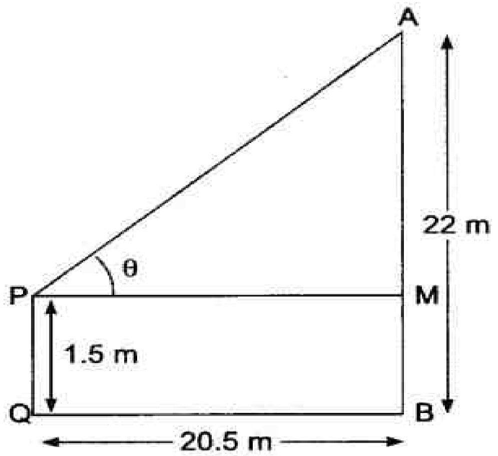


Fig 21.17

Example21.32: A ladder 15 m long makes an angle of 60° with the wall. Find the height of the point where the ladder touches the wall.

$\cos 60^o = \frac{x}{15}$

$\Rightarrow \frac{1}{2} = \frac{x}{15}$

$\Rightarrow x = \frac{15}{2} m = 7.5m$

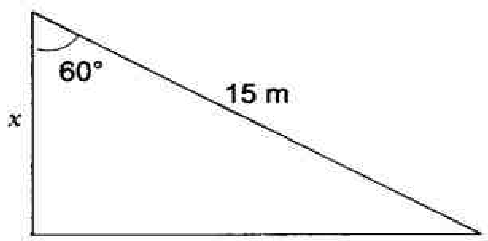


Fig 21.18

Example 21.33: The ratio of the height of a tower and the length of its shadow on the ground is $\sqrt{3}:1$. What is the angle of elevation of the sun?

Given: $\frac{AB}{BC} = \frac{\sqrt{3}}{1}$

Then, $\tan\theta = \frac{AB}{BC} = \sqrt{3}$

$\Rightarrow \theta = 60^o$

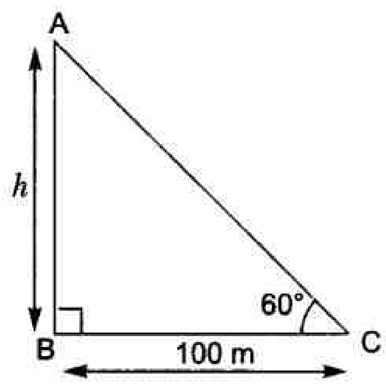


Fig 21.19

Example 21.34: If the angle of elevation of a tower from a distance of 100 m from its foot is 60°, then what will be the height of the tower?

Let *h* be the height of the tower.

$$\tan 60^o = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{h}{100}$$

$$h = 100\sqrt{3}m$$

In Fig. 21.20, AB is a 6 m high pole and CD is a ladder inclined at an angle of 60° to the horizontal and reaches up to a point D of pole. If AD = 2.54 m, find the length of the ladder. (use √3 =1.73) [CBSE Delhi 2016]

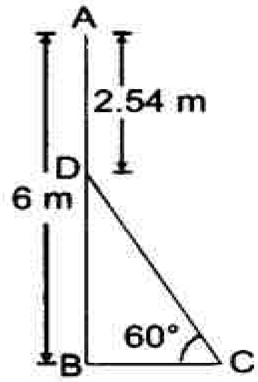


Fig 21.20

$$DB = (6-2.54)m = 3.46$$

$$\text{In } \Delta ABDC, \sin 60^o = \frac{BD}{CD}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{3.46}{CD} \Rightarrow CD = \frac{3.46\times 2}{1.73}$$

$$\therefore DC = 4m$$

Example 21.35: An observer, 1.7 m tall, is $20\sqrt{3}m$ away from a tower. The angle of elevation from the eye of observer to the top of tower is 30°. Find the height of tower.

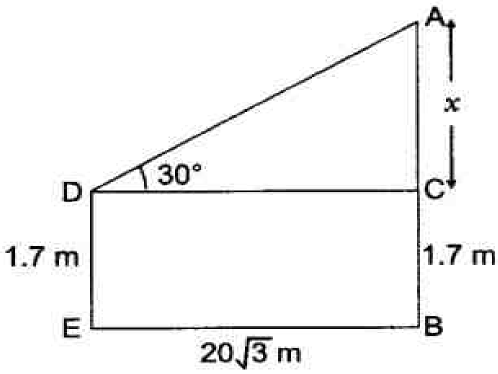


Fig 21.21

Let*AB* be the height of tower and *DE* be the height of observer.

Then in $\Delta ACD, \frac{AC}{DC} = \tan 30^o$

$\Rightarrow \frac{x}{20\sqrt{3}} = \tan 30^o = \frac{1}{\sqrt{3}} \Rightarrow x = 20m$

$\therefore AB = 20 + 1.7 = 21.7m$

Example 21.36: If a tower 30 m high, casts a shadow $10\sqrt{3}m$ long on the ground, the what is the angle of elevation of the sun?

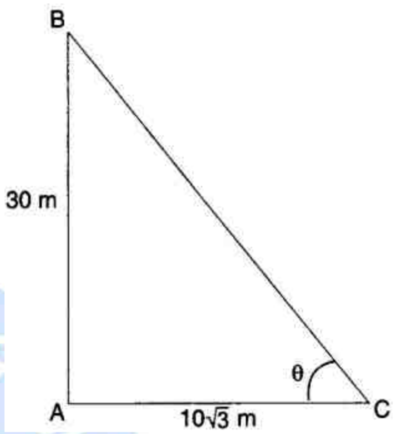


Fig 21.22

In ΔABD

$\tan\theta = \frac{AB}{AC} = \frac{30}{10\sqrt{3}} = \sqrt{3}$

$\Rightarrow \tan\theta = \tan 60$

$\Rightarrow \theta = 60^o$

CHECK YOUR PROGRESS 21.6

1. The shadow of a tower standing on level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

2. A tree breaks due to a storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.

3. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

4. An observer 1.5 metres tall is 20.5 metres away from a tower 22 metres high. Determine the angle of elevation of the top of the tower from the eye of the observer.

5. The angle of elevation of the top of a tower from a certain point is 30°. If the observer moves 20 metres towards the tower, the angle of elevation of the top increases by 15°. Find the height of the tower.

6. A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.

7. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is60° and the angle of depression of its foot is 45°. Determine the height of the tower.

8. If Figure, AB is a 6 m high pole and CD is a ladder inclined at an angle of 60° to the horizontal and reaches up to a point D of pole. If AD = 2.54 m, find the length of the ladder.(use √3=1.73)

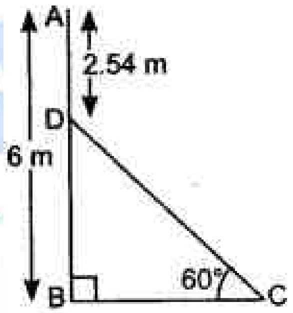


Fig 21.23

9. A ladder, leaning against a wall, makes an angle of 60° with the horizontal. If the foot of the ladder is 2.5 m away from the wall, find the length of the ladder.

10. Express each of the following in terms of trigonometric ratios of angles lying between 0° and 45°.

(i) Sin67° + cos75°

(ii) Cot65° + tan49°

(iii) Sec78° + cosec56°

(iv) Cosec54° + sin72°

11. If cos2θ=sin4θ, where 2θ and 4θ are acute angles, then find the value of θ.
12. If sin 3 A = cos (A − 26°), where 3 A is an acute angle, find the value of A.
13. If tan 2 A = cot (A − 12°), where 2 A is an acute angle, find the value of A.
14. An observer, 1.7 m tall, is 20√3 m away from a tower. The angle of elevation from the eye of observer to the top of tower is 30°. Find the height of tower.

21.8 SOME APPLICATIONS OF TRIGONOMETRY

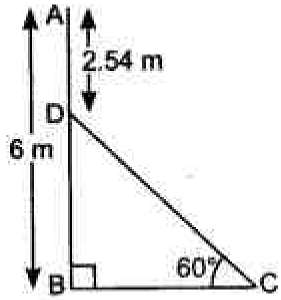


Fig 21.24

Example21.37: If Figure, AB is a 6 m high pole and CD is a ladder inclined at an angle of 60° to the horizontal and reaches up to a point D of pole. If AD = 2.54 m, find the length of the ladder.(use $\sqrt{3}=1.73$)

Solution:

$$BD = AB - AD$$

$$= 6m - 2.54m = 3.46m$$

In ΔABC,

$$\frac{BD}{CD} = \text{Sin } 60°$$

$$\Rightarrow \frac{3.46}{CD} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow CD = \frac{2\times 3.46}{\sqrt{3}} = \frac{2\times 3.46}{1.73} = 2\times 2 = 4m$$

Hence, length of the ladder is 4*m.*

Example21.38: A ladder, leaning against a wall, makes an angle of 60° with the horizontal. If the foot of the ladder is 2.5 *m* away from the wall, find the length of the ladder.

Solution:

Let AC be the ladder of length x.

In ΔABD, $\frac{BC}{x} = \cos 60^o$

$$\Rightarrow \frac{2.5}{x} = \frac{1}{2}$$

$$\Rightarrow x = 2\times 2.5 = 5m$$

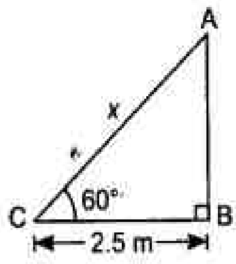


Fig 21.25

Thus, length of the ladder is 5m.

Example21.39: If □A and □P are acute angles such that sin$A$= sin$P$ then prove that □A=□ P

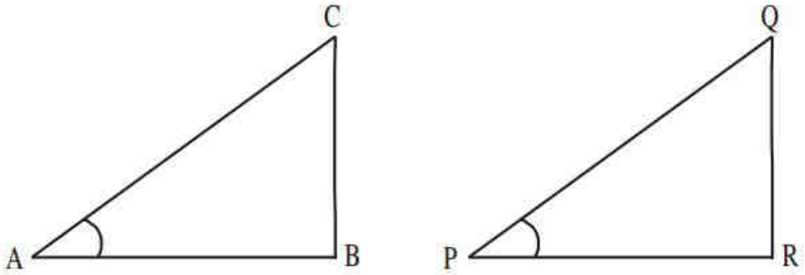
Solution: Given sin$A$ = sin$P$

we have $\sin A = \frac{BC}{AC}$

and $\sin P = \frac{QR}{PQ}$

Then $\sin \frac{BC}{AC} = \frac{QR}{PQ}$

Therefore, $\frac{BC}{AC} = \frac{QR}{PQ} = k$



(i) (ii)

Fig 21.26

By using Pythagoras theorem

Hence, $\frac{AC}{PQ} = \frac{AB}{PR} = \frac{BC}{QR}$ then ΔABC □ ΔPQR

$$\frac{AB}{PR} = \frac{\sqrt{AC^2 - BC^2}}{\sqrt{PQ^2 - QR^2}} = \frac{\sqrt{AC^2 - k^2 AC^2}}{\sqrt{PQ^2 - k^2 PQ^2}} = \frac{AC}{PQ} \text{(From (1))}$$

Therefore, $\angle A = \angle P$

Example 21.40: In ΔABC, right angle is at B, $AB = 5\ cm$ and □$ACB = 30°$. Determine the lengths of the sides BC and AC.

Solution:

Given AB=5 cm and

□ACB=30°.

To find the length of side *BC*, we will choose the trigonometric ratio involving *BC* and the given side *AB*. Since *BC* is the side adjacent to angle *C* and *AB* is the side opposite to angle *C*.

Therefore, $\frac{AB}{BC} = \tan C$

i.e. $\frac{5}{BC} = \tan 30^o = \frac{1}{\sqrt{3}}$

Which gives $BC = 5\sqrt{3}cm$

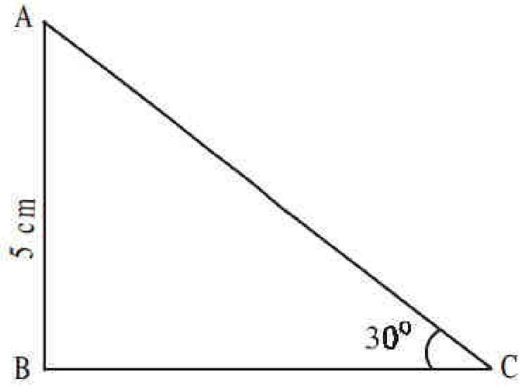


Fig 21.27

Now, by using the Pythagoras theorem

$AC^2 = AB^2 + AC^2$

$AC^2 = 5^2 + 5\sqrt{3}^2$

$AC^2 = 25 + 75$

$AC = \sqrt{100} = 10cm$

RECAPITULATION POINTS

- Trigonometric ratios
  - The certain ratios involving the sides of a right-angled triangle are called Trigonometric ratios.

    Suppose: *b* is the base h is the hypotenuse *p* is perpendicular then,

    $$\sin A = \frac{\text{Perpendicular}}{\text{Hypotenure}} = \frac{p}{h}$$

    $$\cos A = \frac{\text{Base}}{\text{Hypotenure}} = \frac{b}{h}$$

    $$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{p}{b}$$

  - Reciprocals of the ratios are:

    $$\cos ecA = \frac{1}{\sin A} = \frac{h}{p}$$

$$\sec A = \frac{1}{\cos A} = \frac{h}{b}$$

$$\cot A = \frac{1}{\tan A} = \frac{b}{p}$$

Trigonometric /Ratios of some specific angles.

- The specific angles are 0°, 30°,45°, 60°, 90°. These are given on the next page.
- The Trigonometric formulas or Identities are the equations which are true in the case of Right–Angled Triangles. Some of the special trigonometric identities are as given below –
  - Pythagorean Identities

    $\sin^2 \theta + \cos^2 \theta = 1$

    $\tan^2 \theta + 1 = \sec^2 \theta$

    $\cot^2 \theta + 1 = \mathrm{cosec}^2 \theta$

    $\sin^2\theta = 2 \sin \theta \cos \theta$

    $\cos^2\theta = \cos^2 \theta - \sin^2 \theta$

    $\tan^2\theta = 2 \tan \theta / (1 - \tan^2 \theta)$

    $\cot^2\theta = (\cot^2 \theta - 1) / 2 \cot \theta$

  - Sum and Difference identities–

    For angles A and B, we have the following relationships:

    $\mathrm{Sin}(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

    $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

    $\tan(A + B) = \tan(A) + \tan(B)/1 - \tan(A) \tan(B)$

    $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

    $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

    $\tan(A - B) = \tan(A) - \tan(B)/1+\tan(A) \tan(B)$

  - If *A, B* and *C* are angles and *a, b* and *c* are the sides of a triangle, then,

    Sine Laws

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- Cosine Laws

  $c^2 = a^2 + b^2 - 2ab \cos C$

  $a^2 = b^2 + c^2 - 2bc \cos A$

- $b^2 = a^2 + c^2 - 2ac \cos$ B

Specific angle t- ratio	0°	30°	45°	60°	90°
Sin A	0	1/2	1/√2	√3/2	1
Cos A	1	√3/2	1/√2	1/2	0
Tan A	0	1/√3	1	√3	Not defined
Cosec A	Not defined	2	√2	2/√3	1
Sec A	1	2/√3	√2	2	Not defined
Cot A	Not defined	√3	1	1/√3	0

- The value of sin A increases from 0 to 1, as A increases from 0° to 90°
- The value of cos$A$ decreases from 1 to 0, as A increases from 0° to 90°
- The value of tan A increases from 0 to infinity, as A increases 0° to 90°
- √2 = 1.414 and √3 = 1.732
- Trignometric identities
  - $\text{Cos}^2 A + \sin^2 A = 1$
  - $1 + \tan^2 A = \sec^2 A$
  - $\text{Cot}^2 A + 1 = \text{cosec}^2 A$
- Trigonometric ratios of complementary angles

  Two angles are said to be complementary if their sum equals to 90°

  sin (90°–A)= cos A

  cos (90°–A)= sin A

  tan (90°–A)= cot$A$

  cot (90°–A)= tan A

$\sec(90° - A) = \text{cosec} A$

$\text{cosec}(90° - A) = \sec A$

Note

- $\tan 0° = \cot 90° = 0$
- $\sec 0° = \text{cosec } 90° = 1$
- sec 90°, cosec 0°, cot 0° and tan 90° are not defined

TERMINAL EXERCISE

1. A 25 m long ladder is placed against a vertical wall of a building. The foot of the ladder is 7m from base of the building. If the top of the ladder slips 4m, then the foot of the Ladder will slide by how much distance.

2. Express each of the following in terms of trigonometric ratios of angles lying between 0° and 45°.

   (i) Sin67° + cos75°

   (ii) Cot65° + tan49°

   (iii) Sec78° + cosec56°

   (iv) Cosec54° + sin72°

3. If $\cos 2\theta = \sin 4\theta$, where 2θ and 4θ are acute angles, then find the value of θ.

4. If $\tan 2A = \cot(A - 12°)$, where $2A$ is an acute angle, find the value of A.

5. Two men on either side of a 75 m high building and in line with base of building observe the angles of elevation of the top of the building as 30° and 60°. Find the distance between the two men.

6. Choose the correct option:

   (i) In Δ ABC, right–angled at B, AB = 24 cm, BC = 7 cm. The value of tan C is:

   (a) 12/7

   (b) 24/7

   (c) 20/7

   (d) 7/24

(ii) If $\cos X = ⅔$ then $\tan X$ is equal to:

(a) 5/2

(b) $\sqrt{(5/2)}$

(c) $\sqrt{5}/2$

(d) $2/\sqrt{5}$

(iii) If $\cos X = a/b$, then $\sin X$ is equal to:

(a) $(b^2–a^2)/b$

(b) $(b–a)/b$

(c) $\sqrt{(b^2–a^2)}/b$

(d) $\sqrt{(b–a)}/b$

(iv) If $\sin A = 1/2$, then the value of $\cot A$ is

(a) $\sqrt{3}$

(b) $1/\sqrt{3}$

(c) $\sqrt{3}/2$

(d) 1

7. Match the Columns:

1. $\dfrac{\text{Side opposite to angle}\,\theta}{\text{Hypotenuse}}$	(A) $\tan\theta$
2. $\dfrac{\text{Side adjacent to angle}\,\theta}{\text{Hypotenuse}}$	(B) $\sin\theta$
3. $\dfrac{\text{Side opposite to angle}\,\theta}{\text{Side adjacent to angle}\,\theta}$	(C) $\cos\theta$
	(D) $\sec\theta$

(a) 1 – A, 2 – C, 3 – B

(b) 1 – B, 2 – C, 3 – A

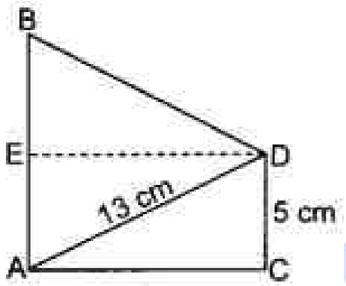
(c) 1 – B, 2 – C, 3 – D

(d) 1 – D, 2 – B, 3 – A

8. In the given figure, if AB = 14 cm, then the value of $\tan B$ is:

(a) $\frac{4}{3}$ (b) $\frac{14}{3}$

(c) $\frac{5}{3}$ (d) $\frac{13}{3}$



9. Match the Columns:

1. $\frac{\text{Hypotenuse}}{\text{Side adjacent to angle}\theta}$
2. $\frac{\text{Side adjacent to angle}\theta}{\text{Side opposite to angle}\theta}$
3. $\frac{\text{Hypotenuse}}{\text{Side opposite to angle}\theta}$

(A) $\cos ec\theta$
(B) $\sec\theta$
(C) $\tan\theta$
(D) $\cot\theta$
(E) $\sin\theta$

(a) 1 – A, 2 – C, 3 – B
(b) 1 – C, 2 – A, 3 – D
(c) 1 – B, 2 – A, 3 – E
(d) 1 – B, 2 – D, 3 – A

10. Given that sin θ = ab then cos θ is equalto

(a) $bb^2-a^2\surd$

(b) ba

(c) $b^2-a^2\surd b$

(d) $ab^2-a^2\surd$

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 21.1

1. $\sin\theta = \frac{12}{13}, \cos\theta = \frac{5}{13}, \tan\theta = \frac{12}{5}, \cot\theta = \frac{5}{12}, \text{cosec}\,\theta = \frac{13}{12}$

2. $\frac{49}{64}$

3. Yes

4. (i) 1, (ii) 0

5. $\sin P = \frac{12}{13}, \ \cos P = \frac{5}{13}, \ \tan P = \frac{12}{5}$

6. i. False, ii. True, iii. False, iv. False, v. False

CHECK YOUR PROGRESS 21.2

1. $\frac{\sqrt{3}-2}{2}$

2. (i) 2

   (ii) 4

4. (i) 20°

   (ii) 45°

5. (i) 0

   (ii) 2

CHECK YOUR PROGRESS 21.3

1. Cos A=4, tan A= 3

2. Sin $A = \frac{15}{17}$, Sec $A = \frac{17}{8}$

3. 0

CHECK YOUR PROGRESS 21.4

4. $x^2b^2 + a^2b^2 = a^2b^2$

9. YES

CHECK YOUR PROGRESS 21.5

1.	(i) 0	(ii) 0	(iii) 0		
2.	$13\sqrt{13}$				
3.	(i) 1	(ii) 1	(iii) 1	(iv) 1	(v) 2
4.	(i) 1	(ii) 0	(iii) 1	(iv) 5	

CHECK YOUR PROGRESS 21.6

1. $20\sqrt{3}$ m 2. 20 m 3. 45° 4. $10(\sqrt{3} + 1)$ m.

5. 10 m 8. $7(\sqrt{3} + 1)$ m. 9. 4m

10. 5m

11. (i) Cos 23°+ Sin 15° (ii) Tan 25° + Cot 41°

 (iii) Cosec 12° + Sec 34° (iv) Sec 36° + Cos 18°

12. 15° 13. 29° 14. 34° 15. 21.7 M

22 DATA AND THEIR REPRESENTATION

INTRODUCTION

Every day in your life, you may be dealing with different types of information. While reading a newspaper or while watching a television programme or attending a function with relatives and friends you may see, observe and use many information. For example, in the newspaper you read a tragic road accident happens to be near your place of residence. In the news it is also mentioned the similar types of accidents have happened in the state in the last one week. These are few examples you may be familiar related with your immediate environment. You may have information about education background of the people from your village, their occupation, sports and games they are playing, etc. How this information could be processed and used for the benefits of our society? Have you ever thought of this? You might have seen that, due to increased number of road accidents at a definite place, government might have taken some steps to reduce the accident by constructing humps or keeping the sign board etc. For taking many decisions in our life, we also may be using such type of information generally we call it as statistical data or statistical information.

Since we all need to use definitely some type of statistical data in our life, it is essential to know in details about the data, their collection process, their representation, analysis and interpretation. The branch of mathematics which deals with these processes is known as STATISTICS.

In this chapter we will try to explain the meaning of data with appropriate illustrations and the process of collecting data along with their representation. The unit will begin by discussing the meaning of the term statistics, then the concept of data and its collection. It will be followed by the detailing of different ways of presentation of data in graphical as well as tabular ways. The chapter will conclude with cumulative frequency distribution, which will be essential for conceptualizing the next chapter. Each sub unit will be explained with the help of examples from our daily life and few practice examples also will be provided.

22.1 LEARNING OBJECTIVES

After completing this lesson, learners will be able to:

- explain the meaning of statistics by taking suitable examples
- define statistical data and different types of data with examples
- illustrate the procedure for collection of data
- explain the procedure of presentation of statistical data in different ways
- define frequency table and differentiate between different types of frequency tables
- draw frequency tables when based on given data
- express the statistical data with the help of diagrams and graphs

22.2 STATISTICS AND STATISTICAL DATA

In the introduction section we mentioned about the importance of information for taking many decisions in our life. How do we take such decisions, of course, the first stage is to gather different types of information then analyze those to arrive at some conclusion.

For example, you are listening the TV news about covid-19 infection cases, the number of recovery and number of deaths regularly, based on the information available for one week, you may be able to analyze thetrend of infection such as whether it is increasing or not, the relationship between the number of cases and number of deaths, etc.

If we look at this example, we have lots of information related with covid cases spreading over weeks and months. Before going to take any conclusion about this information, we may have to organizethat information in a systematic way. This organization will help us in analyzing and interpreting the information to arrive at a conclusion.

The branch of mathematics or in general science which deals with the process of collecting, organizing/ presenting, analyzing and interpreting of information is known as statistics.

The word Statistics is used both in ***Singular*** and ***Plural*** sense.

While defining statistics, you might have seen that, it is ***a branch of mathematics.*** That means it is a subject which deals with collection, organization/presentation, analysis and interpretation of data. In this way statistics is used in singular sense. At the same time, you might have heard about the term ‘selected educational statistics of Sikkim’. What it means? This will consist of data such as the number of Schools, Number of Higher Educational Institutions, Number of Teachers, Number of Students, The rate of drop outs, the pass percentages, etc. All these are numerical information under the selected statistics. In this way statistics is used in plural sense. Hence in plural sense, statistics is used for representing numerical facts or information collected for a particular purpose.

22.2.1 What is Data?

The information we are collecting for a particular purpose are generally known as **data**. For example, consider the information about the enrollment of students in class 1 during a particular year; say for example, 2022 at various schools of Sikkim. This data could be used for calculating the total enrollment during 2022, to compare the enrollment in 2022 with that of 2021, to compare the enrollment in Private as well as Public Schools, etc.

The data may be of two forms.

In the above example related with covid cases, the numbers of infected case, the number of deaths, the number of recovered cases, etc. are all in the form of number or numeral or quantity. Such information is usually known as **quantitative data**.

At the same time, we can have discussion and interview with people from medical fields such as doctors, nurses, pharmacists, etc. to know the quantum of spreading of covid cases in their jurisdiction. That information may not necessarily in the form of quantity. We may get information in the form of their opinion, their appraisals, their daily log books in the form of reports, etc.

in the form of words, phrase, etc. Such information is known as **qualitative data**.

22.3 COLLECTION OF DATA

While defining statistics, you might have noticed that, the first stage in the process of analysis of information is collection of data. Once we have data in hand, we will be able to do further process such as organization or presentation, analysis, interpretation and even the conclusion.

For example, you are interested in supporting the government secondary school nearby your locality in terms of improving the quality. How can you do that? Of course, the first step is to know about the present status of that school in terms of quality parameters. You may need to collect different information about theschool. Few of such information could be collected through visiting the school and interacting with students, teachers, head teacher, other employees, parents, community members, alumni, etc. Even you may be able to conduct some achievement tests to know the ability of students in different subjects, you may be able to observe the school and classroom to know about the facilities etc. In all these processes, you as an investigator himself/herself are taking initiative to collect the data. Such types of data are generally known as ***Primary data.*** Such data are more reliable and relevant due to its originality with the context of the investigation.

Apart from these information or data, you will be able to get some additional data from other sources such as survey conducted by SCERT/DIET, which are available in the form of report, or report of the study con- ducted by another investigator, etc. also. This information is not collected by the investigator, but collected by others for another purpose and are appropriate and hence used for your investigation. Such types of data are called ***Secondary data.***

It is very important to note that, the data collected by you for your investigation is primary but if the same is collected by another investigator for his/her purpose, the data become secondary. Hence when using secondary data care must be given to use that information which are most relevant to your investigation.

CHECK YOUR PROGRESS 22.1

1. Fill in the blanks with suitable word(s).

   (a) Statistics is the branch of mathematics which deals with ________, presentation, ____________, and interpretation of numerical data.

   (b) In __________ sense, statistics is used for representing numerical facts or information collected for a particular purpose.

   (c) The marks secured by the students in an achievement test conducted by an investigator for his/her research purpose is an example of ________ data.

   (d) The data used by an investigator from the published report of UNESCO is an example of ________ data.

   (e) Opinion collected by an investigator through conducting interview of teachers can be considered as __________ data.

2. Mark true or false against each statement.

   (a) The height of the students collected for the purpose of finding Body Mass Index is an example of qualitative data.

(b) In plural sense, statistics is considered as a branch of mathematics.

(c) The marks secured by students in the board examination is an example of quantitative data.

3. Classify the following data into quantitative and qualitativeHeight of the students, Scripts in the student's diary, Percentage of attendance, Age of the students, Interview scripts, Observation notes, Price of the vegetables.

22.4 PRESENTATION OF DATA

Once you collect data using different sources, the next process in the investigation is to organize or present the data in a systematic way so that, the various features or characteristics of data could be studied or under- stood easily. This process of arrangement or organization of data is generally known as ***presentation of data*.**

Let us explain the process with the help of following example.

Let the marks obtained by 25 students of Class IX in a mathematics test are as follows (Out of 50) –

45, 36, 31, 26, 31,

33, 40, 41, 36, 46,

36, 45, 34, 25, 28,

28, 26, 30, 34, 37,

29, 33, 41, 40, 27

The data collected in this form is called **raw data**. You may not be able to tell the highest mark and the lowest mark very quickly. Of course, you can say it by observing these raw data also but it will be a time-consuming process.

Can you provide few information such as how many students scored 26, how many are there with less than 30 marks, etc.? from this data?

Yes, these also you will be able to provide by spending more time. Is there any other way to provide this information quickly as possible? The rearrangement of this raw data, is essential for identifying some of the features of the data.

Hence once you collect data, it is essential to present it in a systematic way to understand more about the collected data.

There are different ways to present the data in more meaningful form. This could be done using tables, diagrams, graphs, etc.

22.4.1 Presentation of Data in Ascending and Descending order

The simplest and a crude way to present the raw data to understand few characteristics are to arrange them according to ascending or descending order of their magnitude.

Consider the raw data given in the example provided above. If we arrange them in an order starting from the lowest score, the data looks as follows –

25, 26, 26, 27, 28,

28, 29, 30, 31, 31,

33, 33, 34, 34, 36,

36, 36, 37, 40, 40,

41, 41, 45, 45, 46

From the above arrangement, you will be able to say many things about the data such as the highest score is 46 lowest score is 25, there are three students with a score of 36, etc. This type of arrangement of raw data is known as data in ***ascending order***.

We could also arrange the data in reverse way as well. That's from highest to the lowest. This will look like the follow –

46, 45, 45, 41, 41,

40, 40, 37, 36, 36,

36, 34, 34, 33, 33,

31, 31, 30, 29, 28,

28, 27, 26, 26, 25

This is known as the arrangement of data in ***descending order.***

The raw data when we present in either ascending or descending order is also called as an ***array or arrayed data.***

Example 22.1: The marks secured by 50 students of class10 in mathematics test is given below –

19, 23, 27, 28, 38, 45, 29, 43, 48, 29,

38, 39, 29, 30, 30, 33, 44, 34, 36, 40,

39, 43, 36, 46, 37, 44, 37, 38, 38, 45,

39, 29, 39, 40, 41, 37, 42, 42, 42, 42,

42, 36, 43, 43, 45, 46, 34, 47, 23, 40

1. Arrange the data in ascending and descending order.
2. What are the maximum and minimum marks secured?
3. How many students secured a mark of 42?

Solution:

1. Ascending order / Descending order

Ascending order	Descending order
19, 23, 23, 27, 28, 29, 29, 29, 29, 30,	48, 47, 46, 46, 45, 45, 45, 44, 44, 43,
30, 33, 34, 34, 36, 36, 36, 37, 37, 37,	43, 43, 43, 42, 42, 42, 42, 42, 41, 40,
38, 38, 38, 38, 39, 39, 39, 39, 40, 40,	40, 40, 39, 39, 39, 39, 38, 38, 38, 38,
40, 41, 42, 42, 42, 42, 42, 43, 43, 43,	37, 37, 37, 36, 36, 36, 34, 34, 33, 30
43, 44, 44, 45, 45, 45, 46, 46, 47, 48	30, 29, 29, 29, 29, 28, 27, 23, 23, 19

2. Maximum mark = 48, Minimum Mark = 19

3. Either from ascending order or from descending order, you can see that the mark 42 repeats 5 times. Hence the number of students secured the mark 42 is 5.

Suppose you are asked to arrange the scores of 100 students in ascending order or descending order. Is it an easy task? You can see that a considerable amount of time is required to arrange the raw data in to arrayed data forms. Hence when the number of data is more, these arrangements will not be economical in time and more over will not provide us much about the data other than the maximum, minimum etc.

To make the data in easier to understand and more informative we can present the same in the form a table.

22.4.2 Frequency Representation of Data

Consider the data of the marks secured by 50 students of class10 in mathematics given in the example 22.1

19, 23, 27, 28, 38, 45, 29, 43, 48, 29,

38, 39, 29, 30, 30, 33, 44, 34, 36, 40,

39, 43, 36, 46, 37, 44, 37, 38, 38, 45,

39, 29, 39, 40, 41, 37, 42, 42, 42, 42,

42, 36, 43, 43, 45, 46, 34, 47, 23, 40

Now, if you arrange these scores in ascending order, we will get the data rearranged in the following way.

19, 23, 23, 27, 28, 29, 29, 29, 29, 30,

30, 33, 34, 34, 36, 36, 36, 37, 37, 37,

38, 38, 38, 38, 39, 39, 39, 39, 40, 40,

40, 41, 42, 42, 42, 42, 42, 43, 43, 43,

43, 44, 44, 45, 45, 45, 46, 46, 47, 48

Look at this arrangement of the data, we could observe the following the number of students with a score of 19 = 1

The number of students with a score of 23 = 2

The number of students with a score of 27 = 1

The number of students with a score of 46 = 2

The number of students with a score of 47 = 1

The number of students with a score of 48 = 1

This could be represented in a tabular form as given below –

Score	No of Students
19	1
23	2
27	1
28	1
29	4
30	2
33	1
34	2
36	3
37	3
38	4

39	4
40	3
41	1
42	5
43	4
44	2
45	3
46	2
47	1
48	1
Total	50

By looking at the table, you will be able to say many more features of the data. Apart from the maximum and minimum score, that you could even say from the ascending order of arrangement, you can see that the score 42 has secured by maximum number of students there are only 1 student with a score 47, there are 4 students with scores of 43, 39, 38 and 29 and so on. This arrangement definitely will give us better scope for analyzing the various properties of the data under consideration.

The second column of the table gives us the number of students who secured a specific mark. These numbers we called as the frequencies corresponding to the specific data or observation. For example, the frequency of the score or observation 19 is 1, frequency of the observation 23 is 2 and so on.

The process of representing raw data in the form of a table based on the frequency of the observation is called a **frequency distribution table.**

In the above example we have taken individual score or observation while developing the frequency distribution table hence it is known as an **ungrouped frequency distribution table.** It is also known as **discrete frequency distribution table.** We can present the data in the form of a **grouped or continuous frequency distribution** table also.

Now let us see how we can form an ungrouped frequency distribution table, when the raw data is given with the following example –

Following are the heights (in cm) of 20 students of class X. Prepare an ungrouped frequency distribution table using the data.

156	145	156	165	168	168	160	169	168	167
160	169	168	159	162	159	149	168	149	139

We may make use of following steps while preparing the ungrouped frequency distribution table

Step 1: Identify the maximum and minimum score (observation) and write all observations in the in the ascending order in the first column.

Step 2: Now look at the first observation in the given raw data and put a bar (vertical line) in the second column against the observation available in the first column. Then move to the second observation in the raw data and put the bar, etc. This process could be repeated till all observations in the given raw data are exhausted. Each of these bars drawn are known as tally marks and to help us to count the total bars under a cell very quickly, it is advisable to use a diagonal tally across the first four. (IIII)

Step 3: Count the number of tally marks under against each observation and write in the third column.

This is known as the ***frequency*** of the observations.

By performing the activities specified under step 1 to step 3, we will be able to form the following table.

Height (in cm)	Tally Marks	No of Students (Frequency)
139	|	**1**
145	|	**1**
149	||	**2**
156	||	**2**
159	||	**2**
160	||	**2**
162	|	**1**
165	|	**1**
167	|	**1**

168	卌	**5**
169	||	**2**
Total		**20**

From the above table you can say that, there are 5 students with a height of 168 cm in the class and there are two tallest students and whose height is 169 cm and the shortest student is with 139 cm.

Example 22.2

The number of people undergone covid vaccination at 20 vaccination centers of a district in the order of the centre code (1-20) on a specific day is as follows –

72 72;50;59;63;72; 93;72;72;81;83;84;53;78;59;84;63;84; 90; 76

1. Construct an ungrouped frequency distribution table
2. What is the maximum vaccination happened on that day in a centre?
3. How many vaccination centres has vaccinated exactly 72 people?

Solution:

1. For constructing ungrouped frequency distribution table, first we have to identify the maximum and minimum so that, we could arrange them in the first column in ascending order.

   Here the minimum is 50 and maximum 93.

Number of People vaccinated	Tally Marks	No of Centres (Frequency)
50	|	1
53	|	1
59	||	2
63	||	2
72	卌	5
76	|	1
78	|	1

81	|	1
83	|	1
84	|||	3
90	|	1
93	|	1
Total		

2. Maximum vaccination happened in a centre is 93.

3. From the above table it can be seen that 72 people each were vaccinated in 5 centres.

When the number of observations is considerably few, we may be able to use the ungrouped frequency distribution table for analysis purpose, but when the number of observations is more, it is tedious to prepare such a table by considering all the observation from the smallest to the highest. In such a condition we could prepare a frequency distribution by clubbing few observations together in a group.

For example, consider the data related with heights (in cm) of 20 students of class X provided earlier.

156	145	156	165	168	168	160	169	168	167
160	169	168	159	162	159	149	168	149	139

In this case the smallest height is 139, and the highest is 169cm. If we group them, like 138-143, 143-148, 148-153, 153-158, 158-163, 163-168 and 168-173, then the entire 20 observation can be classified under these7 groups.

The group 138-143, consists of five observations, 138, 139, 140, 141 and 142.

Similarly, the group 143-148 consists of the five observations 143, 144, 145, 146 and148 and so on.

If we apply step 1 to step 3 for these groups of scores instead of individual observations, we could form the following table.

Group	**Tally marks**	**Total students (Frequency)**
138-143	|	**1**

143-148	|	**1**
148-153	||	**2**
153-158	||	**2**
158-163	卌	**5**
163-168	||	**2**
168-173	卌 ||	**7**
Total		**20**

While drawing tally marks against the group 138-143, we need to put tally for any observation falling from 138 to 142 against this group. That means the group 128 -143 consists only the five observations 138,139,140,141 and 142. The tally mark corresponding to the observation 143 has to put against the group 143-148.

This type of frequency distribution is called **grouped frequency distribution**. Each group formed here such as 138-143, 143-148, 148-153, etc. are known as ***classes***.

Here 138-143 is the first class and the observation 138 is known as the **lower-class limit or simply lower limit** of the class and 143 is known as the **upper-class limit or simply upper limit** of the class.

Hence for the class 158-163, the lower-class limit is 158 and the upper-class limit is 163.

The difference between the upper-class limit and lower-class limit of a class is known as ***class interval*** orclass width or class size.

Hence the class width or class interval for the class 138-143 is 143 − 138 = 5.

You can see that, the class interval of all the classes mentioned above is same and equal to 5. But this may not be the case at all time. You may have frequency distribution with classes having different class intervals as well.

We know that in the class 143-148, there are five observations viz, 143, 144, 145, 146 and 147. If we want to represent all these observation with a single value, we need to find out a representative value between 143 and 148. We call this representative value as ***class mark or mid-point of the class***.

Mid -point will be calculated by adding the upper class limit with lower class limit and dividing the result by 2.

Hence mid-point /class mark of a class $= \dfrac{\text{Upper class limit} + \text{Lower class limit}}{2}$

Therefor the class mark of the class 153 – 158 = (158 + 153) ÷ 2 = = $\frac{311}{2}$ = 155.5

Example 22.3

What is the lower limit, upper limit, class interval and mid- point of the following classes

(i) 120 – 130

(ii) 82 – 88

(iii) 0 – 5

Solution:

(i) 120 – 130 Lower limit = 120

Upper limit = 130

Class interval = 130 – 120 =10

Mid-point = (130 + 120) /2 = 250/2 = 125

(ii) 82 – 88 Lower limit = 82

Upper limit = 88

Class interval = 88 – 82 = 6

Mid-point = (88 + 82) /2 = 170/2 = 85

(iii) 0 – 5 Lower limit = 0

Upper limit = 5

Class interval = 5 – 0 = 5

Mid-point = (5 + 0) /2 = 5/2 = 2.5

Now let us see how we could form a grouped frequency distribution for a given data. For clarity, we may follow the following steps.

Step 1: Identify the minimum and maximum observation and find out the difference between them. We call this difference as the range of the raw data. For example, in the above case the minimum height is 139 and maximum is 169.

Hence the range =169-139 = 30.

Step 2: You may decide the number of groups or classes in to which the raw data could be grouped to form the frequency distribution. There is no hard and fast rule for fixing this number but it is advisable to fix the number of groups in between 5 to 15. In the above example, we have taken it as 7.

Step 3: Determine the class interval. This can be done by dividing the range calculated in the step 1 by the desired number of classes fixed in the step 2 by rounding up. There are two things to be carefully remembered in this step. It is rounding up not off. If the quotient is 4.3 don't round it to 4 but round up to 5. Secondly if the quotient is an integral value, then add one to the class interval or otherwise increase the number of classes by one. In the above example, we have 30/7= 4.3 and rounded up to 5. Hence class interval/width/size of each class will be 5.

Step 4: Form the class limits based on the interval determined in the earlier step. You may start the first class from any data with a condition that the first class should contain the minimum value. Forexample, it the above example the minimum height is 139, hence while deciding the first class, care must be given to include 139 in the first class. Hence, we may start the class with 138 and since the interval is 5, the first class will be 138-143, which consists of 138, 139, 140, 141 and142. You may also be able to start with 136 so that the first class becomes 136-141. But we also need to make sure that our last class should contain the maximum value. That is 169 in this case.

Step 5: Once the classes are finalized, write those in the column 1 starting from the lowest class. Go through each observation from the given data and put a tally mark in the second column against the appropriate class to which belongs. Continue this process for all the observations. Put the tally in groups of five as mentioned earlier using a diagonal stroke after the first four tally marks.

Step 6: Count the total tally marks against each class and mention the number in the third column. This number is known as the frequency of that class. Make sure that the total of all frequencies is equal to the total number of observations.

Now, let us demonstrate the above steps with another example.

The number of covid positive cases reported in the district south of Sikkim during the month of November 2021 is as follows (imagined data).

72	104	95	93	96	76	105	100

88	62	79	78	87	78	89	81
110	68	96	106	80	87	86	84
102	84	96	88	82	78		

Make a grouped frequency distribution table with equal class interval.

Step 1: Maximum number = 110

Minimum number = 62

Hence Range = 110 – 62 = 48

Step 2: Let us decide the number of groups/classes = 5.

Step 3: Class interval = Range /Number of classes $= \frac{48}{5} = 9.6$ and rounded up to 10.

Step 4: Let us decide the classes as 62 – 72, 72 – 82, 82 – 92, 92 – 102, 102 – 112.

Here we can also take classes starting from 61 also. But if we start the classes from 60, the fifth class will be 100-110 and we cannot include the maximum in the last class.

For step 5 and step 6, we need to prepare the table.

Number of covid cases	Tally marks	Frequency
62-72	||	2
72-82	𝍸 |||	8
82-92	𝍸 ||||	9
92-102	𝍸 |	6
102-112	𝍸	5
Total		**30**

In the above example, we have divided the 30 observations into 5 groups or classes and each group is of interval 10. For the group 62-72, the lower limit is 62 and the upper limit is 72 and the same 72 is the lower limit of the next group 72-82, and so on.

Hence, we can say that the upper limit of one group or class is the lower limit of the next group or class. Again, while drawing the tally marks, we have seen that the class 62-72 consists of 10 observations 62, 63, 64, 65, 66, 67, 68, 69, 70 and 71. That means the observation 72 is not included in the class 62-72 and is a part of the class 72-82. Hence the upper limit of the class is not included in that class and is included in the next class. This is one way or method of preparing grouped frequency distribution and is known as ***exclusive method.***

We can construct grouped frequency distribution in another way also. Here we will include both the lower limit and the upper limit of a particular class in that class itself.

Hence the classes formed in the above example can be modified as 62-71, 72-81, 82-91, 92-101, 102-111.

The class interval will be the same 10 and the frequency distribution table will be as follows –

Number of covid cases	Tally marks	Frequency
62-71	||	2
72-81	𝍸 |||	8
82-91	𝍸 ||||	9
92-101	𝍸 |	6
102-111	𝍸	5
Total		**30**

You can see that the frequency of each class is same as that of the earlier classes. The only difference is in the upper limit of the classes.

Here the class 62-71 means, all the observations from 62 to 71 including both 62 and 71.

This method of constructing grouped frequency distribution table is known as ***inclusive method***.

Example 22.4

The weight (in kg) of 25 students of class X is given below. Construct a grouped frequency distribution with equal class width using exclusive and inclusive method.

38, 43, 42, 39, 45,

44, 42, 38, 34, 53,

Solution: Maximum weight = 53

Minimum weight = 30

48, 30, 34, 49, 45,

32, 39, 49, 45, 50

42, 43, 30, 34, 47

Range = 53-30 = 23

Let the number of classes = 6

Class width $= \frac{23}{6}$

= 3.8 and rounded up to 4

Exclusive method

Let classes be 30-34, 34-38, 38-42, 42-46, 46-50 and 50-54.

Classes	Tally Marks	Frequency
30-34	|||	3
34-38	|||	3
38-42	||||	4
42-46	~~||||~~ ||||	9
46-50	||||	4
50-54	||	2
Total		**25**

Inclusive method

Let classes be 30-33, 34-37, 38-41, 42-45, 46-49 and 50-53.

Classes	Tally Marks	Frequency
30-33	|||	3
34-37	|||	3
38-41	||||	4
42-45	~~||||~~ ||||	9
46-49	||||	4

50-53	||	2
Total		**25**

From the above inclusive frequency distribution, we can say that the class 30-33, includes those students whose weights are in between 30 kg to 33 kg including both 30 and 33.

Now the question may come, if a student's weight happens to be 33. 4 kg, in which group, we could include that student?

This question usually comes due to the gap you can see between two consecutive classes. Whether 33.4 kg includes in the class 30-33 or in the class 34-37?

To address these types of concerns and doubts, what we can do is to convert these inclusive classes in to exclusive classes.

For that, firstly we need to identify gap or the difference between the upper limit of a class and the lower limit of the next class. Here it is 34-33 = 1 and can be seen the same gap between all other classes as well.

Now the second step is to find out the half of this difference. Here it is ½ = 0.5.

In the next step, we subtract half of this difference from each lower limit and add half of this difference with each upper limit.

For, example, the class 30-33, can be converted as (30 – 0.5) – (33+ 0.5)ie, 29.5 –33.5.

Similarly, the class 34 –37 can be converted as 33.5 –37.5 and so on. Now the converted frequency distribution will be as follows –

Classes	**Frequency**
29.5-33.5	3
33.5-37.5	3
37.5-41.5	4
41.5-45.5	9
45.5-49.5	4
49.5-53.5	2
Total	25

From the above table, we can say that the class 29.5- 3.5 includes all observations from 29.5 up to 33.5 excluding 33.5. The observation 33.5 will include in the next class 33.5- 37.5 and so on.

Using this process, we have converted the class 30-33 to 29.5- 33.5, where 30 and 33 were the lower limit and upper limit of the original class. We call the lower limit and the upper limit of the revised class 29-5- 33.5 as ***true class limits***. Hence 29.5 is known as the **true lower-class limit** and 33.5 is the **true upper-class limit**. Again, the true upper limit of a class will be the true lower limit of the next class and so on.

Now we can say that, the student with weight 33.4 kg will be included in the class 30-33 or 29.5-33.5. If the weight of another student is 41.5 kg, that student will be included in the class 42-45 or 41.5-45.5?

Example 22.5: A survey of enrollment of students for secondary class at open school from 35 study centres across the country shows the following -

641,155, 219, 935, 218, 430, 594,

500, 225, 244, 541, 520, 583, 750,

112, 635, 270, 414, 681, 300, 170,

920, 730, 183, 655, 713, 768, 493,

271, 261, 163, 785, 265, 508, 122

(i) Organize the data into a grouped frequency distribution of equal size, starting with the class 100- 200 (using inclusive and exclusive method).

(ii) How many classes will be there in the frequency distribution table?

(iii) What is the class interval?

(iv) Which class has maximum frequency?

Solution:

(i) Since the first class is given as 100–200, we can frame the other classes as 200–300, 300–400, 400–500, 500–600, 600–700,800–900 and 900–1000.

Exclusive frequency distribution -

Class	Tally marks	Frequency
100-200	~~||||~~ |	6
200-300	~~||||~~ |||	8
300-400	|	1
400-500	|||	3

500-600	卌 |	6
600-700	||||	4
700-800	卌	5
800-900		0
900-1000	||	2
Total		

1. Inclusive frequency distribution -

Class	**Tally marks**	**Frequency**
100-199	卌 |	6
200-299	卌 |||	8
300-399	|	1
400-499	|||	3
500-599	卌 |	6
600-699	||||	4
700-799	卌	5
800-899		0
900-999	||	2
Total		

2. There are nine classes.
3. Class interval is 100.
4. The class 200-300 has maximum frequency.

CHECK YOUR PROGRESS 22.2

1. Marks secured by 25 students in science unit test are given below:

   40, 43, 46, 39, 33, 46, 25, 35, 36, 41,

   48, 35, 38, 29, 35, 26, 41, 35, 31, 44,

   31, 38, 34, 30, 29

   (i) Present the data in ascending and descending order.

(ii) Find out the range of the data.

(iii) Construct an ungrouped frequency distribution.

2. The number of students enrolled in an online quiz competition organized by SCERT from 30 secondary schools of Sikkim are given as follows:

34	33	48	58	42	63	44	63	52	58
59	38	26	18	22	43	43	24	27	46
37	42	55	40	20	36	28	34	40	45

(i) Construct a grouped frequency distribution of the data using the classes 16-24, 24-32, 32-40, etc.

(ii) Identify the class with maximum frequency.

(iii) Write the lower limit, upper limit and the class mark of the class with maximum frequency.

3. Following is the frequency distribution of experience in years of 48 teachers in a school:

Experience (in years)	No of Teachers
0-4	3
4-8	7
8-12	8
12-16	11
16-20	5
20-24	6
24-28	6
28-32	2
Total	48

(i) What is the class width?

(ii) What is the class mark of the class 16-20?

(iii) Find the upper-class limit of class 24-28.

(iv) Find the lower-class limit of 0-4.

(v) Convert the frequency distribution using inclusive method starting from the class 0-3.

(vi) What is the true class limit of the class 4-11?

22.5 CUMULATIVE FREQUENCY TABLE

The following frequency distribution table gives the weekly wages of 35 employees from a pharmaceutical company.

Weekly wages (in Rs.)	No of employees
1400-1600	5
1600- 1800	3
1800- 2000	5
2000- 2200	6
2200- 2400	8
2400-2600	7
2600-2800	1

How many employees get a weekly wage between Rs.1600 to Rs.1800?

Simply by observing the table, you can say that there are 6 employees with weekly earnings of Rs.2000-Rs.2200.

Can you tell the number of employees who get a weekly wage less than Rs.2000? Yes of course, we can find the number.

What do you mean by the frequency of the class 1400-1600 is 5 means?

It means that, there are 5 employees with weekly wages between Rs.1400 to Rs.1600.

Or in other ways, we can say that, there are 5 employees with weekly wages less than Rs.1600.

In a similar way, it can be said that there are 3 employees with weekly wages in between Rs.1600- 1800. Hence what will be the number of employees with weekly wages less than Rs.1800?

It will be 5 + 3 = 8.

In a similar way we can say that there are 5 + 3 + 5 = 13 employees with weekly wages less than Rs.2000, there are 5+3+5+6 =19 with weekly wages less than Rs.2200 and so on.

The number of employees with wages less than a particular salary (Upper limit of the class) is obtained by adding the number of employees mentioned under that class and the class (es) preceding it. This new number is known as ***cumulative frequency.***

We can convert this in the form a table given below.

Weekly wages (in Rs.)	No of employees	Cumulative Frequency (Less than)
1400-1600	5	**5**
1600- 1800	3	5 + 3 = **8**
1800- 2000	5	5 + 3 + 5 = **13** or 8 + 5 = **13**
2000- 2200	6	5 + 3 + 5 + 6 = **19** or 13 + 6 = **19**
2200- 2400	8	5 + 3 + 5 + 6 + 8 = **27** or 19 + 8 = **27**
2400-2600	7	5 + 3 + 5 + 6 + 8 + 7 = **34** or 27 + 7 = **34**
2600-2800	1	5 + 3 + 5 + 6 + 8 + 7 + 1 = **35** or 34 +1 = **35**

Here the frequencies 5, 8, 13, 19, 27, 34 and 35 are known as the cumulative frequencies for the classes corresponding to it in the table.

Here the cumulative frequency of the class 2400- 2600 is 34 means; there are 34 employees with weekly wages less than Rs.2600.

Note: There are two types of cumulative frequencies, ***less than and greater than*** (more than). In the above example, we constructed the table by considering the total frequencies less than the upper limit of a particular class. Hence this is called as less than cumulative frequency.

In a similar way we can find out the greater than cumulative frequencies as well.

For example, if you are asked to find out the number of employees with weekly wages more than Rs.2000? How can you find the number of employees?

If we start from the last class, ie, 2600-2800, with frequency 1, we can say that, there are 1 employee with weekly wage more than Rs.2600.

Then how many employees are there with weekly wages more than Rs.2400? Clearly, that will be 1 + 7 = 8, In a similar fashion, you will be able to find out the number of employees with weekly wages more thanRs.2200 as 1 + 7 + 8 = 16 and so on.

We will be able to form a table of more than (Greater than) Cumulative Frequency given below –

Weekly wages (in Rs.)	No of employees	Cumulative Frequency (More than)
1400-1600	5	5 + 3 + 5 + 6 + 8 + 7 + 1 = **35**
1600-1800	3	3 + 5 + 6 + 8 + 7 + 1 = 30
1800-2000	5	5 + 6 + 8 + 7 + 1 = 27
2000-2200	6	6 + 8 + 7 + 1 = 22
2200-2400	8	8 + 7 + 1 = 16
2400-2600	7	7 + 1 = 8
2600-2800	1	1

What is the number of employees with weekly wages more than Rs.2000? From the table, you can see this as 22.

Hence by constructing either less than or greater than cumulative frequency table, we will be able to interpret the given data in terms of either less than a particular observation or more than a particular observation.

Note: If we use only cumulative frequency table, then you need to assume that, it is a less than cumulative frequency table.

Let us practice an example.

Example 22.6: The following table gives the marks secured by 40 students in a unit test

Marks	No of Students (Frequency)
20-25	4
25-30	6
30-35	10
35-40	14
40-45	9
45-50	7

Total	50

(i) Construct the less than cumulative frequency table

(ii) How many students secured marks

(a) less than 45?

(b) less than 30?

(iii) Construct the more than cumulative frequency table

(iv) How many students are there with marks

(a) greater than 30?

(b) more than 40?

Solution:

(i) Less than cumulative frequency table is given below

Marks	**No of Students (Frequency)**	**Cumulative Frequency (Less than)**
20-25	4	4
25-30	6	10
30-35	10	20
35-40	14	34
40-45	9	43
45-50	7	50
Total	**50**	

(ii) (a) The cumulative frequency corresponding to the class 40-45 is 43, It means that there are 43 students with marks less than 45.

(b) The cumulative frequency corresponding to the class 25-30 is 10, It means that there are 10 students with marks less than 30.

(iii) Greater than cumulative frequency table is given below –

Marks	**No of Students (Frequency)**	**Cumulative Frequency (Greater than)**
20-25	4	50
25-30	6	46
30-35	10	40
35-40	14	30

40-45	9	16
45-50	7	7
Total	**50**	

(iv) (a) The cumulative frequency corresponding to the class 30-43 is 40, It means that there are 40 students with marks more than 30.

(b) The cumulative frequency corresponding to the class 40-45 is 16, It means that there are 16 students with marks more than 40.

CHECK YOUR PROGRESS 22.3

I. Following is the frequency distribution of number of patients admitted in a hospital during 30 days of April 2021.

Class (No. of Patients)	0-5	5-10	10-15	15-20	20-25	25-30
Frequency (Number days)	2	4	7	6	6	5

(i) Construct a cumulative frequency distribution (both less than and greater than) from the following data.

(ii) How many days the number of patients admitted were more than 20?

(iii) Find out the number of days where the patients admitted is less than 15.

2. Find the values of a, b, c, d, e, f and g from the following frequency distribution.

Class	**Frequency**	**Cumulative Frequency (less than)**
6- 16	4	**a**
16-26	**b**	11
26-36	5	16
36-46	13	**c**
46-56	**d**	40
56-66	**e**	46
66-76	4	**f**
Total	**g**	

22.6 GRAPHICAL REPRESENTATION OF DATA

In the earlier section, we have discussed various ways of representing the raw data in terms of order and tabular forms. We also used tabular forms of arrangements to summarize the raw data in a systematic way and to identify many characteristics of the data in a quicker way. In this section we will be discussing another way of representation of data, with the help of diagram, graph and pictures, which even will help one to understand some properties of data visually. Diagrams, Graphs and Pictures are easy to draw as well as to draw conclusions. There are different ways to present data with the help of diagrams, graphs and pictures. In this section, we will discuss only three important graphical representations, viz, Bar chart (graph), Histogram and Frequency Polygon.

22.6.1. Bar Chart (Graph)

Consider the following example

Following table provides the details regarding the management of Schools available in a particular city.

Management	**No of Schools**
State Government	12
Central Government	1
Local Body	3
Private (Aided)	4
Private (Un aided)	10
Total	30

We can represent this data in the form of a graph given below –

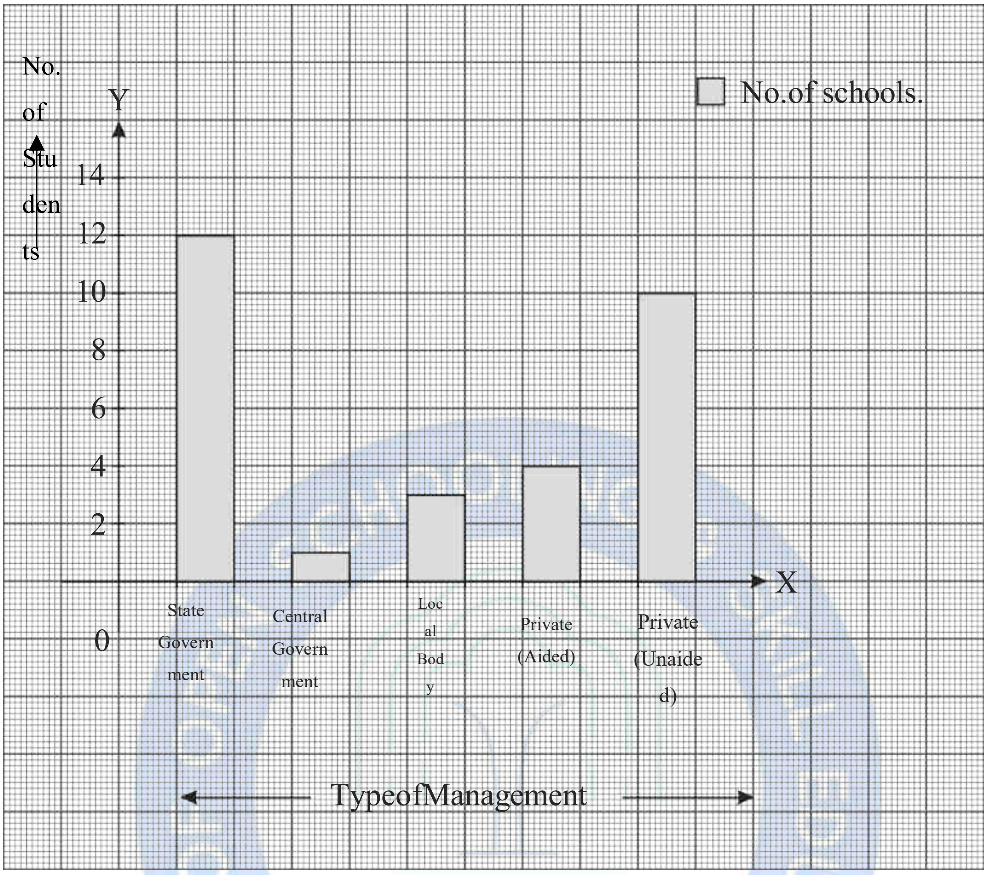


Fig. 22.1

The above figure is a pictorial representation of the number of schools under different management. You can see each management of the school have been represented with the help of bars or rectangles of equal width. The height of each rectangles/bars represents the number of schools or frequencies. The space between the rectangles (bars) also needs to be equal. This type of pictorial or graphical representation of data is known as **Bar Graph or Bar Diagram.**

From the above figure any one can observe that, the maximum number of schools from that city is under state government management whereas the least number of schools are under the central government management.In the above bar diagram, the types of management were mentioned along the horizontal line or along x-axis of the graph, whereas the number of schools was depicted along the vertical axis or y-axis.

We can draw a Bar diagram in other way also, i.e., by taking Management along the y-axis and number of schools along the x-axis. If we do like that, we will get a bar diagram, like below –

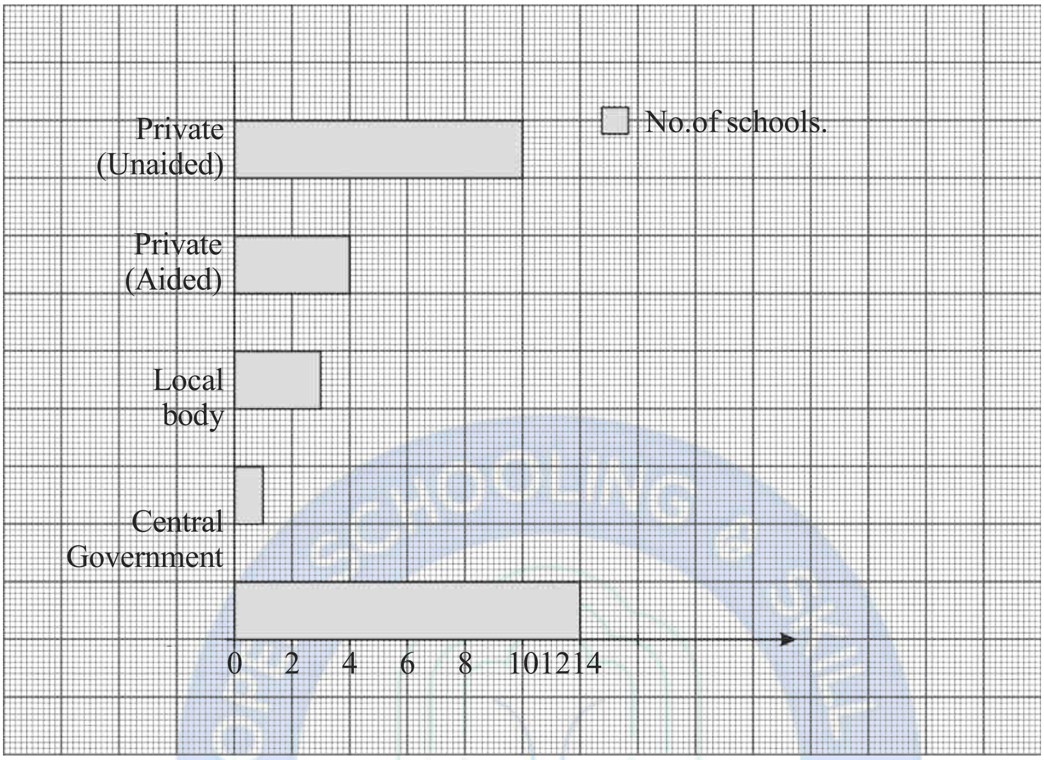


Fig. 22.2

Note: Youarefreetofixthewidthoftherectanglesbut,needtoensurethatallrectanglesofsamewidth while drawing a bar diagram.

Itisuptoyoutodecidewhichtypeofbargraphneedtodraw.Youmaydraweitherverticalbarsorhorizontalbars.Inmostbooksandmaterial,youmayseeverticalbarsbuttheuseofhorizontalbarsalsoavailable in newspapers, TV news etc.

Now,letusdiscussthestep-by-stepprocessofdrawingabardiagramwiththehelpofanexample. Bar chart (graph) is one of the graphical.

The following table provides the price of different fruits per kilogram on a day at Gangtok market.

Fruits	Price /Kg(Rs)
Apple	180
Orange	75
Watermelon	40
Banana	50
Pineapple	65
Grapes	105

Represent the data using a Bar Diagram.

Step1: Drawtwoperpendicularlinesonapaper,preferableonagraphpaper,ie,onewillbevertical(y-axis) and the other horizontal line (x- axis).

Step2: Representthedifferenttypesoffruitsalongthehorizontalline(x-axis)andthepriceperkginRs along the vertical line (y-axis).

Step3: Choose a fixed width for each bar along the horizontal line (eg, 1 unit in the graph paper) and decide the uniform gap between the bars also (eg, 2 unit gap).

Step4: Fix a suitable scale along the vertical axis based on the given data. Here price per kg are given anditrangesfrom40to180.Wemaychoose1unitinthegraphpaperequalstoRs20,sothatfor marking180,wemayrequire9units.(Youmaytake1unit=Rs10,sothat18unitsinthegraph paper required for drawing the bar graph).

Step5: Based on the data given for the prices of different fruits, we may draw bars of equal width and theheightsproportionaltothegivendataandmarkedontheverticalaxis.Keepthegapbetween each bar equal.

The required bar graph is given below –

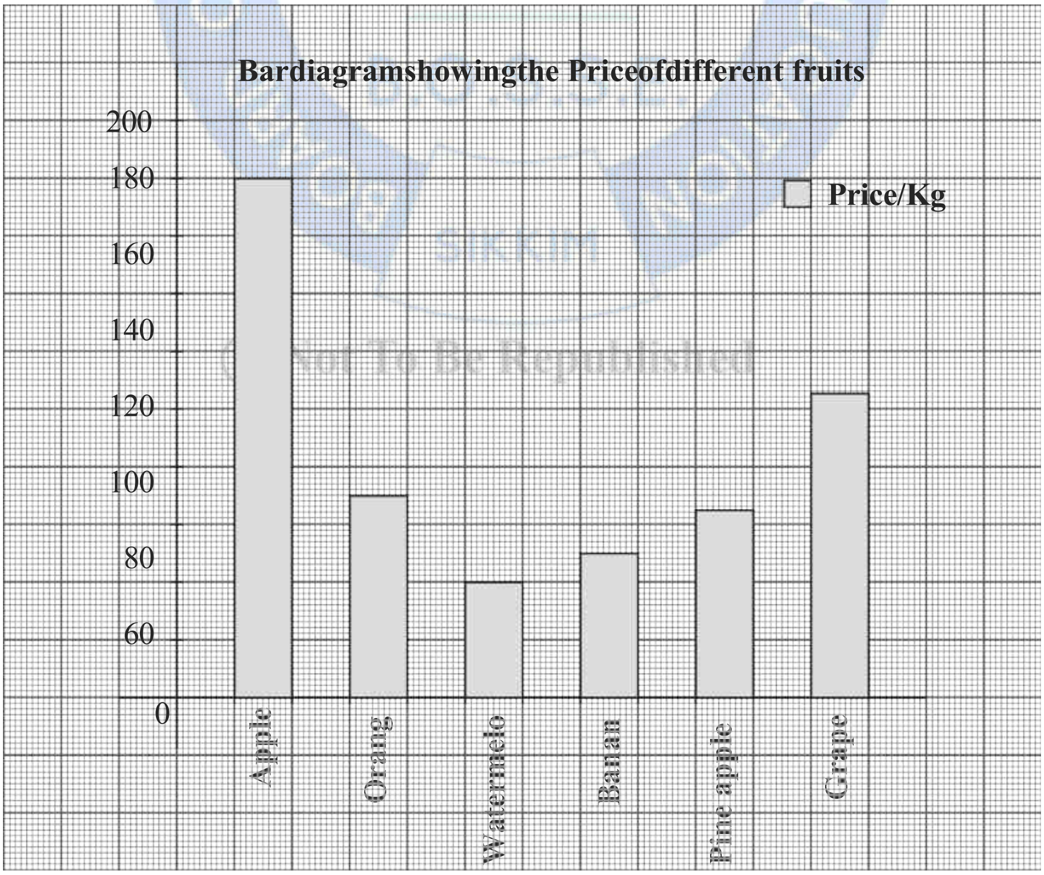


Fig. 22.3

22.6.2 Reading and Interpretation of Bar Graphs

How can we read and interpret the information given through a bar graph? Let us explain this with the help of following example.

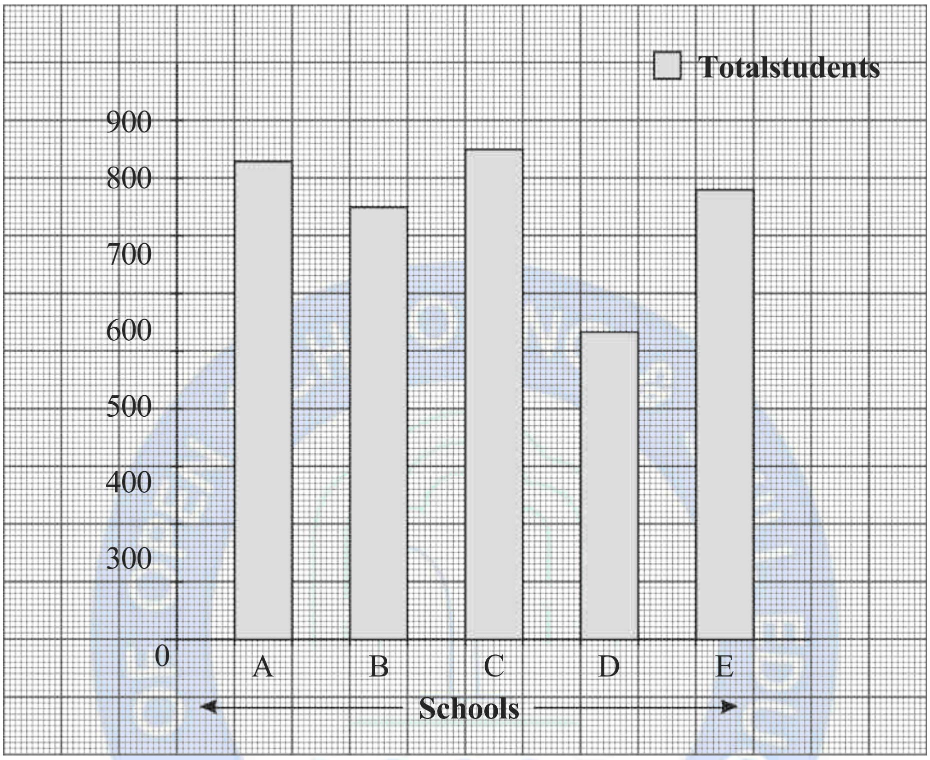


Fig. 22.4

Example 22.7: Read the bar graph given in below and answer the questions given below –

(i) What information has given in the bar diagram?

(ii) Whichschoolhasmaximumstudents?

(iii) Inwhichschoolthetotalstudentsare minimum?

(iv) Ranktheschoolsin termsofthetotal students.

Solution:

(i) Ifyoulookatthehorizontalline,itisshowingtheschoolsandtheverticallineshowingthetotalstudents.Hencethisbargraphdepictsdatarelatedwithtotalstudentsinfivedifferentschoolsnamely,A,B, C, Dand E.

(ii) Fromthefigure itisclear that,theschoolChasmaximum students.

(iii) SchoolD.

(iv) C,A,E, B, D.

22.6.3 Histogram

Histogram is a graphical representation of a grouped frequency distribution. In histogram the class intervals will be marked along the horizontal axis (x-axis) and the corresponding frequencies along the vertical axis (y-axis). Histogram consists of bars or rectangles without spacing between them. The width of each bar/ rectangle will be proportional to the class interval or width and heights proportional to the frequencies of corresponding classes.

Letusexplaintheprocessofconstructingahistogramwiththehelpofanexample The following table gives the marks obtained by 40 students in an examination.

FrequencyTable	
Class	**Frequency**
50-60	1
60-70	4
70-80	12
80-90	15
90-100	8

Whiledrawing the histogramof the above frequencydistribution wewillfollow the following steps –

Step1: Draw two perpendicular lines on a graph paper, one horizontal (x-axis) and the other vertical(y-axis). Mark the point of intersection as 'O'. Let the lines be OX and OY.

Step2: Based on the class marks (values), choose a suitable scale along x-axis and represent the class limits.Ifthelowerlimitofthefirstclassisnotstartingwith'0',startsthemarkingfromthefirst unit and mark a break between 0 and 1.

HerewewilltakeIunitas10mark50willberepresentedatthesecondunitbykeepingfirstunit blank.

Step3: Chooseasuitablescalealongthey-axisbasedonthefrequenciesandmarkaccordingly.Herethe maximum frequency is 15and hence we may take 1 unit = 2 students.

Step4: Drawrectangleswithwidthasclass intervalsandheightastherespective frequencies.

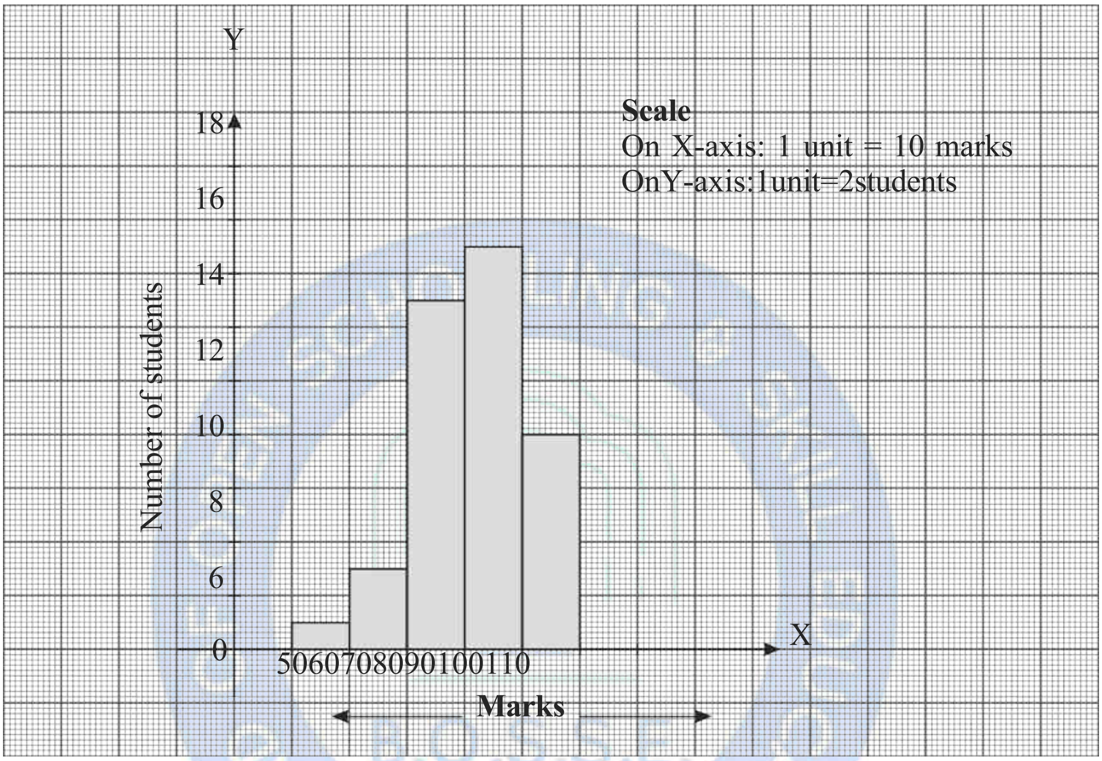


Fig. 22.5

Note: A histogram and the bar graph look very similar; however, they are differentbecause of the data level.

(i) Bar graphs depict data which are categorized into different groups based on characteristics such as typeofmanagement,Jobs,Gender,Yearetc.Buthistogramdepictsthedataprovidedinthefor mof a frequency distribution.

(ii) There will be gaps between bars (rectangles) in a bar graph but in the histogram, the bars are adjacent to each other.

(iii) The width of each bar in a bar graph will be same but that need not be same in a histogram (Histo-gram of a frequency distribution with unequal class interval will have different widths).

Example22.8: Drawahistogramforthefollowingdata.

Monthly income	500-1000	1000-1500	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000
Noof Family	12	15	12	17	11	9	6

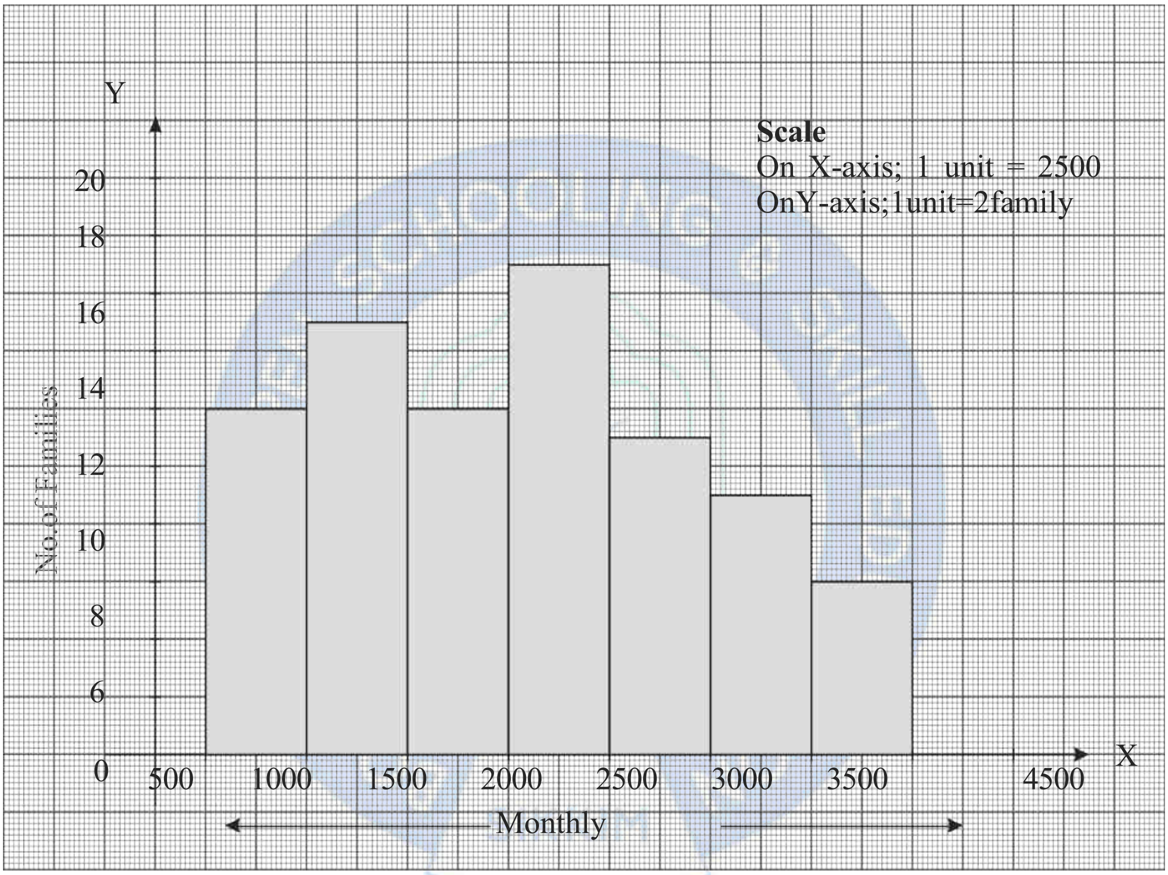


Fig. 22.6

22.6.4 Frequency Polygon

Inthelastsection,wediscussedhowtorepresentafrequencydistributionusinghistogram.Wecanrep resent the same frequency distribution graphically in another way also using frequency polygon.As the name suggests,theshapeofthisgraphwillbelikeapolygon.Wecandrawfrequencypolygonusingtwoways ,viz,by using a histogram and without using the histogram.

Welearnttodrawahistogram,whenfrequencydistributionisgiven After drawing the histogram, we may do the following process.

- Findthemid-points ofthe upperhorizontalsideofeach rectangle.

- Find the mid-points of the class intervals before the first-class interval and the last class intervals(necessarily their frequency will be 0 or they will be on x-axis).
- Then join the mid-points of the adjacent rectangles of the histogram by using dotted line segments and join the both end with the mid points with 0 frequencies.
- The figure we will get using the dotted line will be in the shape of a polygon and is known as frequency polygon.

Considerthe histogram wehave drawn based on the following frequency distribution in the last section.

FrequencyTable	
Class	**Frequency**
50-60	1
60-70	4
70-80	12
80-90	15
90-100	8

First, we need to mark the mid-points of the upper horizontal sides of all the five rectangles. Then, mark the mid- point of the class interval below the class interval 50-60 and above the class interval 90-100. Now join all these mid-points using dotted line –segments. We will get the following figure.

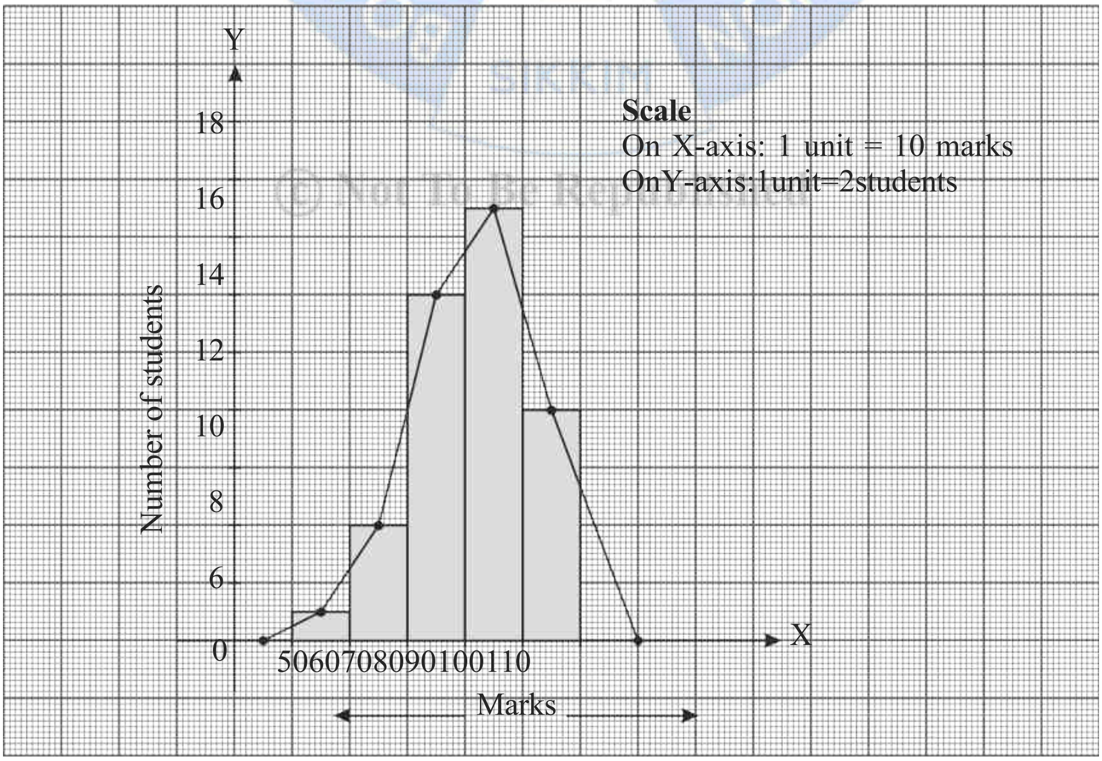


Fig. 22.7

Constructing frequency polygon without drawing a histogram

Let us see how, we can construct a frequency polygon without drawing histogram of a given frequency distribution.

Consider the following frequency distribution.

Age of children (in years)	Frequency
0-3	3
3-6	5
6-9	7
9-12	8
12-15	5
15-18	2

For constructing the frequency polygon, we need to use the following steps or processes

Step 1: While drawing the frequency polygon using a histogram, what we did first is marking the mid- points of upper horizontal sides of each class interval. What it means? We actually calculated the mid-point of each class.

Hence the first step is to find out the mid points (class marks) of each class intervals.

Age of children (in years)	Mid-point (Class mark)	Frequency
0-3	1.5	3
3-6	4.5	5
6-9	7.5	7
9-12	10.5	8
12-15	13.5	5
15-18	16.5	2

Step 2: Draw two perpendicular lines on a graph paper, one horizontal (x-axis) and the other vertical (y-axis). Mark the point of intersection as 'O'. Let the lines be OX and OY.

Step 3: Represent the class marks along the x-axis with suitable scale. Here 1 unit may be taken as 3 years.

Step 4: Along the *y*-axis, frequencies with a suitable scale may be marks. Here 1 unit may be taken as 1 frequency.

Step 5: Mark the points on the graph paper representing the pair (Mid-point, frequency). Here the points are (1.5, 3), (4.5, 5), (7.5, 7), (10.5, 8), (13.5, 5) and (16.5, 2).

Step 6: Mark the mid-point of the class just preceding the first-class interval and the mid-point of the class just succeeding the last class on the *x*-axis. Here the mid points are -1.5 and 19.5.

Step 7: Join the points marked in step 5 by line segments and complete the frequency polygon by joining those lines with the points marked in the step 6.

We will get the following frequency polygon.

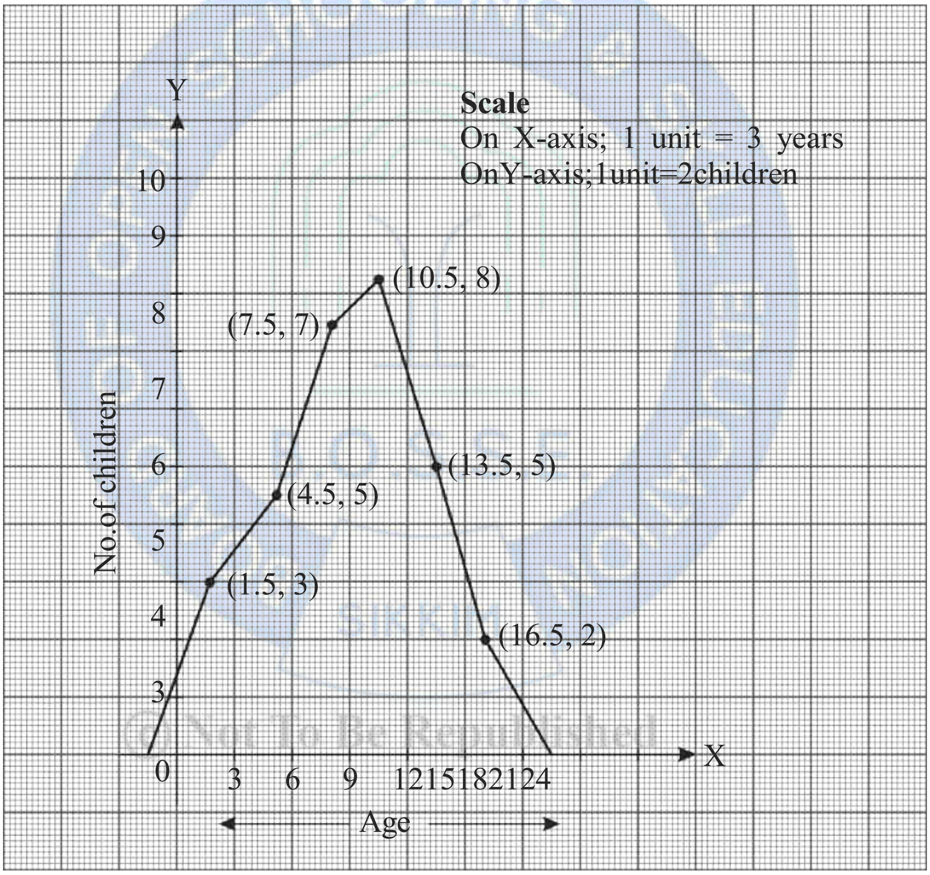


Fig. 22.8

Example 22.9: The daily expense of 100 families (in`) are given in the following table.

Dailyexpense	300-350	350-400	400-450	450-500	500-550	550-600	600-650
No.offamilies	13	8	21	26	17	13	8

Drawa frequencypolygon usingthe above data.

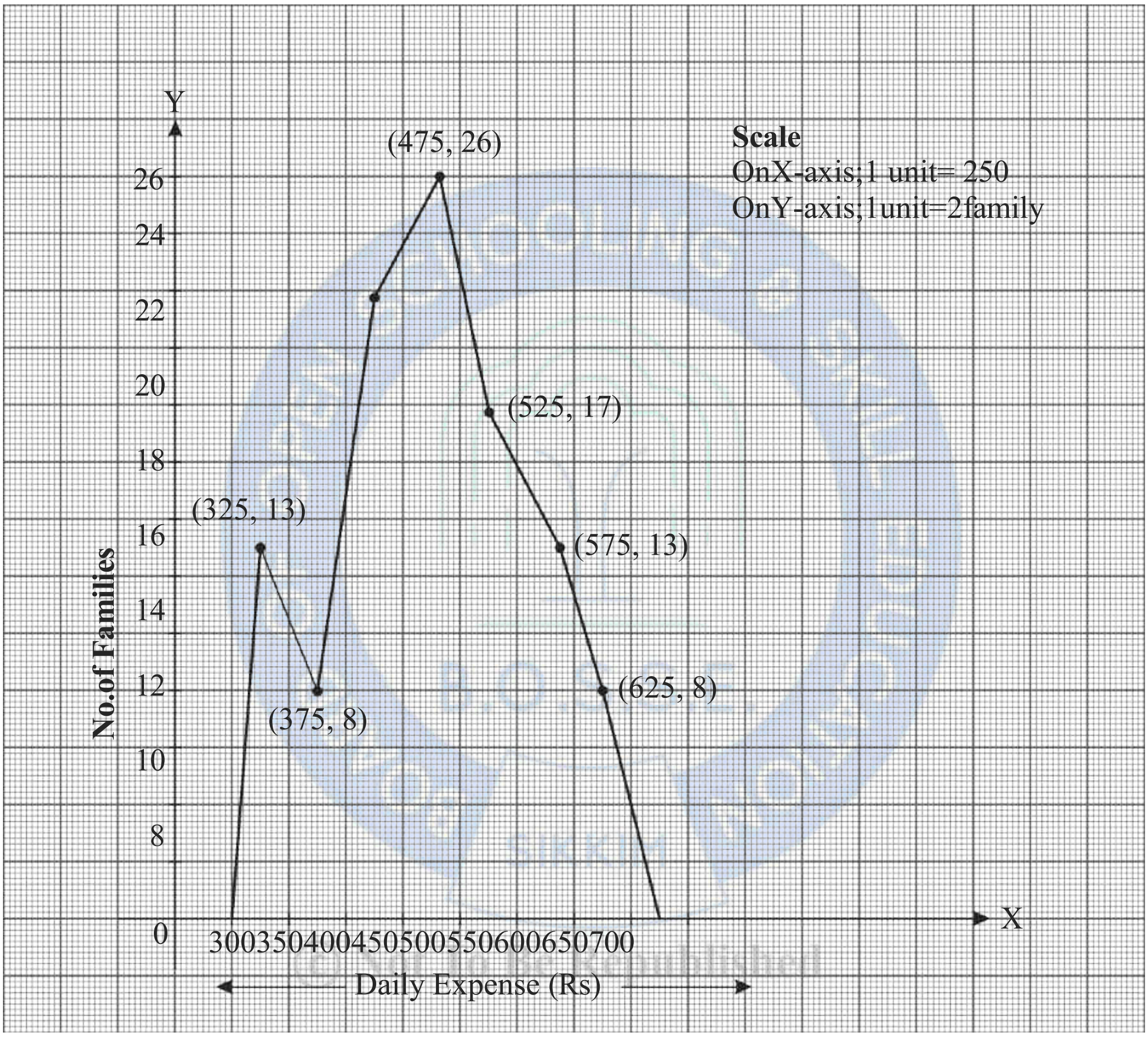


Fig. 22.9

CHECK YOUR PROGRESS 22.4

1. Runsscoredby6 playersin acricket tournamentis given below.

Player	A	B	C	D	E	F
Run	230	210	180	175	205	160

Represent the data using a Bar diagram.

2. Thebargraphgivenbelowrepresentsthenumberofclass10studentsin6secondaryschoolsof a city.

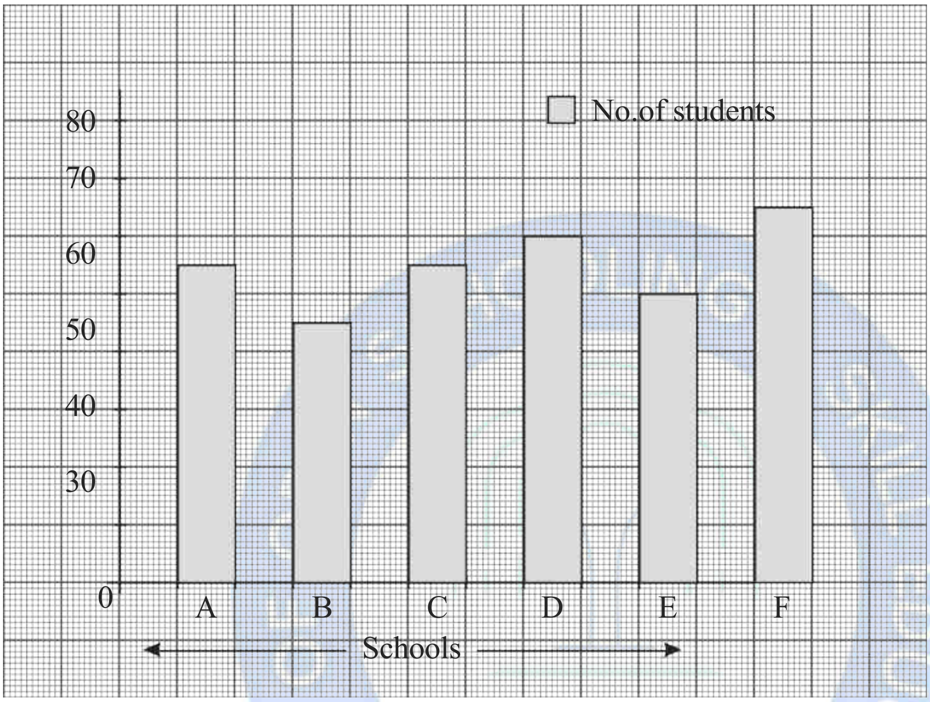


Fig. 22.10

Read the bar graph and answer the following questions –

(i) FindthetotalnumberofstudentsstudyingintheschoolA,DandE.

(ii) Inwhichschoolsamenumbersofstudentsare studying?

(iii) Inwhichschoolthenumberofstudentsis highest?

(iv) HowmanymorestudentsarethereinschoolCthaninschoolB?

3. Thefollowingdatagivestheamountoftime(inminutes)spentdailyinsportsactivitiesby100students of a school.

Timespent	0-15	15-30	30-45	45-60	60-75	75-90	90-105	105-120
Noof students	17	13	20	19	11	8	8	4

Drawa histogram andfrequency polygon torepresent the above data.

RECAPITULATION POINTS

- Statistics is that branch of mathematics which deals with collection, presentation/organization, analysis and interpretation of data. It is used in both plural and singular sense.
- The data collected from a source initially is known as raw data.
- Raw data can be arranged in ascending or descending order of their magnitude is known as arrayed data
- The data collected by an investigator for his/her own purpose is known as primary data
- The data taken from other sources and institutions and agencies and used by the investigator for his/ her purpose is known as secondary data.
- The arrayed data can be presented in the form of a table consisting of data/observation in first column and corresponding frequencies in the second column. This tabular arrangement is known as frequency distribution table for ungrouped data or discrete frequency distribution table.
- The raw data can be presented in the form of a table, where the first column contains a group of continuous observation and the second column contain corresponding frequencies of the group, This type of tabular presentation is called grouped frequency distribution table or continuous frequency distribution table
- The difference between the maximum and minimum observations occuring in the data is called the range of the raw data.
- The total of frequency of a particular class and frequencies of all other classes preceding/succeeding that class is called the cumulative frequency of that class. There are two types of cumulative frequencies, viz, less than and greater than.
- The frequency table showing cumulative frequencies is called cumulative frequency table.
- Using bar graph, data can be represented graphically using number of bars (rectangles) of uniform width, drawn horizontally or vertically with equal space between them
- A grouped frequency distribution can be represented graphically using a histogram.
- A frequency polygon is obtained by first joining the mid points of the top horizontal sides

of the adjacent rectangles in the histogram and then joining the mid-point of first rectangle to the mid-point of the class preceding the lowest class and the last mid-point to the mid-point of the class succeeding the highest class.

- A frequency polygon also can be drawn by taking the mid-points of each class on the x-axis and the frequencies on the y-axis. Points may be marked on the graph paper using mid-point and corresponding frequencies

TERMINAL EXERCISE

1. Fillin theblanks byappropriate words/phrases –

   (i) The difference between the highest and the lowest values observation is known as ____________.

   (ii) The data collected by an investigator 'A' and used by the investigator 'B' for his/her study isknown as ____________ data.

   (iii) The class mark of the class 25-30 is ______________.

   (iv) In a bar graph, the width of each rectangle will be___________.

   (v) The number of times a particular observation occurs in a given data is known as ____________.

2. State true or False

   (i) Tallymarksareusedtofindtherangeofthedata.

   (ii) Mid-pointofa classis thehalfof thesumofthelower limitand theupper limit.

   (iii) In a bar graph the area of each rectangle equal to the frequency.

   (iv) In a histogram class limits are marked along the x-axis.

   (v) Frequencypolygoncannotbedrawnwithouta histogram.

3. Mark the correct alternative in each of the following –

   (i) Thedifferencebetweentheclassmarksoftheclasses38-46and62-70is

   (a) 8 (b) 24

   (c) 42 (d) 32

   (ii) Classwidthof theclass27-33 is

   (a) 7 (b) 30

(c) 33 (d) 6

(iii) In a frequency distribution, the mid-value of a class is 18 and its size is 8.The upper-class limit of the class is

(a) 10 (b) 26

(c) 22 (d) 14

(iv) Markssecuredby6studentsinMathematicsandSocialScienceisgivenbelow

Mathematics:36, 42, 44, 29, 40, 38

Social Science: 37, 40, 39, 38, 43, 35

Thesumoftherangeofthesemarksis

(a) 15 (b) 23

(c) 30 (d) 7

4. The number of members in 20 families of a village are given below 5, 7, 3, 4, 4, 5, 3, 2, 4, 3,4, 6, 5, 4, 3, 4, 6, 5, 3, 2

(i) Construct an ungrouped frequency table for the data.

(ii) Howmanyfamiliesare therewith4 members?

5. Weeklysavingsof40workersinatextilefactoryaregivenbelow –

123, 120, 95, 134, 124, 115, 130, 135, 89, 88,

140,135,124,127,139,99,100,110,107,95,

120,132,134,135,120,105,110,97,101,110,

123,125,130,100,98,124,135,118,120,108

(i) Constructagroupedfrequencydistributionstartingwithclass85-95 (exclusive).

(ii) Construct the cumulative frequency distribution (less than and greater than).

(iii) Draw the histogram of the frequency distribution constructed in question (i).

6. Readthe bargraph shownbelow andanswer the question.

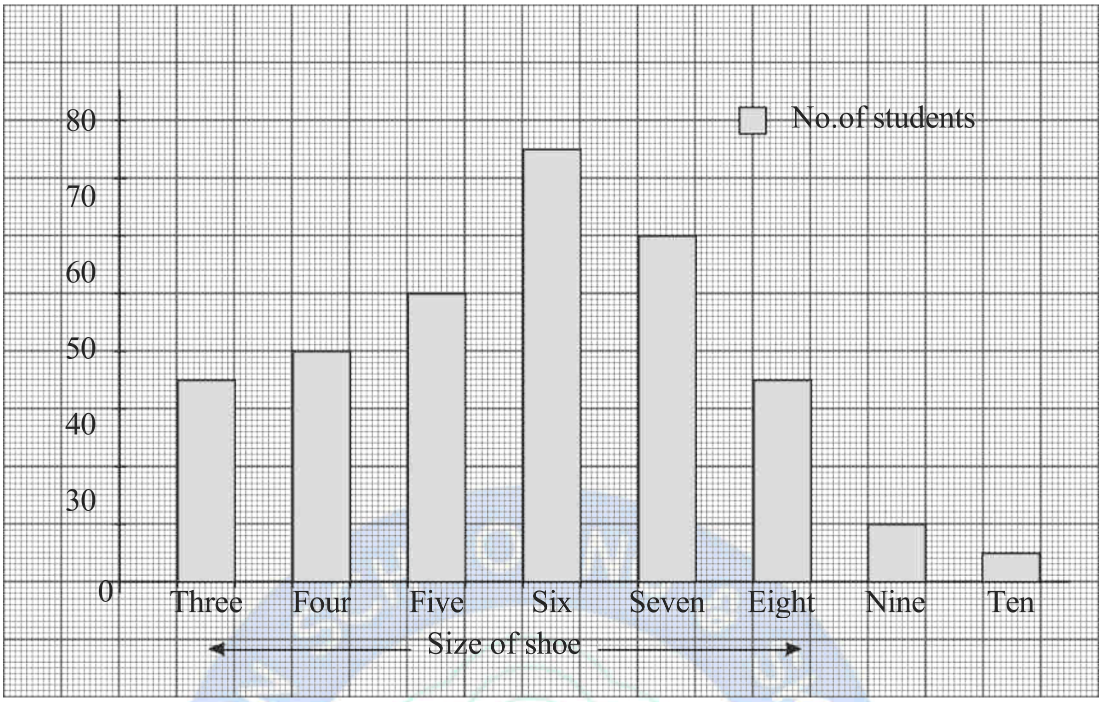


Fig. 22.11

(i) What is the information given in the bar graph?

(ii) Maximum number of students uses shoe of which size?

(iii) How many students' uses shoe of size 10?

(iv) Which shoe sizes used by same number of students?

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 22.1

1. Fillintheblankswithsuitableword(s).

(a) Collection,Analysis

(b) Plural

(c) Quantitative

(d) Secondary

(e) Qualitative

2. Mark true or false against each statement

(a) False

(b) False

(c) True

Quantitative	Qualitative
Height of the students	Scriptsinthestudent'sdiary
Percentage of attendance	Interview scripts
Age of the students	Observationnotes
Price of the vegetables	

CHECK YOUR PROGRESS 22.2

1 (i) Ascendingorder

25, 26, 29, 29, 30, 31, 31, 33, 34, 35,

35, 35, 35, 36, 38, 38, 39, 40, 41, 41,

43, 44, 46, 46, 48

Descendingorder

48, 46, 46, 44, 43, 41, 41, 40, 39, 38,

38, 36, 35, 35, 35, 35, 34, 33, 31, 31,

31, 29, 29, 26, 25

(ii) Maximummark=48

Minimummark=25

Range=Maximum-minimum= 48–25= 23

(iii)

Mark Secured	Tallymarks	Frequency
25	|	1
26	|	1
29	||	2
31	|||	3
33	|	1
34	|	1
35	||||	4
36	|	1
38	||	2
39	|	1
40	|	1

41	||	2
43	|	1
44	|	1
46	|	1
48	||	2
Total		25

2. (i)

Class	Tallymark	Frequency
16-24	|||	3
24-32	||||	4
32-40	~~||||~~ |	6
40-48	~~||||~~ ~~||||~~	9
48-56	|||	3
56-64	~~||||~~	5
Total		30

(ii) Classwithmaximum frequencyis 40-48

(iii) Lower limit = 40 Upper limit = 48

Class mark = (48+ 40)/2 = 88/2 = 44

3. (i) Classwidth=Upperlimit-lowerlimit=4-0=4

(ii) Class mark of 16-20 = (20+16)/2 = 36/2 = 18

(iii) Upperclasslimit of24- 28is 28

(iv) Lowerclasslimitof0-4is0

(v)

Experience (in years)	NoofTeachers
0-3	3
4-7	7
8-11	8
12-15	11
16-19	5

20-23	6
24-27	6
28-31	2
Total	48

(vi) The gap between consecutive classes is 1 and half of the gap is 0.5. Hence the True lower-class limit of the class 4 – 11 is 4 – 0.5 = 3.5.True upper-class limit is 11+0.5=4.5

Therefore, the true class limit of 4 – 11 is 3.5 – 11.5

CHECK YOUR PROGRESS 22.3

1. (i)

Class	Frequency	Cumulative frequency(lessthan)	Cumulativefrequency (greater than)
0-5	2	2	30
5-10	4	6	28
10-15	7	13	24
15-20	6	19	17
20-25	6	25	11
25-30	5	30	5
Total	30		

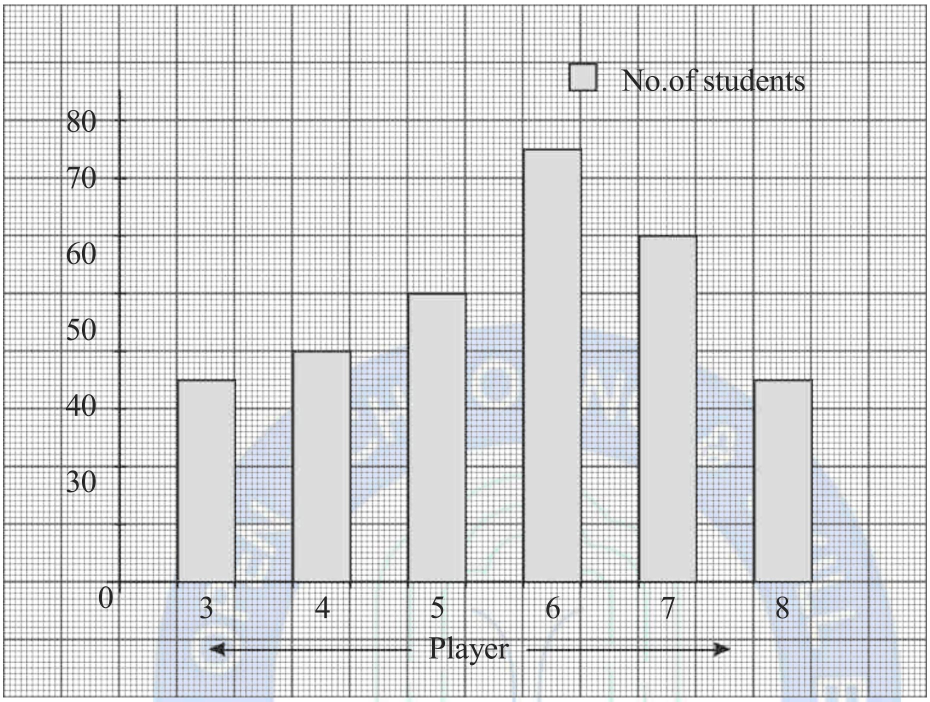
(ii) 11

(iii) 13

2. a=4, b=7, c=29, d= 11, e=6, f= 50 andg = 50

CHECK YOUR PROGRESS 22.4:

1.



BardiagramshowingtheRunscoredby6players

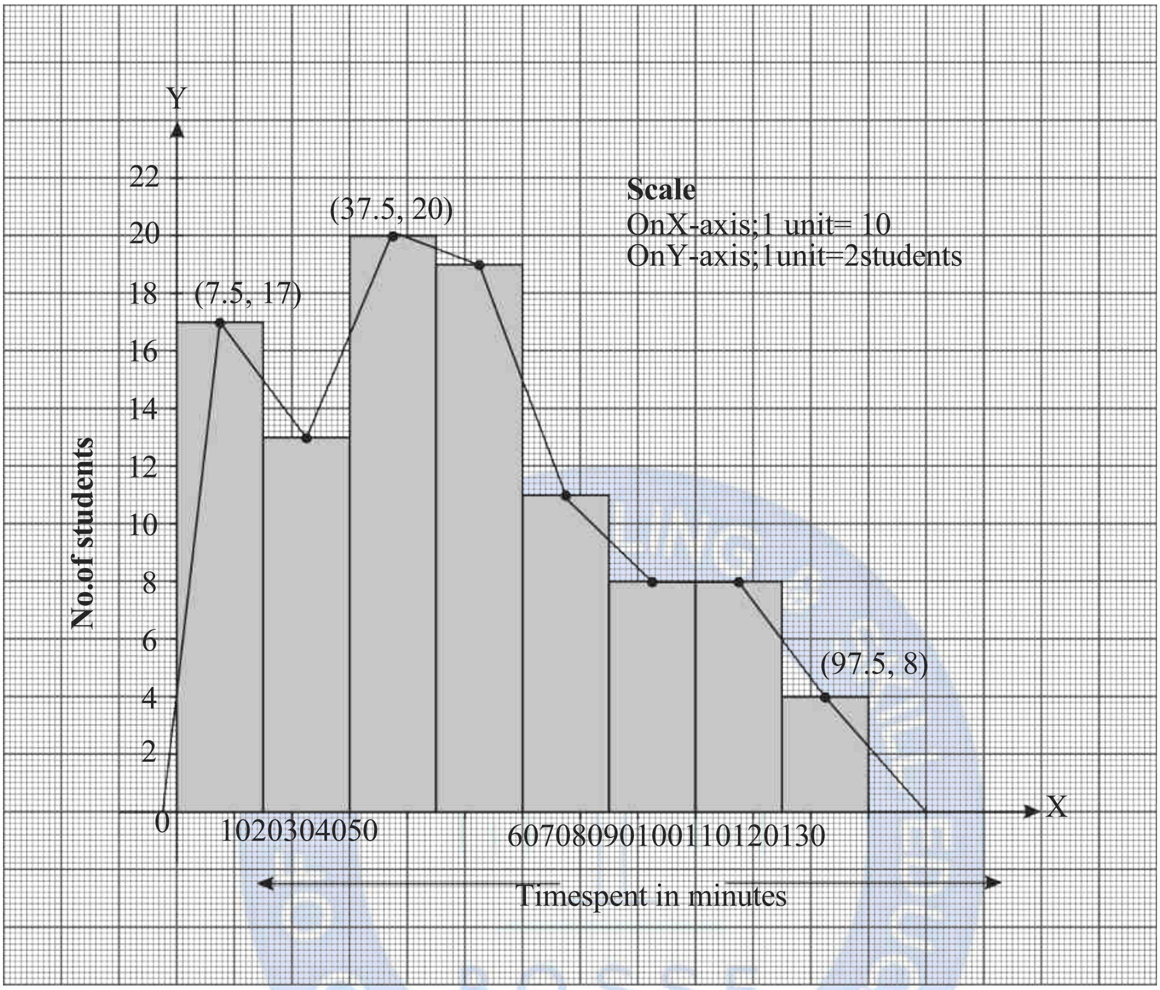
(i) StudentsstudyinginschoolA=55,StudentsstudyinginschoolD=60andStudentsstudyingin school E = 65.

Therefore,studentsstudyinginschoolsA,DandE=180.

(ii) InschoolsAandC,

(iii) SchoolF

(iv) 10



SUPPLEMENTARY STUDY MATERIAL

- NCERT (2021) Mathematics Textbook for Class 9, NCERT, New Delhi
- NCERT (2021) Mathematics Textbook for Class 10, NCERT, New Delhi
- Kathy Chu (Editor, 2013), Elementary Statistics, Rice University, Houston, Texas
- Brian E. Blank(2018), Elementary Statistics, First president university press
- Allan Bluman (2018),Elementary Statistics: A Step By Step Approach, Mc Graw Hill
- https://teachersinstitute.yale.edu/curriculum/units/1986/5/86.05.03.x.html
- https://www.embibe.com/exams/data-representation/
- https://www.kluniversity.in/arp/uploads/2096.pdf
- http://ecoursesonline.iasri.res.in/mod/page/view.php?id=34002

23 MEASURES OF CENTRAL TENDENCY MEAN, MODE AND MEDIAN

INTRODUCTION

Raw data when classified and tabulated is useful for statistical analysis and interpretation. However, this is not alone sufficient for practical purposes. There is a need for further treatment of the data, particularly when we want to compare two or more different distributions data set. We may reduce the entire distribution to one number which represents the distribution using the measures of central tendency. Let us study measures of central tendency in this chapter.

23.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Define measures of central tendency;
- Explain the various types of measures of central tendency;
- Calculate arithmetic mean;
- Calculate median and quartiles;
- Calculate mode;
- Compare the various measures of central tendency.

23.2MEANING OF CENTRAL TENDENCY

The statistical metric that designates one value as the representative of an entire distribution is known as the measure of central tendency. It strives to give a precise account of all the facts. The one value that best characterizes/represents the data is that one. Averages are referred to as measures of central tendency because such typical values, when ordered according to magnitudes, tend to locate centrally within a set of observations. In other words, the central tendency measure condenses the data into a single value that may be used to represent the complete set of data.

Various commonly used average or central tendency measurements include the following:

- Arithmetic Mean
- Median
- Mode

23.3 PURPOSE AND FUNCTIONS OF AVERAGES

- Since an average encompasses all characteristics of a group, conclusions about the entire group can be drawn from it.
- The main characteristics of the entire set of data are succinctly and simply described by the average.
- The data are reduced to a single value via measures of central tendency or averages, which are very helpful for conducting comparison research.
- Averages aid in the growth of a country's economy or the development of a firm's company.

23.4ARITHMETIC MEAN AS A MEASURE OF CENTRAL TENDENCY

A given data collection's arithmetic mean is determined by summing all of its integers and dividing the result by the total number of items in the data set.

For uniformly distributed integers, the middle number serves as the arithmetic mean (AM). Additionally, the AM is estimated using a variety of techniques dependent on the volume and distribution of the data.

The general formula to find the arithmetic mean of a given data is:

$$\textbf{Mean } (\bar{x}) = \frac{Sum\ of\ all\ observations}{Number\ of\ observations}$$

It is denoted by $\bar{x}$, (read as x bar). Data can be presented in different forms.

For n values in a set of data namely as $x_1, x_2, x_3, \ldots xn$, the mean of data is given as:

$$\bar{x} = \frac{x_1 + x_2 + x_3 \ldots\ldots\ldots. x_n}{n} \text{ Or } \bar{x} = \frac{\sum_i^n = 1^x i}{n}$$

For instance, when we receive raw data, such as a student's marks across five subjects, we add the five marks and divide the result by five because there are five topics altogether.

Think of a situation where we have a tonne of information, such as the heights of 40 pupils in a class or the number of people that visited an amusement park on each of the seven days of the week. Will using the aforementioned approach to find the arithmetic mean be practical? NO is the answer! So how can we discover the mean? We put the information in a meaningful and understandable order. Let's see how the arithmetic average is calculated in such circumstances.We will examine how to calculate the arithmetic mean for both grouped and ungrouped data in more detail.

23.4.1 Arithmetic Mean for Ungrouped Data

Here the arithmetic mean is calculated using the formula:

Mean ($\bar{x}$) = $\dfrac{\textit{Sum of all observations}}{\textit{Number of observations}}$

Example 23.1: Compute the arithmetic mean of the first 6 odd, natural numbers.

Solution: The first 6 odd, natural numbers: 1, 3, 5, 7, 9, 11

Mean ($\bar{x}$) = $\dfrac{1+3+5+7+9+11}{6} = 6$

Thus, the arithmetic mean is 6.

23.4.2 Arithmetic Mean for Grouped Data

To determine the arithmetic mean for grouped data, there are three methods: the Direct technique, the Short-cut method, and the Step-deviation method. The numerical value of xi and *fi* determines the method that should be employed.

fi is the total sum of all input frequencies, and xi is the total sum of all data inputs. Summarization is symbolized by the letter "sigma." The direct method will function as long as *xi* and *fi* are both small enough. However, if they are numerically large, we employ the step-deviation approach or the assumed arithmetic mean method. All three methods will be covered in this section along with examples.

23.4.2.1 Direct Method for Finding the Arithmetic Mean

Let $x_1, x_2, x_3 \ldots\ldots x_n$ be the observations with the frequency $f_1, f_2, f_3 \ldots\ldots f_n$.

Then, mean is calculated using the formula:

$$|\bar{x} = (x1f1 + x2f2 + ... + xn_{n}fn) / \sum fi$$

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3.........f_nx_n}{f_1 + f_2 + f_3......f_n} \text{ or } \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Here, $f_1 + f_2 + ...f_n = \sum fi$ indicates the sum of all frequencies.

Example 23.2: Find the mean of the following distribution:

X	10	30	50	70	89
F	7	8	10	15	10

Solution:

x_i	f_i	x_if_i
10	7	10×7 = 70
30	8	30×8 = 240
50	10	50×10 = 500
70	15	70×15 = 1050
89	10	89×10 = 890
Total	$\sum f_i$=50	$\sum x_if_i$=2750

We will add up all the (x_if_i) values to obtain $\sum x_if_i$. Add up all the f_i values to get $\sum f_i$

Now, use the mean formula.

$\bar{x} = \sum x_i f_i / \sum f_i$

$= 2750/50 = 55$

Discrete grouped data can be seen in the problem above.

Now let's look at an illustration where the data is presented **as continuous class intervals.**

Example 23.3: Find the mean of the following frequency distribution:

Class-Interval	15-25	25-35	35-45	45-55	55-65	65-75	75-85
Frequency	6	11	7	4	4	2	1

Solution:

The midpoint of each class, also known as the class mark, is taken into account when calculating the mean when the data is presented in the form of class intervals. The above-stated mean formula is still valid.

$$Class\ mark = \frac{Upper\ limit + Lower\ limit}{2}$$

Class- Interval	**Class Mark (x_i)**	**Frequency (f_i)**	$x_i f_i$
15-25	20	6	120
25-35	30	11	330
35-45	40	7	280
45-55	50	4	200
55-65	60	4	240
65-75	70	2	140

75-85	80	1	80
		$\sum f_i = 35$	$\sum x_i f_i = 1390$

$\bar{x} = \sum x_i f_i / \sum f_i$

$= 1390/35 = 39.71$

Where, $\sum f_i = 35$ and $\sum x_i f_i = 1390$

23.4.2.2 Short-cut Method for Finding the Arithmetic Mean

The short-cut method is called as assumed mean method. The following steps describe this method.

Step1: Determine each class' mid-point (class marks) (x_i).

Step2: Let **A** represent the assumed mean of the data.

Step3: Find deviation (di) = $x_i - A$

Step4: Use the formula:

$\bar{x} = A + (\sum f_i d_i / \sum f_i)$

Example 23.4: Calculate the mean of the following using the short-cut method.

Class-Intervals	45-50	50-55	55-60	60-65	65-70	70-75	75-80
Frequency	5	8	30	25	14	12	6

Solution: Let the assumed mean (A) = 62.5

(A is chosen from the x_i values. Usually, the value which is around the middle is taken).

Class- Interval	**Classmark/ Mid-points (x_i)**	f_i	$d_i = (x_i - A)$	$f_i d_i$

45-50	47.5	5	47.5-62.5 =-15	-75
50-55	52.5	8	52.5-62.5 =-10	-80
55-60	57.5	30	57.5-62.5 =-5	-150
60-65	62.5	25	62.5-62.5 =0	0
65-70	67.5	14	67.5-62.5 =5	70
70-75	72.5	12	72.5-62.5 =10	120
75-80	77.5	6	77.5-62.5 =15	90
		$\sum f_i$=100		**$\sum f_i d_i$= -25**

Now we use the formula,

$\bar{x} = \mathbf{A} + (\sum f_i d_i / \sum f_i)$

$= 62.5 + (-25/100) = 62.5 - 0.25 = 62.25$

∴ Mean = 62.25

23.4.2.3 Step Deviation Method for Finding the Arithmetic Mean

Arithmetic Mean may also be calculated using Step Deviation method. The following steps describe this method:

Step 1: Calculate the class marks of each class (x_i).

Step 2: Let **A** denote the assumed mean of the data.

Step 3: Find $u_i = (x_i - \mathrm{A})/h$, where h is the class size.

Step 4: Use the formula:

$\bar{x} = A + (\sum f_i u_i / \sum f_i)h$

Example 23.5: Find the arithmetic mean of the following using the step-deviation method.

Class Intervals	0-10	10-20	20-30	30-40	40-50	50-60	60-70	Total
Frequency	4	4	7	10	12	8	5	50

Solution: To find the mean, we first have to find the class marks and decide A (assumed mean). Let A = 35. Here h (class width) = 10

C.I.	x_i	f_i	u_i= (xi-A)h	$f_i u_i$
0-10	5	4	-3	4 x (-3)=-12
10-20	15	4	-2	4 x (-2)=-8
20-30	25	7	-1	7 x (-1)=-7
30-40	35	10	0	10 x 0= 0
40-50	45	12	1	12 x 1=12
50-60	55	8	2	8 x 2=16
60-70	65	5	3	5 x 3=15
		∑fi=50		**∑fiui=16**

Using mean formula:

$\bar{x} = A + (\sum f_i u_i / \sum f_i)h = 35 + (16/50) \times 10 = 35 + 3.2 = 38.2$

Mean = 38

CHECK YOUR PROGRESS 23.1

1. State if True or False:
   (a) Mean is rigidly defined.
   (b) Mean is the most commonly used measures of central tendency.
   (c) While computing Arithmetic Mean from a grouped frequency distribution, we assume that the classes are of equal length.
2. Determine the first five prime numbers' arithmetic mean.
3. The following table lists the ages (in years) of 50 players from a school:

Age (in years)	14	15	16	17	18
Number of players	15	14	10	8	3

Calculate the mean age.

23.5 MEDIAN AS A MEASURE OF CENTRAL TENDENCY

The median of the data is the value of the middle observation found after sorting the data in ascending order. At this stage, half of the data are more and half are less. The median makes it possible to express a lot of data points with just one.

The middle data point reflects the median of the data after the data are organised in ascending order for the purpose of calculating the median. The middle value is the median for an odd number of data, and the average of the two middle values is the median for an even number of data. Learn more about the median, how to calculate the median when there are even and odd numbers of data points, and the median formula.

Compute the median for a given set of data: 4, 4, 6, 3, 2

Step 1: Arrange this data in ascending order: 2, 3, 4, 4, 6.

Step 2: Count the number of values. There are 5 values.

Step 3: Look for the middle value. The middle value is the median. Thus, median = 4.

23.5.1 Median for Ungrouped Data

To calculate Median for ungrouped data, the following steps must be followed:

Step 1: Arrange the data in ascending or descending order.

Step 2: Secondly, count the total number of observations 'n'.

Step 3: Check if the number of observations 'n' is even or odd.

(a) When ‘n’ is Odd

Median = $[(n + 1)/2]^{th}$ term

(a) When ‘n’ is Even

Median = $[(^{n}/_{2})^{th}$ term + $((^{n}/_{2}) + 1)^{th}$ term]/2

Example 23.6: Find the median of the data set: 42, 40, 50, 60, 35, 58, 32

Solution:

Step 1: Arrange the data items in ascending order.

= 32, 35, 40, 42, 50, 58, 60

Step 2: Count the number of observations. If the number of observations is odd, then we will use the following formula: Median = $[(n + 1)/2]^{th}$ term

Step 3: Calculate the median using the formula:

Median = $[(n + 1)/2]^{th}$ term

= $(7 + 1)/2^{th}$ term = 4^{th} term = 42

Median = 42

23.5.2 Median for Grouped Data

The following steps are used to determine the median when the data is continuous and represented as a frequency distribution.

Step 1: Find the total number of observations(n).

Step 2: Define the class size(i), and divide the data into different classes.

Step 3: Calculate the cumulative frequency of each class.

Step 4: Identify the class in which the median falls. (Median Class is the class where $^{n}/_{2}$ lies.)

Step 5: Find the lower limit of the median class(L), and the cumulative frequency of the class preceding the median class ($c.f$).

Step 6: To compute Median use the formula: Median $=L+[^n/_2-c.f]/f \times i$

Example 23.7: Calculate the median for the following data:

Marks	0 - 20	20 - 40	40 - 60	60 - 80	80 - 100
Number of students	5	20	35	7	3

Solution:

Marks	**Number of students (*f*)**	**Cumulative frequency (*c.f*)**
0 - 20	5	5
20 - 40	20	25
40 - 60	35	60
60 - 80	7	67
80 - 100	3	70

$n = 70$

$^n/_2 = 70/2 = 35$

Median Class is 40 - 60

$L = 40, f = 35, c.f = 25, i = 20$

Using Median formula:

Median $= L+[^n/_2-c.f]/f\times i$

$= 40 + [(35 - 25)/35] \times 20$

$= 40 + (10/35) \times 20$

$= 40 + (40/7)$

= 45.71

CHECK YOUR PROGRESS 23.2

1. The ages of a soccer team's players are as follows:42,40,50,65,35,58,32. Determine the Median.
2. A student received the following grades in several subjects: 98, 64, 76, 91, 44, and 81. Determine the median.
3. Find the median for the following distribution:

Wages (in Rs.)	0-10	10-20	20-30	30-40	40-50
No. of Workers	22	38	46	35	20

23.6 MODE AS A MEASURE OF CENTRAL TENDENCY

The value that appears more frequently in a particular set of values is referred to as the mode. The value that appears the most frequently is this one. Since it has occured twice in the given collection of data (2, 4, 5, 5, 6, 7), the mode of the data set is 5.

A data set is referred to as bimodal if there are two modes in it.

For instance, since 2 and 5 are repeated three times in the given set, Set A = 2,2,2,3,4,4,5,5,5 has 2 and 5 as its modes.

A data set is referred to as **trimodal** if there are three modes in it.

For instance, the 2, 5, and 8 make up the mode of the set A = 2,2,2,2,3,4,4,5,5,5,7,8,8,8.

23.6.1 Mode for Ungrouped Data

Example 23.8: The number of wickets a bowler claimed throughout the course of 10 games is shown in the following table. Find the mode of the provided data set.

Match No.	1	2	3	4	5	6	7	8	9	10

No. of Wickets	2	1	1	3	2	3	2	2	4	1

Solution: It can be seen that 2 wickets were taken by the bowler frequently in different matches. Hence, the mode of the given data is 2.

Example 23.9: Find the mode of 3, 6, 9, 16, 27, 37, 48.

Solution: A data set lacks a mode if no value or number appears more than once in the collection. As a result, set 3, 6, 9, 16, 27, and 48 do not have a mode.

23.6.2 Mode for Grouped Data

When there is a clustered frequency distribution, the mode cannot be determined just by looking at the frequency. In these situations, the modal class is computed to determine the mode of the data. Modal class contains the concept of mode. The following formula identifies the type of data:

$$Mode = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right) \times h$$

Where,

l = lower limit of the modal class

h = size of the class interval

f_1 = frequency of the modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

Let us take an example to understand this clearly.

Example 23.10: The math marks pupils received out of 50 in a class of 30 students are tallied as follows. Determine the given data's mode.

Marks	10-20	20-30	30-40	40-50
No. of Students	2	12	1	5

Solution: The class interval that corresponds to the maximum class frequency of 12 is 20–30. The modal class so ranges from 20 to 30.

Lower limit of the modal class (l) = 20

Size of the class interval (h) = 10

Frequency of the modal class (f_1) = 12

Frequency of the class preceding the modal class (f_0) = 5

Frequency of the class succeeding the modal class (f_2)= 8

Substituting these values in the formula we get;

$$Mode = \int + \left(\frac{f_{1-}f_0}{2f_{1-}f_{0-}f_2} \right) \times h = 20 + \left(\frac{12-5}{2\times 12-5-8} \right) \times 10 = 26.364$$

Mode can also be calculated the following formula:

Mode = 3 Median – 2 Mean

CHECK YOUR PROGRESS 23.3

1. Calculate the mode for the following data:

Size of the winter coat	38	39	40	42	43	44	45
Total number of shirts	33	11	22	55	44	11	22

2. Given below is the data representing the scores of the students in a particular exam. Find the mode for this:

Class Interval	0–5	5–10	10–15	15–20	20–25
Frequency	5	3	7	2	6

23.7 PRECAUTIONS IN USING CENTRAL TENDENCY

Mean	Median	Mode
Mean is the term for the calculated average of the provided observations	The median is the value that falls in the middle of a particular set of observations.	The term "mode" refers to the number that appears the most frequently in a given set of data.
Divide the total number of terms by the sum of all the numbers.	Sort the numbers in either ascending or descending order.	Derived when a number has frequency occurred in a series.
The average or mean score is the outcome.	Finding the middle number on the list is the next step. It's referred to as the median.	We can either have multiple modes or none at all.

RECAPITULATION POINTS

1. The statistical measure that designates one value as the representative of an entire distribution is known as the measure of central tendency.
2. The arithmetic mean, often known as the mean or arithmetic average, is frequently used. It is determined by adding up each number in a particular data set, then dividing the result by the overall number of items in the data set.The general formula to compute Arthematic Mean is

$$\text{Mean } (\bar{x}) = \frac{\textit{Sum of all observations}}{\textit{Number of observations}}$$

3. There are three ways to determine the arithmetic mean for grouped data: the Directtechnique, the Short-cut method, and the Step-deviation method.
4. Calculation of Arithmetic Mean Using Direct Method

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3.........f_nx_n}{f_1 + f_2 + f_3......f_n}$$

5. The midpoint of each class, also known as the class mark, is taken into account when computing the mean when the data is presented in the form of class intervals.

$$\textit{Class mark} = \frac{\textit{Upper limit + Lower limit}}{2}$$

SW

6. The short-cut method is called as assumed mean method. The formula for finding mean by this method is $\bar{x}$ = A + ($\sum f_i d_i / \sum f_i$)
7. Arthematic Mean may also be calculated using Step Deviation method. The formula for finding the mean by this method is $\bar{x}$ = A + h × ($\sum f_i u_i / \sum f_i$)
8. The median of the data is the value of the middle observation found after sorting the data in ascending order.
9. To calculate Median for ungrouped data we use the formula Median = $[(n + 1)/2]^{th}$ term(when n is odd) and the formula Median = $[(^n/_2)^{th}$ term + $((^n/_2) + 1)^{th}$ term]/2 (when n is even).
10. To calculate Median for grouped data we use the formula: Median =L+$[^n/_2 - c.f]/n \times i$
11. Mode is defined value that has a higher frequency in a given set of values. It is the value that appears the greatest number of times.
12. In the case of grouped frequency distribution, calculation of mode can be done by using the formula:

$$Mode = \int + \left(\frac{f_1 - f_0}{2 f_1 - f_0 - f_2} \right) \times h$$

13. Mode = 3 Median – 2 Mean

TERMINAL EXERCISE

1. What are measures of central tendency?
2. State the importance of various measures of central tendency in Statistical analysis.
3. Define Mean. How is it different from Median and Mode?
4. Find the mean of this set of data:116, 104, 101, 111, 100, 107, 113, 118, 113, 101, 108, 109, 105, 103, and 91.

5. Find the median of this set of data: 34, 31, 37, 44, 38, 34, 43, and 41.
6. Compute the arithmetic mean of the following by direct and short -cut methods both:

Class	20-30	30-40	40-50	50-60	60-70
Frequency	8	26	30	20	16

7. Find the median of the following frequency distribution:

Marks	No. of Students	Marks	No. of Students
Less than 10	15	Less than 50	106
Less than 20	35	Less than 60	120
Less than 30	60	Less than 70	125
Less than 40	84		

8. Find the mode from the following size of shoes:

Size of Shoes	1	2	3	4	5	6	7	8	9
Frequency	1	1	1	1	2	3	2	1	1

9. A student recorded her scores on weekly math quizzes that were marked out of a possible 10 points. Her scores were as follows: 8, 5, 8, 5, 7, 6, 7, 7, 5, 7, 5, 5, 6, 6, 9, 8, 9, 7, 9, 9, 6, 8, 6, 6, 7

   What is the mode of her scores on the weekly math quizzes?

ANSWER TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS 23.1

1. (a) True

   (b) True

   (c) False

2. 5.6

3. 15.4

CHECK YOUR PROGRESS 23.2

1. 42

2. 78.5

3. 24.46

CHECK YOUR PROGRESS 23.3

1. 42

2. 12.22

GLOSSARY

Measures of Central Tendency: A measure of central tendency is a single value that seeks to characterize a collection of data by locating the central location within that collection.

Mean: The arithmetic mean is the total of a set of integers divided by their count. It is also known as the simple average.

Median: The median is an average that, when a set of numbers is ranked, typically denotes the middle number in the sequence.

Mode: The value that occurs more frequently in a particular group of values is referred to as the mode.

SUPPLEMENTARY STUDY MATERIAL

Goel, J.P and Goel, K. (2017). Elementary Statistics. New Delhi: True Friend Publications.

24 PROBABILITY

INTRODUCTION

Our life is full of uncertainties. In our day-to-day life we do make guess and use statements like:

1. **Probably**, it will rain today.

2. Indian cricket team will win first one day test match,**certainly**.

3. It is **almost certain** that my sister will qualify IIT- JEE main examination.

4. **Most probably** my father will be an education minister.

The words used in these statements such as probably, certainly, almost certain and most probably, we have an intuition which enable us to claim that one is more likelyto happen than the other. We do make thesepredictions but at the same time we wish to assess the chances of our predictions coming true. This we can doby measuring the uncertainty involved. The uncertainty of probably etc. can be measurednumerically by the means of what we call probability in many cases. The foundation of theory of probability provides us with the methodology of measuring uncertainty involved. These days probability is being used in the fields of Biological Sciences, Medical Sciences, Economics, life of a product produced by companies etc., while uses of probability are almost endless.

Probability started with the gambling. For example, tossing of a coin, throwing a die or dice (plural of a die), drawing one or more cards from a well shuffled deck of 52 cards or drawing a ball from an urn etc. In fact, probability is the mathematical measurement of uncertainty.

24.1 LEARNING OBJECTIVES

After completing this lesson, you will be able to:

- Gain knowledge of basic terms used such as experiments, event, random experiment negation of an event,equally likely events, favorable event etc. and sample space.
- Understand the theoretical probability of an event.
- Calculate probability of an event.

24.2 SOME BASIC ITEM

In introduction of this lesson, we have used basically three items namely – coins, die and deckof cards. Let us have some basic idea of these.

24.2.1 Coin

Fig. 24.1

We all are very well familiar with Indian coin(s). One face of the coin is known as head and other is tail.

Note: When we toss a coin, the chances of getting head or tail are equally likely.

24.2.2 Die

A **die** is a well-balanced cube with six faces marked with dots ⚀, ⚁, ⚂, ⚃, ⚄, and ⚅.We shall, in thislesson, will denote these dots by their corresponding numbers namely 1,2,3,4,5 and 6.


Fig. 24.2

24.2.3 Deck of Cards

A deck of 52 cards consists of four suits namely Spade (♠), Hearts (♥), Diamonds (♦) and Clubs (♣). Each suit has 13 cards. Four cards are of each suit has an Ace, a King, a Queen and a Jack while other nine cards are marked as 2,3,4,5,6,7,8,9 and 10. The suits of Hearts and Diamond are red in color and suits of Clubs and Spades are black in color. The Ace, King, Queen and Jack of all suits are called **Face Cards**.

Note: In this lesson, we shall be usingonly a fair coin, a fair deck of 52 cards and a fair die or dice only.

Now, we shall first learn some basic concepts/terms used in defining probability.

24.2.4 Experiment

An action or an operation which can produce some well-defined result is called an **experiment**.

24.2.5 Deterministic Experiment

When an experiment is repeated under the identical conditions, and the result (or outcome) each time is same, then such an experiment is called a **Deterministic Experiment**.

Example, "Tossing of a fair coin" is a random experiment because when a fair coin is tossed, we will get a "head" or a "tail". But if we toss a coin again and again the outcome (result) will not be same. i.e., every time either head or tail will come up. "Drawing a card from well shuffled deck of 52 cards", is a random experiment."Throwing an unbiased die" is a random experiment simply because when a die is thrown, we cannot say with certainty which one of the numbers will come up on the upper surface.

24.2.6 Events

The possible outcome of a random experiment is called an **event**.

Example, in an experiment of tossing a coin, the possible outcomes are head (H) or tail (T).

24.2.7 Equally likely events

Events are said to be equally likely when we have no reason to believe that one event is more likely to occur than the other.Example,

1. When a coin is tossed once H or T are equally likely to come up. Hence, we say events H and T are equally likely events.
2. When a die is thrown once, all six faces 1,2,3,4,5 and 6 are equally likely to come up. Hence, we say events 1,2,3,4,5 and 6 are equally likely events.

24.2.8 Sample space

In an experiment, the sample space is the set of all possible outcomes or results of that experiment. It is usually denoted by using set notation, {}, and the possible ordered outcomes are listed as the elements in the set.

Example, when a coin is tossed once, its sample space is {H,T}. Likewise, when a die is tossed, its sample space is {1,2,3,4,5,6}.

Similarly, when two coins are tossed simultaneously, then its sample space is {HH, TT, HT, TH} and when three coins are tossed its sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

You may find it not easy to write this type of sample spaces, if three or more than three coins are tossed.

In the above two examples you have also seen that events are **equally likely events** because chances of their happening are neither less nor greater than the other. Also, they are mutually exclusive because happening of one event does not preludes the happening of another event.

Note: It is easier to prepare sample space for the compound events by using tree-diagrams shown below –

Example,four coins are tossed simultaneously then its tree diagram is:

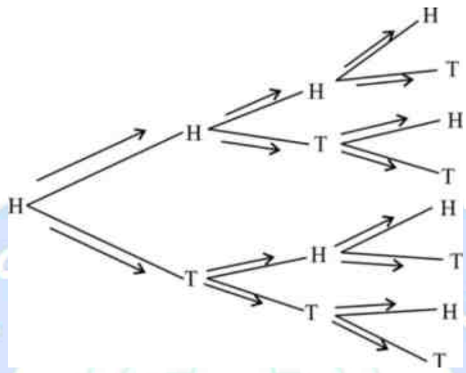


Fig. 24.3

Make a tree diagram as shown above for the event H. Similarly, for T, as given below –

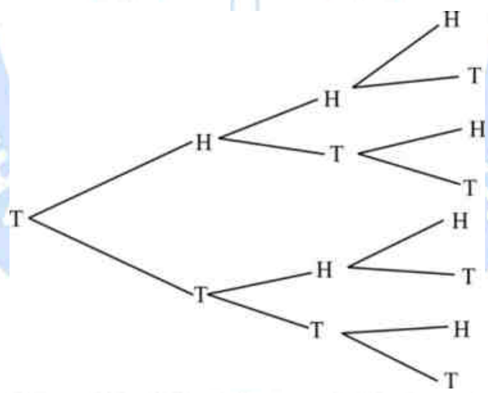


Fig. 24.4

Now move on each branch of the tree and write its sample space as given below –

Sample space = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}.

CHECK YOUR PROGRESS 24.1

1. Define the following terms.

   (i) Experiment.

   (ii) Random Experiment.

(iii) Outcomes.

(iv) Equally likely events.

2. What is a sample space?

3. Write a sample space when two coins are tossed simultaneously.

4. How many times a pair of heads occur together when four coins are tossed simultaneously once?

5. Write a sample space when a die is thrown twice.

6. Two dice one blue and one red are thrown at the same time. How many possible outcomes are there?

7. One card is drawn at random from a well-shuffled deck of 52 card. How many chances are there to:

(i) the Jack of Hearts? (ii) a red face card?

(iii) a face card? (iv) Queen of Diamonds?

(v) a Spade card? (vi) a Club card?

8. If a card is drawn from a well-shuffled deck of 52 cards, then the number of possible outcomes to geta card of spade or an ace are

(a) 8. (b) 10. (c) 12. (d) 16.

9. If a card is drawn from well-shuffled deck of 52 cards, then the number of possible outcomes to get neither a black card nor a red card is

(a) 0. (b) 2. (c) 3. (d) 4.

10. If numbers 1 to 30 written on small papers separately are mixed then the number of possible outcomes to outcomes to have a prime number on its face if one card is drawn are –

(a) 8. (b) 10. (c) 12. (d) 13.

11. Which of the following experiment have equally likely outcomes? Explain.

(i) A trial is made to answer a true-false question. The answer is right or wrong.

(ii) A baby is born. It is a boy or a girl.

(iii) A cricket player attempts to hit a six. He hits or misses a hit.

24.3 RANDOM EXPERIMENT AND ITS OUTCOMES

A **random experiment** is an experiment whose all possible outcomes are known, and which can be repeated under the identical conditions but not possible to predict the outcomes of any particular trail in advance.

Example, tossing of a coin is an example ofa random experiment. The outcomes are two in number, either a head or a tail appears but exact prediction is impossible in any toss.

Similarly,single throw of a die is also a random experiment with six outcomes namely 1,2,3,4,5 and 6 will turn up. But here also exact prediction beforehand is not possible in any throwing.

24.4 SOME SPECIAL EVENTS

24.4.1 Impossible Event

An event is said to be an impossible event when none of the outcomes is favorable to the event.

Example, in a single throw of a die, number of favorable events of getting 9 is 0.

24.4.2 Sure Event

An event is said to be sure event when all possible outcomes are favorable to the event.Example, in a single throw of a die getting a number less than 7 is a sure event.

24.4.3 Complement of an Event

Corresponding to every event say *E,* associated with the random experiment, there is an event 'not *E*' which occurs only when the event *E* does not occur.The event $\bar{E}$or E' is usually representing 'not *E*' is called complement of the event *E*.

24.4.4 Compound Event

An event associated with a random experiment is compound event if it is obtained by combining two or more events associated to the random experiment.

Example, in tossing a coin, the event head or tail is compound event. Since it can be decomposed into two simple events "head" and the event tail. Similarly, in tossing two coins, the event "one head and one tail" is a compound event consisting of events (HT) and (TH).

24.4.5 Favorable Events

An event is said to be favorable to compound event A, if it satisfies the definition of compound event A.

Example, consider a random experiment of throwing a pair of dice and compound event A defined by "get a doublet". We find that event A occurs if we get any one of following event as outcome.

(1,1) (2,2) (3,3) (4,4) (5,5) and(6,6).

Thus, there are 6 events favorable to event A.

24.5 PROBABILITY

In a random experiment, if the events are equally likely, and mutually exclusive,then ratio of favorable events to the total number of the events of the experiment is called the probability.i.e.,

$$\textit{Probability of an event} = \frac{\text{Number of favorable events}}{\text{Total number of events}}$$

Example, for a single throw of a die we have only 2 outcomes then its sample space will be {H,T}. Now number of favorable outcomes for the occurrence of head = 1

Also, number of favorable outcomes for the occurrence of tail = 1

Thus, probability of occurrence of head P(H) = $\frac{1}{2}$and probability of occurrence of tail P(T)= $\frac{1}{2}$

Note that,

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

In other words, P(H) + P($\bar{H}$) = 1, where P($\bar{H}$) or P(H′) (as per the usual notation)denotes 'not H′, i.e., P($\bar{H}$) or P(H′) = 1 – P(H).

24.6 AN IMPORTANT RESULT

Let m be the number of favorable outcomes of an event E and n be the total number of outcomes. Then, $0 \le m \le n$.

i.e. $0 \leq \frac{m}{n} \leq \frac{n}{n}$,

or $0 \leq \frac{m}{n} \leq 1$,

or $0 \leq P(E) \leq 1$.

Thus, probability of an event always lies from 0 to 1.

Example 24.1: Which of the following cannot be the probability of an event?

(a) $\frac{2}{3}$ (b) –1.5 (c) 25% (d) 0.8

Solution: Since probability of an event cannot be negative, thus (b) -1.5 cannot be the probability of an event.

Example 24.2: An event is unlikely to happen. Its probability to closest to

(a) 0.00002. (b) 0.01. (c) 0.1. (d) 0.001.

Solution: Note that the smallest number among the given numbers is 0.00002. Thus, the probability of event unlikely to happen is close to (a) 0.00002.

Example 24.3:A die is thrown once. What is the probability of getting

(a) a prime number?

(b) an even number?

(c) a prime number or an even number?

(d) a number divisible by 3?

Solution: In a single throw of a die our sample space is {1,2,3,4,5,6}.i.e.,total number of outcomes is 6.

(a) Prime numbers in a single throw are 2,3 and 5.

$\therefore$ Favorable outcomes = 3

Hence, probability of getting a prime number $= \frac{3}{6} = \frac{1}{2}$.

(b) The even number in a single throw are 2,4 and 6.

$\therefore$ Favorable events = 3

Hence, probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$.

(c) In a single throw of a die, a prime number or an even number or an even number are 2,3,4,5 and 6.

$\therefore$ Favorable outcomes = 5

Its probability is $= \frac{5}{6}$.

(d) Numbers divisible by 3 in a single throw of a die are 3 and 6.

Therefore, favorable events = 2

Hence, probability of getting a number multiple of 3 = 2

Hence, required probability $= \frac{2}{6} = \frac{1}{3}$

Example 23.4: In a simultaneous throw of a pair of dice, find the probability of getting

(a) an even number on the first die.

(b) a sum of more than 7.

(c) a doublet of odd numbers.

(d) neither 9 or 11 as the sum of numbers on the faces.

Solution: (a) We know that when two dice are thrown simultaneously its sample space is

{ (1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)}

Here total number of events are 36.

The favorable events for getting "an even number on the first die" are

{ (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

Therefore, favorable events of getting "an even number on the first die" are 18.

Hence, required probability = $\frac{18}{36} = \frac{1}{2}$.

(b) The favorable events for getting "a sum more than 7" are

{ (2,6) (3,5) (3,6) (4,4) (4,5) (4,6) (5,3)

(5,4) (5,5) (5,6) (6,2) (6,3) (6,4) (6,5) (6,6) }

Number of favorable outcomes = 15

Hence, required probability = $\frac{15}{36}$

(c)The favorable events for getting "a doublet of odd numbers is"

{ (1,1) (3,3) (5,5) }

Therefore, number of favorable outcomes = 3

Hence, the required probability = $\frac{3}{36} = \frac{1}{12}$.

(d) The favorable events for "9 or 11" as the sum of numbers on the faces are

{ (3,6) (4,5) (5,4) (6,3) (5,6) (6,5) }.

Therefore, number of favorable events for 9 or 11 as sum are 6.

Hence, probability of getting 9 or 11 as the sum of numbers on the faces= $\frac{6}{36} = \frac{1}{6}$

We know that,

P(E) + P(Ē) =1

i.e., P(Ē) = 1-P(E)

$= 1 - \frac{1}{6} = \frac{5}{6}$

Thus, probability of events "neither getting 9 or11 as the sum of the numbers" = $\frac{5}{6}$.

Example 24.5: A card is drawn from a well-shuffled deck of 52 cards. Find the probability that the card drawn is

(a) a black King.

(b) either a Jack or an Ace.

(c) a card of Club or an Ace.

(d) either a King or a black Queen.

Solution:(a) Total number of possible outcomes = 52

Total number of black King in a deck of 52 cards (a Club King and a Spade King) =2

Therefore, probability of getting a black King = $\frac{2}{52} = \frac{1}{26}$

(b) Now total number of jack or an ace are (4+4) = 8

Total number of events = 52

Thus, the probability that the card drawn is either a Jack or an Ace = $\frac{8}{52} = \frac{2}{13}$.

(c) Total number of possible outcomes = 52

Favorable outcomes i.e., to have a card of Club or an Ace = (13+3) = 16

Therefore, probability that the card drawn is a card of club or anace = $\frac{16}{52} = \frac{4}{13}$.

(d) Here also total number of outcomes = 52

In a deck of 52 cards, there are 4 Kings and 2 black Queens.

Therefore, total numbers of favorable outcomes = 6

Hence, the required probability =$\frac{6}{52} = \frac{3}{26}$

Example 24.6: A bag contains 5 red balls, 7 black balls and 10 white balls. If one ball is drawn at random from the bag, then find the probability that the ball drawn is

(a) red or white.

(b) not red.

(c) neither white nor black.

(d) only black.

Solution: Total number of balls in a bag = 22

So, we can drawa ball from the bag in 22 ways.

i.e.,total numbers of possible outcomes = 22

(a) Total number of red or white balls = 15

$\therefore$ One ball can be drawn in 15 ways.

i.e., total number of favorable outcomes = 15

Hence, the required probability = $\frac{15}{22}$.

(c) Now, total number of balls other than red balls are 17.

$\therefore$ One ball can be drawn in 17 ways.

i.e.,total number of favorable outcomes =17

Hence, the required probability is = $\frac{17}{22}$.

(d) Now, total number of balls other than white and black are 5.

i.e., one ball can be drawn in 5 ways.

Hence, the required probability = $\frac{5}{22}$.

(e) Now black balls in a bag are 7.

$\therefore$ One ball can be drawn in 7 ways.

$\therefore$ Total number of favorable events = 7

Thus, required probability = $\frac{7}{22}$.

Example 24.7: Cards marked with numbers 2 to 101 are placed in the box and mixed thoroughly. One card is drawn at random from the box. Find the probability that the number on the card is

(a) an even number.

(b) a number less than 44.

(c) a number which is a perfect square.

(d) a number whose square root is 3.

Solution: Cards are marked as 2,3,4,5,6,7,8,9……100,101.

Total number of cards =100

Total number of the sample space = 100

(a) Number of even numbers from 2 to 100.

i.e., favorable events = 50

Thus, the required probability = $\frac{50}{100} = \frac{1}{2}$.

(b) Total numbers less than 44 from 2 to 43, are 42.

Therefore, favorable events number = 42

Thus, the required probability = $\frac{42}{100} = \frac{21}{50}$.

(c) Total number from 2 to 101 which are perfect squares is9.

i.e., {4,9,16,25,36,49,46,81,100}.

Therefore, the favorable events are 9.

Thus, the required probability = $\frac{9}{100}$.

(d) In this case, the favorable events will be {9}.

i.e., only one favorable event.

Thus, the required probability = $\frac{1}{100}$.

Example 24.8: Three coins are tossed together. Find the probability of getting

(a) at least 2 head.

(b) only tails.

(c) one head and two tails.

Solution: We know that the sample space when three coins are tossed simultaneously is

{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Total number of events = 8

(a) Favorable events = {HHH, HHT, HTH, THH}

i.e., number of favorable events =4

Thus, the required probability = $\frac{4}{8} = \frac{1}{2}$.

(b) Favorable event is {TTT}.

i.e., number of favorable event = 1

Thus, the required probability = $\frac{1}{8}$.

(c) Favorable events {HTT, THT}.

i.e., Number of favorable events = 2

Thus, the required probability = $\frac{2}{8} = \frac{1}{4}$.

RECAPITULATION POINTS

- **Experiment:**An action or operation which can produce some well-defined result is called an experiment.
- **Event:** Collection of possible outcomes of an experiment is called a sample space.
  - Probability of a sure event is 1.
  - Probability of an impossible event is 0.
  - The probability of an event E, written as p(E), is defined by

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Total number of outcomes of the experiment}}$$

  - $P(E) + P(\bar{E}) = 1$.
  - The probability or an event E is a number P(E) such that$0 \leq P(E) \leq 1$.

TERMINAL EXERCISE

A. Multiple Choice Questions:

1. If an event cannot occur, then its probability is

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) 1

2. An event is very unlikely to happen. Its probability is

(a) 0.01 (b) 0.001 (c) 0.0001 (d) 1

3. If the probability of winning a game is 0.6 then the probability of losing it is

(a) 0.5 (b) 30% (c) $\frac{4}{10}$ (d) 70%

4. When a die is thrown once, the probability of getting at least number 2 is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c)1 (d) $\frac{5}{6}$

5. A card is selected from a deck of 52 cards. The probability of its being a black face card is

(a) $\frac{3}{13}$ (b) $\frac{1}{13}$ (c) $\frac{3}{26}$ (d) $\frac{3}{52}$

6. If p(E) denotes the probability of the event E then

(a) $-1 \leq P(E) \leq 1$ (b) $P(E) \leq 0$ (c) $0 \leq P(E) \leq 1$ (d) $P(E) > 1$

7. A single letter is selected from the word 'PROBABILITY'. The probability that it is a vowel is

(a) $\frac{4}{11}$ (b) $\frac{3}{13}$ (c) $\frac{2}{11}$ (d) $\frac{5}{11}$

8. The probability of getting a bad egg in a lot of 400 is 0.035. Then the number of good eggs in the lot is

(a) 335 (b) 386 (c)285 (d) 14

9. A girl calculates that the probability of her winning the first prize in a lottery is 0.08. if 6000 are sold, how many tickets has she bought?

(a) 40 (b)240 (c)840 (d) 480

10. Someone is asked to take a number from 1 to 100. The probability that it is not a prime number is

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{5}$ (d) $\frac{13}{50}$

B. Very Short Type Questions

(i) What is the probability of a sure event?

(ii) What is the probability of not getting any head in a throw of two coins simultaneously once?

(iii) What is the probability that there are 53 Wednesdays in a leap year?

(iv) A coin is tossed two times. Find the probability of getting at the most one head.

(v) Two dice are thrown at the same time. Find the probability of getting same number on both the dice.

C. Matching Type Questions

1. Match the following columns:

Column I	Column II
(i) A die is thrown once. The probability of getting a number less than 2 is	(a) $\frac{1}{26}$.
(ii) The probability of falling a ceiling fan on the ground is	(b) 1.
(iii) One card is drawn at random from a well-shuffled deck of 52 cards. The probability of getting an ace of spade is	(c) $\frac{1}{52}$.
(iv) If E_1 and E_2 are complementary events then $P(E_1) + P(E_2)$ is	(d) Undefined.

(v) A card is drawn from a well-shuffled deck of 52 cards then the probability of getting a black King is (e) $\frac{1}{6}$.

D. Short Answer Questions with Reasoning

(i) When two coins are tossed simultaneously, there are four possible outcomes. Therefore, probability of a head or a tail is $\frac{1}{4}$. Justify your answer.

(ii) When four coins are tossed together,the possible outcomes are no head, 1 head, 2 heads, 3 heads and 4 heads. So, I say that probability of no head is $\frac{1}{16}$. What is wrong with this statement?

(iii)The probability of falling a ceiling fan on the ground is zero. What is wrong with this statement?

(iv)In a family having three children, there may be no girl, one girl, two girls, three girls. So, the probability of each is $\frac{1}{4}$. Is this correct? Justify your answer.

(v) A girl says that if you throw a die, it will show 1 or not 1. Therefore, the probability of 'getting 1' or 'not 1' each is equal to $\frac{1}{2}$. Is this correct?

E. Long Answers Type Questions

(i) A bag contains 10 red, 5 blue and 7 green balls. A ball is drawn at random. Find the probability that the ball being a

(a) red ball. (b) green ball. (c) not a blue ball.

(ii) All face cards are removed from a deck of 52 cards. The remaining cards are well shuffled and then one card is drawn at random. If an ace is given value 1,and the value of other cards remains the same, find the probability that the card has the value

(a) 7. (b) Greater than 7. (c) less than 7.

(iii)A committee of 25 members is formed from 15 women and 10men. To elect a chairperson, names of all persons are written on the cards and put in abox. Onecard is then drawn from the box. What is the probability the name written on the card is that of

(a) a woman. (b) a man.

(iv) A bag contains 15 red, 20 orange, 35 green and 30 white balls. If a ball is drawn at random, what is the probability that it will be

(a) red or green. (b) green or orange. (c) not white.

(v) Two dice are thrown simultaneously. Find the probability that the sum of two numbers appearing on the top is less than or equal to 10.

F. HOTs

(i) A bag contains 5 red and some blue balls. If the probability of drawing the blue ball is double that of a red ball, then find the number of blue balls in the bag.

(ii) There are 2 boys more that the number of girls in a chess club. A member of the club is to be chosen at random to participate in a tournament. Each member is equally likely to be chosen. If the probability that a girl is chosen is $\frac{3}{7}$, then determine the number of boys and girls in the chess club.

(iii) Fig 24.3, shows a square target of side 10 cm. A dart is thrown and lands on the target. What is the probability that the target will land on the shaded region?

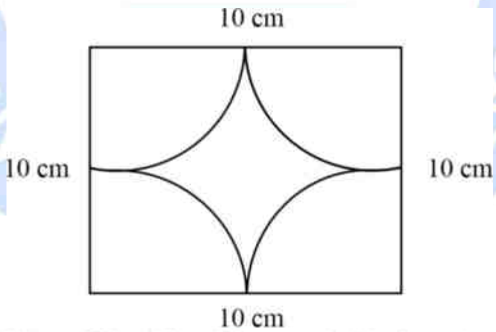


Fig. 24.5

(iv) Sum of two-digit number is 10.

(a) Write the sample space of such numbers.

(b) Find the probability that such a number is divisible by 3.

(c) Find the probability that such a number is a composite number.

(v) What is the probability that a leap year has 53 Sundays and 53 Mondays?

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 24.1

3. {HH, HT, TH, TT}

4. 7

5. { (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

6. 36

7. (i) 1 (ii) 6 (iii) 12 (iv) 13 (v) 13 (vi) 13

8. d 9. a 10. b

11. (i) It is very well known that the answer will be either right or wrong. Thus, the outcome right or wrong are equally likely to occur.

(ii) The outcomes are equally likely.

(iii) Even the expert players cannot hit a six most of the time. Thus, the outcomes are not equally likely.

25 PERCENTAGE AND ITS APPLICATION

INTRODUCTION

Mathematics teacher generally asked the students in a new class about the marks obtained by theminMathematics.Heconvertedthesemarksinpercentage:

1. Manik $\frac{45}{90} = 50\%$
2. Sulekha $\frac{60}{80} = 75\%$
3. Raj Kumar $\frac{42}{70} = 60\%$
4. Nitin $\frac{23}{50} = 46\%$
5. Rukmani $\frac{32}{40} = 80\%$

Fromthemarkssecuredbythestudentsoneisnotabletocomparetheirknowledge properlybut frompercentagewecaneasilydrawtheconclusionofcomparison.

Theshopkeeperannounces20%discountonparticularitems.Interestratearegiveninpercentage –

Period	**Rateofinterest**
Fixeddepositfor15days	4%
Fixeddepositfor6months	4.2%
Fixeddepositfor1year	4.5%
Fixeddepositfor 2years	5%

Thus,thewordpercentplaysanimportantrole in daily life.Theword'percent'isderivedfrom Latinterm'*per centum*'.'*Per*' means '*out of*' and '*centum*' means '*one hundred*'. Therefore, percent means out of a hundred.

Inthislesson,weshallstudypercentasfractionoradecimal.Weshallstudyits applicationinsimpleandcompoundinterest,profitandlossetc.

25.1 LEARNING OBJECTIVES

After completing this lesson, learners will be able to:

- illustrate the concept of percentage;
- convert a fraction into percent and vice versa;
- convert a decimal into percent and vice versa;
- calculate the percent of a quantity or a number;
- solve problems based on profit and loss;
- calculate simple interest and amount for a given sum of money for a particular period and rate of interest;
- calculate amount and compound interest on a given sum with given rate of interest and time;
- problems based on both the interests i.e.,compound interest and simple interest.

25.2 PERCENT

RecallmarkssecuredbyfivestudentsinMathematicsas $\frac{45}{90}, \frac{60}{80}, \frac{42}{70}, \frac{23}{50}$ and $\frac{32}{40}$.

Theabovefractionsarewitho ut100asdenominator.Theabovemarkscanbewrittenas50%,75%,60 %,46%and80%.

The symbol % is used for the term percent. A ratio where second term is 100 is alsocalledapercent,so53:100isequalto53%.

Considerthemarkssecuredbyfivestudents andweconvertthemintosimpleform:

$$\frac{45}{90}=\frac{1}{2};\frac{60}{80}=\frac{3}{4};\frac{42}{70}=\frac{3}{5};\frac{23}{50}=\frac{23}{50};\frac{32}{40}=\frac{4}{5}$$

Convertabovesimplifiedfractionwithcommondenominatoras100:

$$\frac{1}{2}=\frac{50}{100}=50\%;\frac{3}{4}=\frac{75}{100}=75\%;\frac{3}{5}=\frac{60}{100}=60\%;\frac{23}{50}=\frac{46}{100}=46\%;\frac{4}{5}=\frac{80}{100}=80\%$$

Now we can compare them easily and find out the best scorer or less scorer. Conversion of a fraction into percent and vice versa.

For converting fraction into percent, we change the fraction into simplest form. Further the simplified fraction is changed into equivalent fraction with denominator as 100 and then attach the symbol % with the changed numerator. For example:

$$\frac{28}{35}=\frac{4}{5}=\frac{4\times 20}{5\times 20}=\frac{80}{100}=80\% \text{ and}$$

$$\frac{42}{56}=\frac{3}{4}=\frac{3\times 25}{4\times 25}=\frac{75}{100}=75\%$$

To change a fraction into percent, we may multiply the fraction by 100, simplify it and assign the % symbol.

Example 25.1: $\frac{7}{20}=\frac{7}{20}\times 100=35\%$

Conversely,

To write a percent as a fraction, we multiply the number by $\frac{1}{100}$ (or divide the number by 100) and simplify it. For example:

(i) $48\% = 48\times\frac{1}{100}=\frac{48}{100}=\frac{12}{25}$

(ii) $33\% = 33\times\frac{1}{100}=\frac{33}{100}$

(iii) $7\% = 7\times\frac{1}{100}=\frac{7}{100}$

(iv) $320\% = 320\times\frac{1}{100}=\frac{320}{100}=\frac{32}{10}=\frac{16}{5}$

(v) $y\% = y\times\frac{1}{100}=\frac{y}{100}$

25.3 CONVERSION OF A DECIMAL INTO PERCENT AND VICE VERSA

We first convert decimal into fraction with denominator as 10, 100, 1000 etc. depending an integer after decimal. Multiply the fraction with 100 and simplify by inserting decimal if necessary and get the result as percentage.

Example 25.2:

$$3.7=\frac{37}{10}=\frac{37}{10}\times 100=370\%$$

$$0.83=\frac{83}{100}=\frac{83}{100}\times 100=83\%$$

$$0.269=\frac{269}{1000}=\frac{269}{1000}\times 100=\frac{269}{10}=26.9\%$$

$$0.0037 = \frac{37}{10000} = \frac{37}{10000} \times 100 = \frac{37}{100} = 0.37\%$$

Conversely,

To change the percent into decimal, we drop the % sign and put the decimal point two places to the left. For example:

$7\% = 0.07$	$18\% = 0.18$	$67\% = 0.67$	$118\% = 1.18$
$257\% = 2.57$	$4325\% = 43.25$	$0.8\% = 0.008\%$	$0.85\% = 0.0085$
$3.7\% = 0.037$			

Let us take some practical questions.

Example 25.3: Hemlata scored 35 runs out of 50 balls thrown to her. What was her percentage of score?

Total balls thrown = 50

Runs scored = 35

Score fraction = $\frac{35}{50}$

Percentage of score = $\frac{35}{50} \times \frac{2}{2} = \frac{70}{100} = 70\%$

or $\frac{35}{50} \times 100 = 70\%$

Example 25.4: 550 items out of 2000 items in a store were sold on discount. What percentage of items were not sold on discount?

Total items = 2000

Items sold on discount = 550

Fraction of items sold on discount = $\frac{550}{2000}$

Percentage of items sold on discount = $\frac{550}{2000} \times 100$

$$= \frac{55}{2} = 27.5\%$$

Percentage of items not sold on discount = (100 – (27.5) = 72.5%

Example 25.5: In an election 1200 persons out of 2500 voted. What percentage of persons did not vote?

Total voters = 2500

Numbers of person who voted = 1200

Number of people who did not vote = 2500 – 1200 = 1300

Fraction of people who did not vote = $\frac{1300}{2500}$

Percentage of people who did not vote = $\frac{1300}{2500} \times 100 = 52\%$

Example 25.6: In a class of 45 students, 15 are boys and rest are girls. Find the percentage of girls in the class.

Total students in the class = 45

Number of boys = 15

Number of girls = 30

Fraction of girls = $\frac{30}{45} = \frac{2}{3}$

Percentage of girls = $\frac{2}{3} \times 100 = \frac{200}{3} = 66\frac{2}{3}\%$

CHECK YOUR PROGRESS 25.1

1. Write the following percentage as fraction:

(i) 42% (ii) $16\frac{2}{3}\%$ (iii) 8.625% (iv) 3.75%

(v) 8% (vi) $17\frac{1}{7}\%$ (vii) 0.25% (viii) 53.45%

(ix) 0.0055% (x) 80% (k) 325%

2. Convertthefollowingfractionintoapercent:

(i) $\frac{1008}{750}$ (ii) $\frac{27}{20}$ (iii) $\frac{147}{180}$ (iv) $\frac{98}{700}$

(v) $\frac{33}{1100}$ (vi) $\frac{39}{2600}$ (vii) $\frac{110}{125}$ (viii) $\frac{115}{35}$

3. Write the following percent as decimal:

(i) 13% (ii) 145% (iii) 6% (iv) 38% (v) 95%

(vi) 320% (vii) 105.2% (viii) 0.065% (ix) 11.35% (ix) 0.0035%

4. Write each of the following decimal as a percent:

(i) 0.63 (ii) 0.0750 (iii) 5.3875 (iv) 0.125 (v) 0.3

(vi) 5.0025 (vii) 3.09 (viii) 0.08 (ix) 3.75 (ix) 385.231

5. In a class of 45 students, 18 students are going out on a picnic. What percentage of students did not opt for picnic?

6. Rs. 1800 is spent on conveyance out of a total income of Rs. 14400. What percent of income is spent on conveyance?

7. Namchin earns Rs. 54000 per month. He spends Rs. 27000 for house hold expenses, Rs. 10800 on the education of children, Rs. 5400 for his personal expenses and rest he saves. What percentage of income does he save?

8. In an election a candidate get 24000 votes out of total votes of 60,000. How much percentage of votes he got? How much percentage of votes did his opponent got if these are only two candidates in the election.

9. A group pf 75 person, 35 persons are illiterate. Find the percentage of literate persons.

10. In a class of 52 students 20 students like to see cartoon film, 19 students like to see adventure film and rest like to play a game on mobile. What percentage of students use mobile phone for a game?

25.4 FINDING THE WHOLE NUMBER USING A GIVEN PERCENTAGE

To calculate the wholenumber, we first change percentage to a fraction or a decimal than we multiply it with the given quantity or number. For example:

Example 25.7: A class ate 35% of 140 banana. How many bananas did they eat?

Solution: Total banana = 140

Eaten by a class = 35%

No. of Bananas eaten = $\frac{35}{100} \times 140 = 49$

Example 25.8: Calculate: (i) 40% of 130 (ii) 70% of 250 (iii)160% of 20 kg

Solution: (i) $\frac{40}{100} \times 130 = 52$ (ii) $\frac{70}{100} \times 250 = 175$ (iii) $\frac{160}{100} \times 20 = 32$ kg

Example 25.9: During pandemic $16\frac{2}{3}\%$ students out of 900 students left the school.

Find the numbers of students who left school.

Solution: Total students = 900

Students who left the school = $16\frac{2}{3}\%$

No. of students who left the school = $900 \times \frac{50}{3} \times \frac{1}{100} = 150$

Example 25.10: A distance of 90 km was to be covered. A person covered 60% of distance by scooter and 30% by bicycle and rest on foot. How much distance was covered on foot?

Solution: Total distance = 90 km

Distance covered by scooter = – 60% Distance covered by bicycle = 30%

Distance covered by foot = 100 –(60 + 30) = 10%

Distance covered on foot = $90 \times \frac{10}{1000} = 9$ km

Example 25.11: 15% rice was taken out of a sack of 80 kg of rice. Find the quantity of rice left in the sack.

Solution: Total rice = 80 kg

Rice taken out = 15%

Quantity of rice taken out = $80 \times \frac{15}{100} = 12$ kg

Rice left in the sack = 80 – 12 = 68 kg

Example 25.12: A family spends 50% of its income more on food than on education. Education expenditure is 40% more than his saving. If the total income of the family is 90,000 p.m. Find its monthly saving.

Solution: Let saving = Rs. x

Education expenditure = $x + \frac{40x}{100} = x + \frac{2x}{5} = \frac{7x}{5}$

Food expenditure = $\frac{7x}{5} \times \frac{150}{40} = \frac{21x}{10}$

Total income = $\frac{21x}{10} + \frac{7x}{5} + \frac{x}{1} = \frac{21x + 14x + 10x}{10} = \frac{45x}{10}$

$\frac{45x}{10} = 90,000$

$x = \frac{90,000 \times 10}{45} =$ Rs. 20,000

Monthly saving = Rs. 20,000

Example 25.13: A house rent is increased by 15% after 3 years. It is Rs. 34500 at percent. Find the rent 3 years ago.

Solution: Let the rent 3 yrs ago = Rs. x

% Rent increase in 3 yrs = $15\% = \frac{15}{100}$

Rent increased = $x \times \frac{15}{100} = \frac{15x}{100}$

Present rent = $x+\frac{15x}{100}=\frac{115x}{100}$

$\frac{115x}{100}=34500$

$x = 34500 \times \frac{100}{115} = 30,000$

Rent 3 years ago = Rs. 30,000

Example 25.14: What percent of 480 is 216?

Solution: Let 216 be x% of 480.

$\frac{480\times x}{100}=216$

$x=\frac{216\times 100}{480}=45\%$

or $\frac{216}{480}\times 100=45\%$

Example 25.15: The strength of students of class X is increased from 125 to 150. Find the increase percent.

Solution: Old strength = 125

New strength = 150

Increase = 25

Increase in fraction = $\frac{25}{125}$

Increase in percent = $\frac{25}{125}\times 100 = 20\%$

Example 25.16: The population of a city increased by 7%. If the population at present is 25680. Find the population last year.

Solution: Let last year population = x

Increase = $x\times\frac{7}{100}=\frac{7x}{100}$

Present population = $x + \frac{7x}{100} = \frac{107x}{100}$

$\frac{107x}{100} = 25680$

$x = \frac{25680 \times 100}{107} = 24000$

Last year population = 24000

Example 25.17: A reduction of 10% in the price of oil enables a dealer to purchase 2 kg more oil for Rs. 1800. Find the reduced price of oil per kg. Also find the original price.

Solution: 10% of 1800 = $\frac{10}{100} \times 1800$ = Rs. 180

Reduced price of 2 kg of sugar = Rs. 180

Reduced price per kg = $\frac{180}{2}$ = Rs. 90

Original price = $90 \times \frac{100}{90}$ = Rs. 100

Example 25.18: In an election there are two contestants. One candidate got 40% of the total votes and lost by 2000 votes. Find the total number of votes cast.

Solution: Let total votes = x

One candidate got = $x \times \frac{40}{100} = \frac{4x}{10}$

Second candidates got = $x \times \frac{60}{100} = \frac{6x}{10}$

Difference = $\frac{6x}{10} - \frac{4x}{10} = \frac{2x}{10}$

Difference in votes = 2000

$\therefore \frac{2x}{10} = 2000 \Rightarrow x = 2000 \times \frac{10}{2} = 10,000$

Total votes cost = 10,000

CHECK YOUR PROGRESS 25.2

1. Find: (i) 18% of 750 (ii) 47% of 2000

2. 80 is reduced to 60. Find reduction in percentage.

3. In a group of 800 students 40% read Math for 2 hours at, 32% read Mathematics for $1\frac{1}{2}$ has and rest read Mathematics for 1 hour. How many students read Mathematics for 1 hour.

4. A scooter was bought at Rs. 54,000. Its value depreciated at the rate of 8% per annum. Find its value after one year.

5. A man got 10% increase in his salary. if his new salary is Rs. 1,54,000. Find his original salary.

6. A shopkeeper gives a discount of 20% on a pair of shoes. If he pays Rs. 1600, find the original price.

7. Joseph is earning 10% more than his brother George. Find what percent George income is less than Joseph income.

8. A fruit seller purchased some fruits. He found that 15% of the fruits were rotten. He sold 60% the remaining fruits. He is now left with 17 fruits. Find the total fruits purchased.

9. Out of the total 20,000 people, it was found that 30% like to listen to Hindi film music, 20% like to listen to English music, 15% of classical music and the rest like to listen to other kind of music. Find the exact number of people who like each type of music.

10. Pass marks in Mathematics in an examination are 40%. A student got 68 marks and failed by 12 marks. Find the total marks of Mathematics in the examination.

11. A student gets 35% marks in 1st terminal and 65% in the 2nd terminal. How much percent should he get in third terminal so as to get 60% as overall score. Total marks of each terminal are equal.

12. Price of Petrol was increased by 15% and then decreased by 10%. What are net increases or decrease?

13. 80 is reduced 65 what is decrease percent?

14. Yashi earns 10% more than Tannia and Tannia earn 20% more than Salma. If Salma earns Rs. 3200 less than Yashi. Find the earning of each.

15. 40,000 students appeared at Mathematics Olympiad, out of this 45% are girls and rest boys, 10% of the boys and 15% of the girls scored merit certificate. Find the percentage of students who scored merit certificate.

25.5 APPLICATION OF PERCENTAGE

In our day-to-day life we come across a number of situations where we use the concept of percent. At times we have to borrow money for our needs. We pay some extra money after a certain period. The payment of extra amount is called interest. The interest is of two types i.e., Simple and compound interest. In the calculation of interest, we use percentage. First of all, we take up applications on simple interest and after that on compound interest and profit and loss.

Simple Interest: Rajeev borrowed Rs. 5,000 from his friend for medical treatment of her mother for two years. After two years he paid Rs. 5600. The money borrowed is Rs. 5,000 and extra payment is Rs. 600.

The money borrowed is called Principal usually denotes by P and the extra money paid is called interest usually denoted by S.I. The total money paid back in called amount denoted by A.

Thus $A = P + S.I.$

The interest is calculated at a certain rate which is expressed in percent per annum. Thus, interest depends on P (amount borrowed), T (Time) and R [(rate percent) = $r\% = \frac{r}{100}$]

Thus interest = $\frac{P \times R \times T}{100}$

The interest thus calculated is called simple interest. Let us solve some questions to understand the concept of simple interest.

Example 25.19: A man borrowed Rs. 12,000 at 5% per annum for 2 yrs. Find the simple interest and amount.

Solution: Here P = Rs. 12,000, T = 2 years, r = 5% per annum

Simple interest (S.I.) = $\frac{P \times R \times T}{100}$

$$= \frac{12000 \times 5 \times 2}{100} = \text{Rs. } 1200$$

Amount (A) = 12000 + 1200 = Rs. 13200

Example 25.20: Meena borrowed Rs. 20,000 from a bank at the rate of 6.9% per annum for $2\frac{1}{2}$ years at simple interest. Calculate simple interest and amount on the sum 2 borrowed.

Solution 25.21: Sum borrowed (P) = Rs. 20,000

Rate = 6.9%

$$\text{Time} = 2\frac{1}{2}\text{ yrs.} = \frac{5}{2}\text{ yrs.}$$

$$\text{S.I.} = \frac{P \times R \times T}{100} = 20{,}000 \times \frac{6.9}{100} \times \frac{5}{2} = 20{,}000 \times \frac{69}{1000} \times \frac{5}{2}$$

= Rs. 3450

Amount (A) = Rs. 20,000 + Rs. 3450 = Rs. 23,450

Calculation of Principle/Rate/time if S.I. is given

Example 25.22: Find at what rate of simple interest per annum will Rs. 4000 amount to Rs. 4840 in 3 years.

Solution: Here A = Rs. 4840, P = Rs. 4000, T = 3 years

S.I. (A – P) = (Rs. 4840 – Rs. 4000) = Rs. 840

$$\text{S.I.} = \frac{P \times R \times T}{100} = \text{S.I.} \times 100 = P \times R \times T$$

$$\therefore R = \frac{\text{S.I.} \times 100}{P \times T} = \frac{840 \times 100}{4000 \times 3} = 7\%$$

$\therefore$ r = 7% per annum

Example 25.23: In how many years will a sum of Rs. 6000 annual to Rs. 6975 at 6.5% per annum at simple interest.

Solution: Amount = Rs. 6975

Principal = Rs. 6000

S. Interest = Rs. (6975 – 6000) = Rs. 975

Time =?, Rate = 6.5% = $\frac{13}{2}\%$ per annum

$$\text{S.I.} = \frac{P \times R \times T}{100}$$

S.I. × 100 = P × R × T

$$T = \frac{\text{S.I.} \times 100}{P \times R}$$

$$\therefore \frac{975 \times 100}{6000 \times \frac{13}{2}} = \frac{975 \times 100 \times 2}{6000 \times 13} = \frac{5}{2} \text{ years}$$

$$\therefore \text{ Time} = \frac{5}{2} = 2\frac{1}{2} \text{ years}$$

Example 25.24: A sum borrowed amount to Rs. 8625 in 2 years at 7.50% per annum. Find the sum.

Solution: Amount = Rs. 8625, R = 7.5% = $\frac{15}{2}\%$ per annum

Time = 2 years

Let Set Principal = Rs. 100

S.I. = S. Interest = $\frac{15}{200} \times \frac{2}{1}$ = Rs. 15

Amount = Rs. 100 + Rs. 15 = Rs. 115

If amount is Rs. 115, Principal = Rs. 100

If amount is Rs. 1 Principal = $\frac{100}{115}$

If amount is Rs. 8625 Principal = $\frac{100}{115} \times 8625$ = Rs. 7500

or 115:100:: 8625: P

$$P = \frac{8625 \times 100}{115} = \text{Rs. } 7500$$

Example 25.25: At what rate of simple interest will amount be double the principal in 10 years.

Solution: Let P = Rs. 100, Amount = Rs. 200, Time = 10 years

S.I. = Amount – Principal = Rs. 200 – Rs. 100 = Rs. 100

We know that,

$$R = \frac{P \times 100}{100 \times T}$$

$$R = \frac{100 \times 100}{100 \times 10} = 10\%$$

Example 25.26: Out of Rs. 70,000 to interest for one year, a man invests Rs. 30,000 at 4% and Rs. 20,000 at 3% per annum simple interest. At what rate percent should he lend the remaining money, so that he gets 5% interest on total amount?

Solution: Interest on total amount at 5% for one year

$$= \text{Rs. } 70{,}000 \times \frac{5}{100} \times 1 = \text{Rs. } 3500$$

Interest on Rs. 30,000 at 4% for 1 year

$$= \text{Rs. } 30{,}000 \times \frac{4}{100} \times 1 = \text{Rs. } 1200$$

Interest on Rs. 20,000 at 3% for 1 year

$$= 20{,}000 \times \frac{3}{100} \times 1 = \text{Rs. } 600$$

Interest on remaining Rs. 20,000 for 1 year

= [3500 – 1200 – 600]

= Rs. 1700

$$1700 = 20{,}000 \times \frac{R}{100} \times 1$$

or $$R = \frac{1700 \times 100}{20{,}000 \times 1} = 8.5\%$$

The remaining amount should be invested at 8.5% per annum.

Example 25.27: A certain sum of money at simple interest amount to Rs. 3720 in 4 years and Rs. 4080 in 6 years. Find the sum and rate percent.

Solution: A = P + Interest for 6 years

Rs. 4080 = P + Int for 6 years

Rs. 3720 = P + Int for 4 years

Subtracting Rs. 360 = Int for 2 years

Rs. 180 = Int for 1 years

Interest for 4 years = 180 × 4 = Rs. 720

(A)Rs. 3720 = P + Rs. 720 (Int for 4 years)

Rs. (3720 – 720) = P

P = Rs. 3000

$$R = \frac{Int + 100}{P \times T} = \frac{720 \times 100}{3000 \times 4} = 6\%$$

Example 25.28: In which case, is the simple interest earned more:

(a) Rs. 5000 at 6% for 4 years (b) Rs. 5000 at 4% for 6 years

Solution: In case of (a):

P = Rs. 5000, Time = 4 years, R = 6%

$$\text{Interest} = \frac{5000 \times 4 \times 6}{100} = \text{Rs. } 1200$$

In case (b):

P = Rs. 5000, Time = 6 years, Rate = 4%

$$\text{Int} = \frac{5000 \times 6 \times 4}{100} = \text{Rs. } 1200$$

Interest in both cases is same.

Example 25.29: A certain sum of money double itself in 8 yrs. In how many time will it become 4 times of itself half rate of interest?

Solution: Let P = Rs. x

A = 2 (x) = Rs. 2x

Interest = A – P = Rs. $(2x - x)$ = Rs. x

Time = 8 years $\therefore$ Rate = $\frac{\text{Int}+100}{\text{P}\times\text{T}} = \frac{x\times 100}{x\times 8} = 12\frac{1}{2}\%$

Again A = $4x$

Interest = 4x – x = 3x (Int. = A–P)

Rate = $\frac{1}{2}\left(\frac{25}{2}\%\right) = \frac{25}{4}\%$

Time = $\frac{\text{Int}+100}{\text{P}\times\text{R}} = \frac{3x\times 100}{x\times\frac{25}{4}} = \frac{3x\times 100\times 4}{x\times 25}$ = 48 years

CHECK YOUR PROGRESS 25.3

1. In what time will simple interest be half the principal at 5% per annum.
2. Deepa borrowed Rs. 2500 from his friend at 7.5% per annum at simple in text. He returned the money after 2 yrs. How much money did he pay?
3. Yakub Rs. 30,000 to his two friends. He gave Rs. 1800 at 8% per annum to one of his friends and the remaining amount to other friend at 11% per annum. How much interest did he earn after 2 years and 6 months.
4. A certain amount of money became $1\frac{1}{2}$ time in 5 years. In how much time will it become 4 times of itself at the same rate of interest.
5. Mrs. Prema deposited Rs. 8,000 in a financial company at $7\frac{1}{2}\%$ 1% and received back 2 Rs. 9800 after a certain time. Find the time for which money was at financial company.
6. At what rate of interest will simple interest be $\frac{2}{5}$ the principal in 8 years?
7. A sum of money amounts to Rs. 7260 in 3 years and Rs. 8100 in 5 years at simple interest. Find the sum and rate percent.
8. A man has Rs. 60,000 with him. He invests Rs. 25,000 at 4% per annum and Rs. 15,000 at 6% per annum. At what percent should he lend the remaining money so as to get 5% interest on the total after one year.

9. Mrs. Kanath invested by 40,000 in a company for 2 yrs. on S.I. and received Rs. 45360 in all. Find the rate of simple interest.

10. Kulwant Kaur lent Rs. 20,000 to his two friends. She gave 12,000 at 8.5% per annum to of his one friend and the remaining to other at 9.5% per annum. How much interest did she earn after 2 years?

25.5.1 Compound Interest

In the previous section we read about simple interest. Simple interest is paid after a year. It is calculated as S.Interest = $\frac{P \times R \times T}{100}$ where rate is R% per annum.

If interest is not paid after a year, it is added to principal. The next year interest is calculated on the amount (A) = Principal + Interest. This amount for one year is called Principal for next year. Similarly Amount for 2 years becomes Principal for third year and so on. The process of calculating interest (Amount – Principal) is called compound interest.

The period for calculations can be half yearly, quarterly etc. Let us clear it by taking some examples:

Example 25.30: Find the compound interest on Rs. 5,000 for two years at 10% per annum.

Solution: Here P = Rs. 5,000, Time = 2 years, Rate = 10% yearly

Interest on Rs. 5000 for 1 years = $\frac{5,000 \times 1 \times 10}{100}$ = Rs. 500

Amount (after 1 year) = P + R = Rs. 5000 + Rs. 500 = Rs. 5500 Principal for 2nd year = Rs. 5500

Interest for 1 year (2nd year) = $5500 \times \frac{10}{100} \times 1$

= Rs. 550

Amount = Rs. (5500 + 550) = Rs. 6050

Compound interest = Rs. 6050 – Rs. 5000 = Rs. 1050 or (Rs. 500 + Rs. 550) = Rs. 1050

Formula for compound interest Let sum borrowed = Rs. P

Rate = r% per annum

Time = n years

Interest for one year = $\mathrm{P}\times\frac{\mathrm{r}}{100}\times 1=\frac{\mathrm{Pr}}{100}$

Amount = $\mathrm{P}+\frac{\mathrm{Pr}}{100}=\mathrm{P}\left(1+\frac{\mathrm{r}}{100}\right)$

Principal for second year = Amount of 1st year = $P\left(1+\frac{r}{100}\right)$

Interest for second year = $P\left(1+\frac{r}{100}\right)\times\frac{r}{100}\times 1$

$=\frac{\mathrm{Pr}}{100}\left(1+\frac{r}{100}\right)$

Amount after 2nd year = $P\left(1+\frac{r}{100}\right)+\frac{\mathrm{Pr}}{100}\left(1+\frac{r}{100}\right)$

$=P\left(1+\frac{r}{100}\right)\left[1+\frac{r}{100}\right]$

$=P\left(1+\frac{r}{100}\right)^2$

Similarly amount after 3 years = $P\left(1+\frac{r}{100}\right)^3$

Amount after n years = $P\left(1+\frac{r}{100}\right)^n$

Compound Interest = A – P = $P\left(1+\frac{r}{100}\right)^n-P$

Example 25.31: Calculate compound interest on Rs. 20,000 for 3 years at the rate of 7.5% per annum.

Solution: Here P = Rs. 20,000, R = 7.5% and T = 3 year

Amount = $P\left(1+\frac{r}{100}\right)^n$

$$\text{Amount} = 20,000\left(1+\frac{7.5}{100}\right)^3$$

$$= 20,000\times\left(1+\frac{3}{40}\right)^3$$

$$= 20,000\times\frac{43}{40}\times\frac{43}{40}\times\frac{43}{40}$$

= Rs. $\frac{397535}{16}$ approx. = Rs. 24846

Compound interest = Rs. 24846 – 20,000 = Rs. 4846

Calculation of compound interest when compound interest is calculated half yearly (similarly) or quarterlyrate = 12% annually, T = 2 years

If Interest is compounded half yearly then applicable Rate = $\frac{12}{2}$ = 6%,

Time = 2 years = 4 half years

If interest is compounded quarterly then applicable Rate = $\frac{12}{4}$ = 3%,

Time = 2 years, = 8 quarters

Example 25.32: Calculate the amount and compound interest on Rs. 25,000 for $1\frac{1}{2}$ yrs. at the rate of 8% per annum compounded annually.

Solution: Here P = Rs. 25,000, R = 8% per annum and $n = 1\frac{1}{2}$ years

Rate of interest for full year (1 year) = 8%

Rate of interest for half year = $\frac{8}{2}$ = 4%

$$\text{Amount} = P\left(1+\frac{r}{100}\right)$$

= 25000 $\left(1+\frac{8}{100}\right)$ for 1 year $\left(1+\frac{4}{100}\right)$ (for half year)

= Rs. 28080

Compound interest = Rs. (28080 – 25000)

= Rs. 3080

Example 25.33: Find the compound interest on Rs. 15000 at 12% per annum for 6 months where interest is compounded quarterly:

Solution: Here P = Rs. 15000, R = 12% annually, Time = 6 month

Interest is calculated quarterly

$\therefore$ Rate = $\frac{12}{4}$ = 3% per quarter

Time = 6 months = 2 quarters

$\therefore$ Amount = $P\left(1+\frac{r}{100}\right)^t$

$= 1500\left(1+\frac{3}{100}\right)^2$

$= 1500\times\frac{103}{100}\times\frac{103}{100} = \text{Rs.}\frac{31827}{2}$

= Rs. 15913.50

Compound interest = Rs. 15913.50 – 15000 = Rs. 913.50

Example 25.34: Find the difference between simple interest and compound interest on Rs. 12000 for $1\frac{1}{2}$ years at 10% per annum compounded half yearly.

Solution: Here P = Rs. 12000, Time = $1\frac{1}{2}$ year or 3 half years

R = 10% per annum or 5% half yearly

Simple interest = $\frac{12000\times10\times3}{100\times2}$ = Rs. 1800

Amount (for compound interest) = $P\left(1+\frac{r}{100}\right)^t$

$= 12000\left(1+\frac{5}{100}\right)^3$ = Rs. 13891.50

Compound interest = Rs. (13891.50 – 12000) = Rs. 1891.50 Difference between compound interest and simple interest

= Rs. (1891-50 – 1800) = 91.50

Example 25.35: At what rate percent will a sum of Rs. 16000 becomes Rs. 21296 in three years when interest is compounded annually?

Solution: Here A = Rs. 21296, Time = 3 years, rate =?, P = Rs. 16000

Let rate = r% per annum

$$\therefore \text{A} = P\left(1+\frac{r}{100}\right)^t$$

$$21290 = 16000\left(1+\frac{r}{100}\right)^3$$

$$\left(1+\frac{r}{100}\right)^3 = \frac{21296}{16000} = \frac{1331}{1000} = \left(\frac{11}{10}\right)^3 = \left(1+\frac{1}{10}\right)^3$$

$$\therefore \frac{r}{100} = \frac{1}{10}$$

$\therefore$ r = 10

$\therefore$ Rate = 10% per annum

Example 7 : At what rate percent per annum will a sum of Rs. 4000 yield compound interest of Rs. 665.60 in 2 years.

Solution : A = P + C. I. = Rs. (4000 + 665.60) = Rs. 4665.60

Time = 2 yrs and rate = r% p.a.

$$\text{A} = P\left(1+\frac{r}{100}\right)^n$$

$$\frac{4665.60}{100} = 4000\left(1+\frac{r}{100}\right)^2$$

$$\left(1+\frac{r}{100}\right)^2 = \frac{46656}{10} \times \frac{1}{4000} = \frac{11664}{10000}$$

$$\left(1+\frac{r}{100}\right)^2=\left(\frac{108}{100}\right)^2$$

$$\left(1+\frac{r}{100}\right)=\frac{108}{100}=1+\frac{8}{100}$$

$$\frac{r}{100}=\frac{8}{100}$$

$r = 8$

$\therefore$ Rate of interest = 8%

Example 25.35: In how much time will Rs. 6000 amount to Rs. 6945.75 at 5% per annum compound interest?

Solution: Here P = Rs. 6000, r = 5% p.a.

A = Rs. 6945.75

We know $A = P\left(1+\frac{r}{100}\right)^t$

$$6945.75 = 6000\left(1+\frac{5}{100}\right)^t$$

$$\left(1+\frac{1}{20}\right)^t=\frac{694575}{600000}=\frac{9261\times 75}{8000\times 75}=\frac{9261}{8000}$$

$$\left(\frac{21}{20}\right)^t=\left(\frac{21}{20}\right)^3 \text{ or } t=3$$

Time = 3 years

Example 25.36: Find the difference between compound interest and simple interest on a sum of Rs. 25000 for 3 years at 4% per annum.

Solution: Here P = Rs. 25000, r = 4%, time = 3 years

$$\text{S.I.} = \frac{P\times r\times T}{100}=\frac{25000\times 4\times 3}{100}=\text{Rs. } 3000$$

$$\text{Amount of C.I.} = 250000\left(1+\frac{4}{100}\right)^3=\frac{140608}{25}$$

Compound interest = $\frac{140608}{5} - 25000 = \frac{15608}{5}$ = Rs. 3121.60

Difference between C.I. and S.I. = Rs. 3121.60 – Rs. 3000 = Rs. 3121.60

Example 25.37: If the simple interest on a sum of money for 2 years at 5% per annum is Rs. 1200. What will be the compound interest on that sum at the same rate and for the same period?

Solution: S.I. = Rs. 1200, R = 5% p.a., time = 2 years

$$P = \frac{S.I. \times 100}{R \times T}$$

$$P = \frac{1200 \times 100}{5 \times 2} = \text{Rs. } 12000$$

How P = Rs. 120000, t = 2 years, r = 5% p.a.

$$A = P\left(1+\frac{r}{100}\right)^t = 12000\left(1+\frac{5}{100}\right)^2$$

$$= 12000 \times \frac{105}{100} \times \frac{105}{100}$$

Compound interest = Rs. (13230 – 12000) = Rs. 1230

CHECK YOUR PROGRESS 25.4

1. Find the amount at the end of 2 years on Rs. 2500 at 4% p.a. at compound interest.
2. Vimal gets a loan of Rs. 64000 from a company at 2.5 paisa per rupee per annum. calculate the compound interest payable after 3 years.
3. At what rate percent will a sum of Rs. 1000 amount to Rs. 1102.50 in 2 years at compound interest?
4. What sum of money amounts to Rs. 23152.50 at 10% per annum for $1\frac{1}{2}$ years, interest being compounded half yearly?
5. The compound interest is Rs. 1800 at 10% per annum for a certain period of time is Rs. 378. Find the time in years.

6. Find the difference between compound interest and simple interest on Rs. 1,20,000 for $1\frac{1}{2}$ years at the rate of 4% per annum. 2

7. Compound interest on a certain sum of money at the rate of 5% becomes Rs. 1537.50 in 2 years. Find the simple interest on the same sum for same time at same interest rate.

8. At what sum of money the difference between compound interest and simple interest at 5% per annum for 3 yrs is Rs. 45.75?

9. In how much time will Rs. 5,000 amount to Rs. 6655 at 10% per annum compound interest?

10. Aneesh took a loan of Rs. 25000 at 8% per annum. The interest is compounded half yearly. What amount will be pay after $1\frac{1}{2}$ years?

The application of percentage is in different fields. Two such fields are simple interest and compound interest. We studied simple interest and compound interest in last few pages. Now we shall study the use of percentage in Profit and Loss.

25.5.2 Profit & loss

A fruit seller Vendor purchases fruit and sells it. Let us suppose that the Vendor bought fruit for Rs. 2500 and sold the whole for Rs. 2950. Thus he earns Rs. 450.

The price at which the goods are purchased is called Cost Price (C.P.)

The price at which the goods are sold is called Sales Price (S.P).

Profit (Gain) when S.P. > C.P. then there is profit Profit = S.P. – C.P.

Loss: When S.P. < C.P. then there is loss

Loss = C.P. – S.P.

In the above example of vendor (fruit seller)

C.P. = Rs. 2500

S.P. = Rs. 2950

S.P. > C.P.

∴ Profit (gain) = 2950 – 2500 = Rs. 450

$$\text{Profit \%} = \left(\frac{\text{Profit}}{\text{C.P.}} \times 100\right)\%$$

$$\text{Loss \%} = \left(\frac{\text{Loss}}{\text{C.P.}} \times 100\right)\%$$

If gain percent is given, to find S.P. or C.P.

$$\text{S.P.} = \frac{\text{C.P.} \times (100 + \text{gain\%})}{100}$$

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{(100 + \text{gain})}$$

If loss percent is given, to find S.P. or C.P.

$$\text{S.P.} = \frac{\text{C.P.} \times (100 - \text{loss\%})}{100}$$

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{100 - \text{loss\%}}$$

Note: Gain % or loss % is always calculated on C.P.

Let us learn the application of these formulae in the following examples:

Example 25.38: A man buys a shirt for Rs. 320 and sells it for Rs. 280. Find his gain or loss percent.

Solution: Here C. P. = Rs. 320 and S. P. Rs. 280 We see that C.P. > S.P.

∴ there is a loss

Loss = C.P. – S.P. = Rs. (320–280) = Rs. 40

$$\text{Loss \%} = \frac{\text{loss} \times 100}{\text{C.P.}}$$

$$= \frac{40 \times 100}{320} = \frac{25}{2} = 12\frac{1}{2}\%$$

25.5.2.1 Over Head Expenses

We know that a retailor purchases goods from some manufacturer or wholesaler and stored various items in his shop. He sells these items to the consumers. For transporting the goods from manufacturer or wholesaler he spends some extra money. Sometimes a seller has to spend some money on repair etc. of the good to be sold (specially like use car or scooter etc.) These expenses incurred by the seller are called 'overhead' expenses.

For the purpose of computing profit or loss on various transactions overhead expenses are added to C.P. of the items. The cost thus obtained is called the Actual cost of the item. In such cases actual cost is treated as cost price of the item.

Example 25.39: Anuradha purchased an old scooter for Rs. 56,450 and spent Rs. 3,550 on its repair. He sold it for Rs. 66,000. Find his gain or loss on this transaction.

Solution: C. P. of scooter = Rs. 56,450

Repair cost (overhead expenses) = Rs. 3,550

Actual cost of scooter = Rs. (56,450 + 3,550) = Rs. 60,000

S.P. of scooter = Rs. 66,000

S.P. > C.P. $\therefore$ gain = S.P. – C.P.

= Rs. (66,000 – 60,000)

= Rs. 6,000

$$\text{gain } \% = \frac{\text{gain}}{\text{C.P.}} \times 100$$

$$= \frac{6000}{60,000} \times 100 = 10\%$$

$\therefore$ Profit/Gain% = 10%

Example 25.40: A shopkeeper buys oranges at 20 for Rs. 80 and sells them at 25 for Rs. 125. Find his gain or loss percent.

Solution: C.P. of 20 oranges = Rs. 80

$$\text{C.P. or 1 orange} = \text{Rs. } \frac{80}{20} = \text{Rs. } 4$$

S.P. of 25 oranges = Rs. 125

$$\text{S.P. of 1 orange} = \text{Rs. } \frac{125}{25} = \text{Rs. } 5$$

S.P. > C.P.

$\therefore$ gain = S.P. – C.P.

gain = Rs. (5 – 4) = Rs. 1

gain % $\dfrac{\text{gain}}{\text{C.P.}}\times 100 = \dfrac{1}{4}\times 100 = 25\%$

gain = 25%

Example 25.41: Raju sold two cycles for Rs. 3960 each. On selling one cycle he had a gain of 10% and on selling other cycle, he had a loss of 10%. Find his gain or loss in the whole transaction.

Solution: S.P. of one cycle = Rs. 3960

Gain = 10%

C.P. of cycle = $\dfrac{\text{S.P.} \times 100}{(100 + \text{gain})} = \dfrac{3960\times 100}{(100+10)}$

$= \dfrac{3960\times 100}{110}$ = Rs. 3600

S.P. of 2nd cycle = Rs. 3960

Loss = 10%

C.P. of 2nd cycle = $\dfrac{\text{S.P.} \times 100}{100 - \text{loss}} = \dfrac{3960\times 100}{100-10}$

$= \dfrac{3960\times 100}{90} = 4400$

Total C.P. of two cycles = 3600 + 4400 = Rs. 8000

Total S.P. of two cycles = Rs. (3960+3960) = Rs. 7920

C.P. > S.P.

$\therefore$ Loss = C.P. – S.P. = Rs. (8000 – 7920) = Rs. 80

Loss percent = $\dfrac{80}{8000}\times 100 = 1\%$

Example 25.42: A shopkeeper sold an item for Rs. 720 at a loss of 10%. at what price should be sell so as to gain 5%?

Solution: S.P. of item = Rs. 720

loss = 10%

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{100 - \text{loss}} = \frac{720 \times 100}{90} = \text{Rs. } 800$$

Now required gain = 5%

$$\text{S.P.} = \frac{\text{C.P.} \times (100 + \text{gain})}{100} = \frac{800 \times 105}{100} = \text{Rs. } 840$$

Example25.43: A shopkeeper bought 200 pens for Rs. 15 each. He found 20 pens not usable. He sold the remaining pens so as to gain 20% on whole transaction. at what price should he sell each pen?

Solution: Numbers of pens bought = 200

C.P. of 1 pen = Rs. 15

C.P. of 200 pens = 200 × 15 = Rs. 3000

Number of unusable pens = 20 Pens to be sold = 200 – 20 = 180 Required gain = 20%

$$\text{S.P. of 180 pens} = \frac{\text{C.P.} \times (100 + \text{gain }\%)}{100}$$

$$= \frac{3000 \times 120}{100} = \text{Rs. } 3600$$

$$\text{S.P. of 1 pen} = \frac{3600}{180} = \text{Rs. } 20$$

Example 25.44: The cost price of 10 articles is equal to selling price of 8 articles. Find the gain percent.

Solution: Let C.P. of 10 articles = Rs. 100

$$\text{C.P. of 1 article} = \frac{100}{10} = \text{Rs. } 10$$

$$\text{S.P. pf 8 articles} = \frac{100}{1}$$

$$\text{S.P. of 1 article} = \frac{100}{8} = \frac{25}{2}$$

S.P. > C.P.

$$\therefore \quad \text{gain} = \left(\frac{25}{2} - 10\right) = \text{Rs.}\frac{5}{2}$$

$$\text{gain\%} = \frac{\text{gain}}{\text{C.P.}} \times 100 = \frac{5}{2 \times 10} \frac{1}{} \times 100 = 25\%$$

Example 25.45: A watch was sold at a profit of 15% that it been sold for Rs. 50 more, the profit would have been 20%. Find the cost price of watch.

Solution: Let C.P. of watch = Rs. x

$$\text{S.P.} = \frac{\text{C.P.}(100 + \text{gain})}{100} = \frac{x(100+15)}{100} = \text{Rs.}\frac{115x}{100} = \text{Rs.}\frac{23x}{20}$$

New profit = 20%

$$\text{New S.P.} = \frac{x(100+20)}{100} = \text{Rs.}\frac{6x}{5}$$

$$\text{Difference in two S.P.} = \frac{6x}{5} - \frac{23x}{20} = \frac{120x - 115x}{100}$$

$$= \text{Rs.}\frac{5x}{100} = \text{Rs.}\frac{x}{20}$$

If difference is $\frac{x}{20}$, C.P. = Rs. x

If difference 1 is = Rs. $x \times \frac{20}{x}$

If difference 50 is = $\frac{x}{1} \times \frac{20}{x} \times 50$ = Rs. 1000

or $\frac{x}{20} : x :: 50 : \text{C.P.}$

$$\text{C.P.} \times \frac{x}{20} = x \times 50$$

$$\text{C.P.} = x \times 50 \times \frac{20}{x} = \text{Rs. } 1000$$

Example 25.46: By selling 45 oranges for Rs. 160, a woman loses 20%. How many oranges should she sell for Rs. 112 to gain 20% on the whole.

Solution: S.P. of 45 oranges = Rs. 160

Loss = 20%

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{(100 - \text{Loss})} = \frac{160 \times 100}{100 - 20} = \text{Rs. } 200$$

gain = 20%

$$\text{S.P.} = \frac{\text{C.P.} \times (100 + \text{gain})}{100}$$

$$= 200 \times \frac{120}{100} = \text{Rs.}240$$

S.P. of 45 oranges = Rs. 240

For Rs. 240, oranges = 45

$$\text{Rs. 1, oranges} = \frac{45}{240}$$

$$\text{For Rs. 112, oranges} = \frac{45}{240} \times \frac{112}{1} = 21$$

∴ For Rs. 112 oranges to be sold = 21

Example 25.47: Anuj bought a chair for Rs. 300 and sold it to Raman at a profit of 20%. Raman sold it to Surjeet at a loss of 10%. Find the money paid by Surjeet for the chair.

Solution: C.P. of Anuj for chair = Rs. 300

gain = 20%

$$\text{S.P.} = \frac{\text{C.P.} \times (100 + \text{gain})}{100}$$

$$= \frac{300(100 + 20)}{100} = \text{Rs. } 360$$

C.P. of Raman = Rs. 360

Loss = 10%

$$\text{S.P. of Raman} = \frac{360 \times 90}{100} = \text{Rs. } 324$$

C.P. of Surjeet = S.P. of Raman = Rs. 324

CHECK YOUR PROGRESS 25.5

1. Thalvi bought an almirah for Rs. 10500 and sold it for Rs. 12075. Find her gain or loss%.
2. Yusuf sold a shirt for Rs. 658 at a loss of 6%. Find cost price of shirt.
3. A wholesale dealer purchased goods worth Rs. 16000 from a manufacturer. He spent Rs. 4000 on transporting the goods to his store. He sold entire goods and got Rs. 20500. Find his gain or loss percent.
4. A man bought a refrigerator for Rs. 8800 and spent Rs. 400 on transportation. At what price should he sell so as to gain 12%
5. Jaya bought two sarees for Rs. 1200 each. She sold one at a gain of 5% and other at a loss of 6%. Find her gain or loss on the whole transaction.
6. Yukub sold a study table for Rs. 4462 at a loss of 4\3%. At what price should he sell so as to gain 5%.
7. A vendor buys lemon at the rate of 5 for Rs. 7 and sells them at Rs. 1.75 per lemon. Find his gain percent.
8. A man buys banana at 6 for Rs. 5 and an equal number at Rs. 15 per dozen. He mixes the two lots and sold them at Rs. 14 per dozen. Find his gain or loss in transaction in percentage.
9. Roy sold two tape recorders for Rs. 5940 each. One selling on tape recorder he had a gain of 10% and on selling other, he had a loss of 10%. Find his gain or loss percent in whole transaction.

RECAPITULATION POINTS

- Percent means ‘per hundred’.
- Decimal can be written as percent and vice versa.
- Fraction can be written as percent and vice versa.
- To write fraction as a percent, we multiply the fraction by 100, simplify it and put the sign % after it.
- Purchase price is called Cost price (C.P.) and Selling Price as S.P.

- Gain or profit = S.P. – C.P.; Loss = C.P. – S.P.
- Gain % $= \frac{\text{gain}}{\text{C.P.}} \times 100$; Loss % $= \frac{\text{Loss}}{\text{C.P.}} \times 100$
- S.P. = C.P. $\times \frac{(100+\text{gain\%})}{100}$; S.P. = C.P. $\times \frac{(100-\text{loss\%})}{100}$
- Sum borrowed is called principal written as P; Time as T years and Rate as R% per annum.
- S.I. $= \frac{P \times R \times T}{100}$
- In case of compound interest
- Amount (A) = $P\left(1+\frac{R}{100}\right)^t$ where P is principal, R = rate % and time = t
- Rate and time are adjusted as per conversion period (compound interest is charged annually, half yearly or quarterly.
- In case of half yearly/quarterly time is converted to half years or Quarters and Rate (if given per annum) is divided by 2 for half yearly and 4 for quarterly.
- The compound interest is more than simple interest except for 1st conversion period.

TERMINAL EXERCISE

Choose the correct option:

1. The population of a town decreased from 50000 to 45000. The percent decrease in population is:

   (a) 10% (b) 11.1% (c) 12.2% (d) 15%

2. An article was bought for Rs. 80 and sold for Rs. 95. The gain percent is:

   (a) 15% (b) $17\frac{2}{5}$% (c) $48\frac{1}{2}$% (d) $18\frac{3}{4}$%

3. A cooler was sold for Rs. 3168 at a loss of 12%. The cost price of the cooler is:

   (a) Rs. 3500 (b) Rs. 2800 (c) Rs. 3600 (d) Rs. 2788

4. Kajal bought a cycle for Rs. 2650 and spent Rs. 350 on its repair and sold it for Rs. 2550 the loss % is:

(a) 10% (b) 15% (c) 12% (d) 18%

5. The interest on Rs. 10,000 at the rate of 10% per annum for 1 year compounded half yearly is:

(a) Rs. 10,000 (b) Rs. 11025 (c) Rs. 1000 (d) Rs. 1025

6. Rajesh earns Rs. 80,000 per month and saves 25%. 75% of the remaining is spent on food and clothes. The amount spent on food and clothes is:

(a) Rs. 45,000 (b) Rs. 50,000 (c) Rs. 60,000 (d) Rs. 15,000

7. The simple interest (S.I.) and compound interest (C.I.) for 1st year on the same sum at the same rate per annum is:

(a) S.I. less the C.I. (b) C.I. is less than S.I.

(c) S.I. is equal to C.I. (d) No relation

Fill in the blanks:

8. In case of compound interest, amount of a particular year becomes _______ for the next year.

9. When compound interest is calculated half yearly the period of $2\frac{1}{2}$ year is ________ half year.

10. C.P. = S.P. + __________ (gain/loss)

11. ________ % of 50 is 15.

12. To find actual C.P. of an article, overhead expenses are __________ to the price at which article is purchased. (added/subtracted)

13. Find the number whose $8\frac{1}{3}\%$ is 4.

14. In an examination, 96% of the candidate passed and 80 failed. How many candidates appeared?

15. After having 30% increase in the salary, the present salary of the person is 5200. Find his original salary.

16. An article selling price is Rs. 1200. He sold the article at a loss of Rs. 300. Find loss percent.

17. Nitin is left with Rs. 1200 after spending 60% of his pocket money. How much pocket money Nitin had in the beginning?

18. x% of 120 is 80, find x.

19. Reema invested Rs. 64000 for 3 years at the rate of 10% per annum compounded annually. Sonali invested the same amount at the same rate and for the same period at simple interest. Who got more interest and by how much?

20. The population of a village is 64000. It is increasing at the rate of 5% annually. What will be the population after 2 years

21. Calculate the difference between compound interest and simple interest on Rs. 1200 for $1\frac{1}{2}$ year at 10% per annum when the interest is compounded half yearly.

22. Raj sold two scooters for Rs. 40,000 and Rs. 36000 respectively. He sells first one at a gain of 25% and second at a loss of 10%. Find his overall gain or loss percent in the whole transaction?

23. The difference between compound interest and simple interest on a certain sum of money at 15% per annum for 2 years is Rs. 45. Find the sum.

24. Find the amount Reena will get on Rs. 81920 if she kept it for 18 months at $12\frac{1}{2}$% per annum interest being compounded semi-annually.

25. The population of a town in the year 2002 was 22050. The population grows the rate of 5% p.a. What was population in the year

(a) 2000 (b) will be in 2004.

26. By selling 90 pens for Rs. 160, a person loses 20%. How much pens should be sold for Rs. 96 so as to gain 20% on it?

27. A man bought two consignments of eggs first at Rs. 18 per dozen and an equal number at Rs. 20 per dozen. He sold the mixed eggs at Rs. 23.75 per dozen. Find his gain percent.

28. The price of sugar rises by 25%. By how much percent should a household reduce the consumption of sugar so as not to increase his expenditure on sugar?

29. 56% of the students of a class are boys. If the number at boys more than girls is 6, find the total students in the class.

30. Find the sum of money which will amount to Rs. 13230 in six months at 20% per annum when interest is compounded quarterly.

31. In what time will Rs. 2700 yield the same interest at 4% per annum as at Rs. 2250 in the years at 3% per annum. (At S.I.)

32. Simple interest on a sum of money is $\frac{1}{3}^{rd}$ of the sum itself and the number of years is thrice the rate percent. Find the rate of interest?

33. A Vendor bought bananas at 12 for Rs. 60 and sold them at 3 for Rs. 18. Find his gain or loss percent.

34. At what rate percent per annum will a sum of Rs. 12000 amount to Rs. 15972 is three years at C.I., when interest is compounded annually?

35. A sum of money becomes the Rs. 26620 in 3 years and Rs. 29282 in 4 years at compound interest compounded annually. Find the sum and rate of interest.

ANSWERS TO 'CHECK YOUR PROGRESS'

CHECK YOUR PROGRESS 25.1

1. (i) $\frac{42}{100} = \frac{21}{50}$ (ii) $\frac{1}{6}$ (iii) $\frac{69}{800}$ (iv) $\frac{3}{80}$

   (v) $\frac{8}{100}$ (vi) $\frac{6}{35}$ (vii) $\frac{1}{400}$ (viii) $\frac{1069}{2000}$

   (ix) $\frac{11}{200000}$ (x) $\frac{4}{5}$ (xi) $\frac{13}{4}$

2. (i) $\frac{2016}{15}\%$ (ii) 135% (iii) $32\frac{2}{3}$ (iv) 14%

   (v) 3% (vi) $\frac{3}{2}\%$ (vii) 88% (viii) $\frac{2300}{7}$

3. (i) 0.13 (ii) 1.45 (iii) 0.06 (iv) 0.38

   (v) 0.95 (vi) 3.2 (vii) 1.052 (viii) 0.00065

(ix) 0.1135 (x) 0.000035

4. (i) 63% (ii) 7.5% (iii) 538.75% (iv) 12.5%
   (v) 30% (vi) 500.25% (vii) 309% (viii) 8%
   (ix) 375% (x) 38523.1%

5. 60%

6. $12\frac{1}{2}\%$

7. 20%

8. 40%, 60%

9. $53\frac{1}{3}\%$

10. 25%

CHECK YOUR PROGRESS 25.2

1. (a) 135 (b) 940

2. 25%

3. 224

4. Rs. 49680

5. Rs. 1,40,000

6. Rs. 2000

7. $9\frac{1}{11}\%$

8. 50

9. Hindi film music = 6000;

   English music = 4000,

   Classical music = 3000;

   Other kind of music = 7000

10. 200

11. 80%

12. increase 3.5%

13. 12%

14. Yashi = Rs. 13320, Tannia = Rs. 12000, Salma = Rs. 10,000

15. $12\frac{1}{4}\%$

CHECK YOUR PROGRESS 25.3

1. 10 years
2. Rs. 2825
3. Rs. 6900
4. 30 years
5. 3 yrs.
6. 5%
7. 7%, sum = Rs. 6000
8. 5.5%
9. 6.7%
10. Rs. 1780

CHECK YOUR PROGRESS 25.4

1. Rs. 204
2. Rs. 5889.60
3. R = 5% per annum
4. Rs. 20,000
5. 2 years
6. Rs. 96
7. Rs. 1500
8. Rs. 6000

9. 3 years
10. Rs. 3080

CHECK YOUR PROGRESS 25.5

1. 15% gain
2. Rs. 700
3. gain 25%
4. Rs. 10304
5. Loss Rs. 12
6. Rs. 4830
7. 25%
8. gain 12%
9. loss 1%