BBA-501

QUANTITATIVE TECHNIQUES

DIRECTORATE OF DISTANCE EDUCATION

SWAMI VIVEKANAND

SUBHARTI UNIVERSITY

Meerut (National Capital Region Delhi)

CONTENTS

Syllabus

BBA 3rd Year Semester 5th Semester

Quantitative Techniques

Course Code: BBA- 501		
Course Credit:	**Lecture: 04**	**Tutorial: 1**
Course Type:	**Core Course**	**01**
Lectures delivered:	**30 L+ 10 T**	

End Semester Examination System

Maximum Marks Allotted	**Minimum Pass Marks**	**Time Allowed**
70	**28**	**3:00 Hours**

Continuous Comprehensive Assessment (CCA) Pattern

Minor Tests(marks)	**Assignment/ Tutorial/ Presentation**	**Attendance**	**Total**
15	**5**	**10**	**30**

Course Objective: (3 to 4 lines or 2 to 3 points)
To acquaint the students how to make better decisions in complex scenarios by the application of a set of advanced analytical methods. It couples theories, results and theorems of mathematics, statistics and probability with its own theories and algorithms for problem solving.

UNIT	Content	Hours
I	**Operations Research Introduction** Introduction, Historical Background, Scope of Operations Research , Phases of Operations Research, Types of Operations Research Models, Limitations of Operations Research	10
II	**Linear Programming Problem & Transportation Problem** Linear programming: Mathematical formulations of LP Models for product-mix problems; graphical and simplex method of solving LP problems; duality. Transportation problem: Various methods of finding Initial basic feasible solution-North West Corner Method, Least Cost Method &VAM Method and optimal solution-Stepping Stone & MODI Method, Maximization Transportation Problem	14
III	**Assignment model & Game Theory** Assignment model: Hungarian Algorithm and its applications, Maximization Assignment Problem. Game Theory: Concept of game; Two-person zero-sum game; Pure and Mixed Strategy Games; Saddle Point; Odds Method; Dominance Method and Graphical Method for solving Mixed Strategy Game.	12
IV	**Sequencing & Queuing Theory** Sequencing Problem: Johnsons Algorithm for n Jobs and Two machines, n Jobs and Three Machines, Two jobs and m - Machines Problems. Queuing Theory: Characteristics of M/M/I Queue model; Application of Poisson and Exponential distribution in estimating arrival rate and service rate; Applications of Queue model for better service to the customers.	12
V	**Replacement Problem & Project Management** Replacement Problem: Replacement of assets that deteriorate with time, replacement of assets which fail suddenly. Project Management: Rules for drawing the network diagram, Applications of CPM and PERT techniques in Project planning and control; crashing of operations.	12

Course Outcomes: (up to 4 to 6 bulleted points)

1. Understand the basic operations research concepts and terminology involved in optimization techniques
2. Understand how to interpret and solve business-related problems

3. Apply certain mathematical techniques in getting the best possible solution to a problem involving limited resources
4. Apply the most widely used quantitative techniques in decision making
5. Identify project goals, constraints, deliverables, performance criteria, control needs, and resource requirements in order to achieve project success

Text Books: (02)

1. R. Panneerselvam - Operations Research (PHI, 2nd Edition)
2. Sharma J K - Operations Research (Pearson, 3rd Edition
3. R K Gupta- Quantitative techniques for managers (Krishna prakashan media (P) Ltd.

Reference Books: (5 to 6 maximum)

1. Apte-Operation Research and Quantitative Techniques (Excel Books)
2. S Kalawathy-Operation Research (Vikas IVth Edition)
3. Natarajan- Operation Research(Pearson)
4. Singh & Kumar—Operation Research(UDH Publisher edition 2013)
5. Taha Hamdy - Operations Research - An Introduction (Prentice-Hall, 9th edition)
6. Vohra - Quantitative Techniques in Management (Tata McGraw-Hill, 2nd)
7. Kothari - Quantitative Techniques (Vikas 1996, 3rd Edition).

CHAPTER 1 OPERATIONS RESEARCH

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★ STRUCTURE ★

1.1 HISTORY AND BACKGROUND OF OPERATIONS RESEARCH

In the books of management one often finds a specific period of the development of management thought, called the *Period of Scientific Management*. It was in 1885 that Fredrick W. Taylor, "father of scientific management", developed the scientific management theories. It was also called the Modern era when rapid development of concepts, theories and techniques of management took place. During World War II, production bottlenecks forced the government of Great Britain to look up to scientists and engineers to help achieve maximum military production. These scientists and engineers created mathematical models to find the solution of the problems about increasing production of military equipments. This branch of study was called *Operations Research* (OR). Since, it was used in the research in war operations of armed forces. These problems of the armed forces seemed to be similar to those that occurred in production systems. Because of the success of OR in military operations and approach to war problems it began to be used in industry as well.

1.2 WHY STUDY OPERATIONS RESEARCH ?

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We basically helps in determining the best (optimum) solution (course of action) to problems where decision has to be taken under the restriction of limited resources. It is possible to convert any real life problem into a mathematical model. The basic feature of OR is to formulate a real world problem as a mathematical model. Since in the production industry, most of the manufacturers want to lower their labour or production costs to achieve higher profits, OR can be very usefully implied to real life production problems.

OR should be seen as a problem-solving technique. Like management, OR is also a both Science and an Art. The Science part of OR is using mathematical techniques for solving decision problems. The Art part of OR is the ability of the OR team to develop good rapport with those supplying information and those who have to implement the recommended solutions. It is important that both the Science and the Art parts of the OR are understood properly as a system of problem-solving.

In India, OR society was formed in 1950's. The Journal of Operations Research has the mission to serve entire OR community including practitioners, researchers, educators and students. It celebrated its 50th Anniversary of Operations Research and published Anniversary issue in Jan.-Feb. 2001. Industry has become quite aware of the potential of OR as a technique and many industrial and business houses have OR teams working to find solutions to their problems. Particularly, Railways, Indian Airlines, Defence Forces, Telco, DCM, etc., are using OR to their advantage. As a matter of fact, some techniques of OR like Programme Evaluation and Reviewing Techniques (PERT) and Critical Path Method (CPM) are frequently used by many organizations for effective planning and control of the construction projects.

OR helps in taking decisions which optimize (maximize) the interest of the organization, it is a decision-making tool and should be seen as such. Many individuals and organizations see it as a management fad, which has limited use to them. At the same time, tendency of some organizations to force-fit OR to prove that they use managerial techniques whether their functioning needs demand OR or not must be curbed. However, this is also true that complex real life problems can be solved for the advantage of the organizations by using OR techniques.

1.3 DEFINITION OF OPERATIONS RESEARCH

Many authors have given different interpretation to the meaning of Operations Research as it is not possible to restrict the scope of Operations Research in a few sentences. Students must understand that there is no need to single definition of Operations Research which is acceptable to everyone. Two of the widely accepted definitions are provided below for understanding the concept of Operations Research.

"Operations Research is concerned with scientifically deciding how best to design and operate man-machine system usually under conditions requiring the allocation of scarce resources."

—Operations Research Society of America

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The salient features of the above definition are :

(*a*) It is a scientific decision-making technique.

(*b*) It deals with optimizing (maximizing) the results.

(*c*) It is concerned with man-machine systems.

(*d*) The resources are limited.

"Operations Research is a scientific approach to problem solving for executive management."

—HM Wagner

The above definition lays emphasis on :

(*a*) OR being a scientific technique.

(*b*) It is a problem-solving technique.

(*c*) It is for the use of executives who have to take decisions for the organizations.

A close observation of the essential aspects of the above two definitions will make it clear that both are in reality conveying the same meaning. Other definitions of OR also converge on these essential features. One need not remember the definitions word by word but understand the true meaning of the definition provided by different authors. The emphasis has to be on the application of technique so that organizations are benefitted. Hence, the real work of any managerial technique is the ability of the organizations to take advantage for meeting their objectives.

1.4 SALIENT FEATURES OF OPERATIONS RESEARCH

After having understood the basic concept of OR and the need, one can easily understand its salient features.

1. **System Approach :** OR is a systematic approach as is clear from the conceptual model of OR explained above. It encompasses all the sub-systems and departments of an organization. Since it is a technique that effects the entire organization, optimizing results of one part of the organization is not the proper use of OR. Before applying OR techniques the management must understand its impact and implications on the entire organization.
2. **OR is both a Science and an Art :** OR has the scientific orientation because of its inherent methodology and scientific methods are used for problem-solving. But its implementation needs the art of taking the entire organization along. OR does not perform experiment but helps in finding out solutions. OR must take into account the human factor which is the most important factor in implementing any technique/methods of problem-solving.

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3. **Interdependency Approach :** Problem of organizations could be related with economics, engineering, infrastructure related with markets, management of human resources and so on. If OR has to find a solution to problems related to diverse fields, the OR team must be constituted of members with background disciplines of science, management and engineering etc. Only then, practical solutions which can be implemented, can be found to the advantage of organizations.
4. **Management Decision-Making :** Management of any organization has to make decision, which has, impact on its profitability. All business organizations exist to make profits. Non-business organizations like hospitals, educational institutions, NGOs etc., generate profits by reducing the inputs and increasing the outputs through effective and efficient management. Decision-making involves generating different alternatives and selecting the best under the given situation. OR helps in making the right decisions.
5. **Quantitative Technique :** OR is a quantitative technique, which uses mathematical models and finds rational quantitative solutions to the managerial problems. The management may use the OR inputs and take into account the quantitative analysis of the problem in finding the solution in the best interest of the organization.
6. **Use of Information Technology (IT) :** OR extensively uses the IT for complex mathematical problems to its advantage. OR approach to decision-making depends heavily on the use of computers.

1.5 OPERATION RESEARCH MODELS

What is a Model ?

It is very difficult to represent the exact real life situations on a piece of paper. A model attempts to represent reality of the situation by identifying all factors of situation and by establishing some relationship between them. In real life situations, there are so many uncertainties and complexities, which cannot be exactly reproduced. Model helps in identifying such uncertainties and complexities in terms of different factors.

Types of Operation Research Models

Following are some important types of operations research models:

Symbolic or Mathematical Models

This is the most important type of model. Mathematical modelling focuses on creating a mathematical representation of management problems in organisations. All the variables in a particular problem are expressed mathematically. The model then provides different outcomes, which will result from the different choices the management wishes to use. The best outcome in a particular situation will help the management in decision-making. These models use set of mathematical symbols and hence are also called *symbolic models*.

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The variables in many business and industry situations can be related together by mathematical equations. To understand the concepts of symbolic or mathematical model, visualise a balance sheet or profit and loss account as a symbolic representation of the budget. Similarly, the demand curve in economics can be seen as symbolic representation of the buyers' behaviour at varying price levels.

Simulation Models

In simulation model, the behaviour of the system under study is 'initiated over a period of time'. Simulation models do not need mathematical variables to be related in the form of equations, normally, these models are used for solving such problems that cannot be solved mathematically. Simulation is a general technique, which helps us in developing dynamic models, which are similar to the real process. Developing good simulation models is difficult because 'creating' a real life situation to perfection is extremely difficult.

Iconic Models

These models represent the physical simulations to the real life system under study. Physical dimensions are scaled up or down to simplify the actual characteristics and specifications of the system. Preparation of prototype models for say, an automobile or 3-D plant layout are some examples of iconic models.

These are the physical replica of a system and are based on a smaller scale than the original. The models have all the operating features of the actual system. Flight simulators, missile firing simulators, etc., are also examples of iconic models.

Analog Models

They are not the exact replica. Like the iconic models, these are smaller, simple physical systems as compared to the real life systems which are complex. These models are used to explain an actual system by analogy.

Deterministic Models

When the change of one variable has a certain or definite change in the outcome, the model is called a *Deterministic model*. In fact, everything is absolutely clearly defined and the results are known. Economic Order Quantity (EOQ) is a deterministic model, as economic lot size can be exactly known, with change in one of the variables in the EOQ formula.

1.6 METHODOLOGY OF OPERATION RESEARCH

There are many steps involved in application of OR. The methodology to be adopted involves the following :

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1. **Observations of the Operating Environment :** OR is a problem-solving technique. First step in solving the problem is the formulation of the problem. This is done through observation of the system and its environment. As much of information regarding the problem as possible is generated by using the researchers, observers etc.
2. **Formulation of the Problem :** Any problem has many interconnected factors related to the situation. The factors may or may not be in the control of the management. The factors which are relevant to the situation of the problem and are under the control of management must be identified. Once the problem area is known, different variables considered responsible for the problem are listed. Now, it is possible to define the problem in terms of the variables and their relationship.
3. **Selecting and Developing a Suitable Model :** At this stage, a suitable model which best represents the real life situation has to be selected. The model is developed to show the relations and interrelation between a cause and effect. Normally, the model is fully tested and modified to ensure that OR technique applied is able to solve the problem.
4. **Collecting the Data :** Next step is to collect the data required by the selected model. The process of the OR model in finding the solution depends to a large extent on the quantity and quality of data. More the data and lesser the errors in data, the quality of managerial decisions will be better. The required information can be obtained through observations or from recorded data or even based on experience and maturity of the OR team.
5. **Finding the Solution :** Once the model has been developed, it is possible to find the solution. OR solutions are under a particular situation and under certain assumptions. Many assumptions have to be made by the OR team to simplify the model. The solution is valid only under these assumptions. Once the solution by the OR techniques is found, certain input variables are changed to see the output. By this method the best possible solution can be found.
6. **Presenting the Solution to the Management :** The OR team has to present the solution to the management in a proper manner. The conditions under which the solution can be used and the conditions under which solution cannot work must be explained to the management. The assumptions made at arriving the solution and the weakness of the solutions should also be explained to the management.
7. **Implementing the Solution :** This is the last step in the OR application methodology. The solution provided by the OR technique is scientific but the application of this technique involves many behavioural aspects. This is the 'art' part of OR and is of utmost importance. Any gap between the perception of the management and the approach of OR team must be removed.

1.7 TOOLS OF OPERATION RESEARCH

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Operation Research is a very versatile science and has many tools/techniques, which can be used for problem solving. However, it is not possible to list all these techniques as everyday new methods in the use of OR are being developed. Some of the tools of OR are discussed in the succeeding paragraphs :

1. **Linear Programming (LP) :** Most of the industrial and business organisations have the objectives of minimizing costs and maximizing the profits. LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model*. It is a mathematical technique used to allocate limited resources amongst competing demands in an optimal manner. The application of LP requires that there must be a well-defined objective function (like maximizing profits and minimizing costs) and there must be constraints on the amount and extent of resources available for satisfying the objective function.
2. **Queuing Theory :** In real life situations, the phenomenon of waiting is involved whether it is the people waiting to buy goods in a shop, patients waiting outside an Out Patient Department (OPD), vehicles waiting to be serviced in a garage and so on. Because in general, customer's arrival and his service time is not known in advance; hence a queue is formed. Queuing or waiting line theory aims at minimizing the overall cost due to servicing and waiting. How many servicing facilities can be added at what cost to minimize the time in queue is the aim in the application of this theory.
3. **Network Analysis Technique :** A network can be used to present or depict the activities necessary to complete a project. This helps us in planning, scheduling, monitoring and control of large and complex projects. The project may be developing a new battle tank, construction of dam or a space flight. The project managers are interested in knowing the total project completion time, probability that a project can be completed by a particular time, and the least cost method of reducing the total project completion time. Techniques like Programme Evaluation and Reviewing Technique (PERT) and Critical Path Method (CPM) are part of network analysis. These are popular techniques and widely used in project management.
4. **Replacement Theory Model :** All plants, machinery and equipment needs to be replaced at some point of time, either because there is deterioration in their efficiency or because new and better equipment is available and the old one has become obsolete. Sooner or later the equipment needs to be replaced. The decision to be taken by the management involves consideration of the cost of new equipment which is to be purchased and what can be recovered from the old equipment through its sale, or its scrap value, the residual life of the old equipment and many other related aspects. These are important decisions involving investment of capital and need to be taken very carefully.

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5. **Inventory Control :** Inventory includes all the stocks of material, which an organization buys for production/manufacture of goods and services for sale. It will include raw material; semi-finished and finished products, spare parts of machines, etc. Managers face the problems of how much of raw material should be purchased, when should it be purchased and how much should be kept in stock. Overstocking will result in locked capital not available for other purposes, whereas under-stocking will mean stock-out and idle manpower and machine resulting in reduced output. It is desirable to have just the right amount of inventory at the right time. Inventory control models can help us in finding out the optimal order size, reorder level, etc., so that the capital resources are conserved and maximum output ensured.
6. **Integer Programming :** Integer programming deals with certain situations in which the variable assumes non-negative integer (complete or whole number) values only. In LP models the variable may take even a fraction value and the figures are rounded off to the nearest integer to get the solution, *i.e.*, number of vehicles available in a problem cannot be in fractions. When such rounding off is done the solution does not remain an optimal solution. In integer programming the solution containing unacceptable and fractional values are ruled out and the next best solution using whole numbers is obtained. An integer programming may be called *mixed* or *pure depending* on whether some or all the variables are restricted to integer values.
7. **Transportation Problems :** Transportation problems are basically LP model problems. This model deals with finding out the minimum transportation cost for transporting the single commodity from a number of sources to number of destinations. Typical problem involves transportation of some manufactured products (say cars in 3 different plants) and these have to be sent to the warehouses of various dealers in different parts of country. This may be understood as a special case of simplex method developed for LP problems, allocating scare resources to competing demands. The main purpose of the transportation is to schedule the dispatch of the single product from different sources like factories to different destinations as total transportation cost is minimized.
8. **Decision Theory and Games Theory :** Information for making decisions is the most important factor. Many models of OR assume availability of perfect information which is called *decision-making under certainty*. However, in real life situations, only partial or imperfect information is available. In such a situation we have two cases, either decision under risk or decision under uncertainty. Hence from the point of view of availability of information, there are three cases, certainty and uncertainty, the two extreme cases and risk is the "in-between" case.

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Games theory is concerned with decision-making in a conflict situation where two or more intelligent opponents try to optimize their own decision. In Games theory, an opponent is referred to as a player and each player has a number of choices. The Games theory helps the decision maker to analyse the course of action available to his opponent. In decision theory, we use decision tree which can be graphically represented to solve the decision-making problems.

9. **Assignment Problems :** We have the problem of assigning a number of tasks to a number of persons who may use machines. The objective is to assign the jobs to the machines in such a way that the cost is least. This may be considered a special case of LP transportation model. Here jobs may be treated as 'services' and machines may be considered the 'destinations'. Assignment of a particular job to a particular person so that all the jobs can be completed in shortest possible time hence incurring the least cost, is the assignment problem.

10. **Markov Analysis :** Markov analysis is used to predict future conditions. It assumes that the occurrence of a future state depends upon the immediately preceding state and only on it. It is based on the probability theory and predicts the change in a system over a period of time if the present behaviour of the system is known. Predicting market share of the companies in future as also whether a machine will function properly or not in future, are examples of Markov analysis.

11. **Simulation Techniques :** Since all real life situations cannot be represented mathematically, certain assumptions are made and dynamic models which act like the real processes are developed. It is very difficult to develop simulation models which can give accurate solutions to the problems, but this is a good method of problem solving, when the problems are very complex and cannot be solved otherwise.

1.8 IMPORTANT APPLICATIONS OF OPERATION RESEARCH

In today's world where decision-making does not depend on intuition, managerial techniques are widely used. All the applications of OR cannot be listed because OR as a tool finds new application everyday. It finds typical applications in many activities related to work planning.

Some important applications of OR are :

1. **Manufacturing/Production**
   - Production planning and control
   - Inventory management.

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2. **Facilities Planning**
   — Design of logistic systems
   — Factory/building location and size decisions
   — Transportation, loading and unloading
   — Planning warehouse locations.
3. **Accounting**
   — Credit policy decisions
   — Cash flow and fund flow planning.
4. **Construction Management**
   — Allocation of resources to different projects in hand
   — Workforce/labour planning
   — Project management (scheduling, monitoring and control).
5. **Financial Management**
   — Investment decisions
   — Portfolio management.
6. **Marketing Management**
   — Product-mix decisions
   — Advertisement/Promotion budget decisions
   — Launching new product decisions.
7. **Purchasing Decisions**
   Inventory management (optimal level of purchase), Optimal re-ordering.
8. **Personnel Management**
   — Recruitment and selection of employees
   — Designing training and development programmes
   — Human Resources Planning (HRP).
9. **Research and Development**
   — Planning and control of new research and development projects.
   — Product launch planning.

1.9 PITFALLS IN THE USE OF OPERATION RESEARCH FOR DECISION-MAKING

The first stage of OR application after collecting data/information through observation is the formulation of the problem. It is the most important and most difficult task in OR application. Have the OR team been able to identify the right problem for

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finding the solution ? Has the problem been accurately defined in unambiguous manner ? Selecting and developing a suitable model is not an easy task. The model must represent the real life situation as far as possible. Collection of data needs a lot of time by a number of people. It is time-consuming and expensive process. Collection of data is done either by observation or from the previous recorded data. When a system is being observed by the OR team, it effects the behaviour of the persons performing the task. The very fact that the workers know that they are being observed is likely to change their work behaviour. The second method of data collection, the records, are never reliable and do not provide sufficient information which is required.

As OR problem-solving techniques is very time-consuming, the quality of decision-making may become a causality. The management has to make a decision either way. Decision based on insufficient or incomplete information will not be the best decision. A reasonably good solution without the use of OR may be preferred by the management as compared to a slightly better solution provided by the use of OR which is very expensive in time and money.

Due to the above reasons, many OR specialists try and fit the solution they have, to the problem. This is dangerous and unethical and organizations must guard against this.

1.10 LIMITATIONS OF OPERATIONS RESEARCH

Operation Research is an extremely powerful tool in the hands of a decision-maker and to that extent the advantage of OR techniques are immense. Some of them are :

(*a*) *It helps in optimum use of resources.* LP techniques suggest many methods of most effective and efficient ways of optimally using the production factors.

(*b*) *Quality of decision can be improved by suitable use of OR techniques.* If a mathematical model representing the real life situation is well-formulated representing the real life situation, the computation tables give a clear picture of the happenings (changes in the various elements *i.e.*, variables) in the model. The decision-maker can use it to his advantage, specially if computerised software can be used to make changes in variables as per requirement.

The limitations of OR emerge only out of the time and cost involved as also the problem of formulating a suitable mathematical model, otherwise, as suggested above, it is a very powerful medium of getting the best out of limited resources. So, the problem is its application rather than its utility, which is beyond doubt. Some of the limitations are :

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(a) *Large number of cumbersome computations*. Formulation of mathematical models which takes into account all possible factors which define a real life problem is difficult. Because of this, the computations involved in developing relationships in very large variables need the help of computers. This discourages small companies and other organisations from getting the best out of OR techniques.

(b) *Quantification of problems*. All the problems cannot be qualified properly as there are a large number of intangible factors, such as human emotions, human relationship and so on. If these intangible elements/variables are excluded from the problem even though they may be more important than the tangible ones, the best solution cannot be determined.

(c) *Difficult to conceptualize and use by the managers.* OR applications is a specialist's job, these persons may be mathematicians or statisticians who understand the formulation of models, finding solution and recommending the implementation. The managers really do not have the hang of it. Those who recommend a particular OR technique may not understand the problem well enough and those who have to use may not understand the 'why' of that recommendation. This creates a 'gap' between the two and the results may not be optimal.

1.11 TIPS ON FORMULATING LINEAR PROGRAMMING MODELS

(a) *Read the statement of the problem carefully.*

(b) *Identify the decision variables.* These are the decisions that are to be made. What set of variables has a direct impact on the level of achievement of the objectives and can be controlled by the decision-maker ? Once these variables are identified, list them providing a written definition (*e.g.*, x_1 = number of units produced and sold per week of product 1, x_2 = number of units produced and sold per week of product 2).

(c) *Identify the objective.* What is to be maximised or minimised ? (*e.g.*, maximize total weekly profit from producing product 1 and 2).

(d) *Identify the constraints.* What conditions must be satisfied when we assign values to the decision variables? You may like to write a verbal description of the restriction before writing the mathematical representation (*e.g.*, total production of product 1 > 100 units).

(e) *Write out the mathematical model.* Depending on the problem, you might start by defining the objective function on the constraints. Do not forget to include the non-negativity constraints.

1.12 SUMMARY

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- Operations Research is concerned with scientifically deciding how best to design and operate man-machine system usually under conditions requiring the allocation of scarce resources.
- Operations Research is a scientific approach to problem solving for executive management.
- Economic Order Quantity (EOQ) is a deterministic model, as economic lot size can be exactly known, with change in one of the variables in the EOQ formula.
- Operation Research is a very versatile science and has many tools/techniques, which can be used for problem solving.
- LP deals with maximizing a given objective. Since the objective function and boundry conditions are linear in nature, this mathematical model is called *Linear Programming Model.*
- Operation Research is an extremely powerful tool in the hands of a decision-maker and to that extent the advantage of OR techniques are immense.

1.13 REVIEW QUESTIONS

1. What is the concept of Operation Research ? Write a detailed note on its development.
2. Discuss significance and scope of OR in business and industry.
3. What are the different phases of OR ? How is OR helpful in decision-making ?
4. Discuss briefly various steps involved in solving an OR problem. Illustrate with one example from the functional area of your choice.
5. Explain applications of Operations Research in business.
6. What is the significance and scope of Operation Research in the development of Indian Economy ?
7. What is the role of OR in modern day business ? Give examples in support of your answer.
8. Discuss the meaning, significance and scope of Operations Research. Describe some methods of OR.
9. Illustrate and explain various features of OR.
10. Define Operations Research in your own words and explain various tools of OR.
11. What is a model ? What are the types of models you are familiar with ? What are the advantages and pitfalls of models ?

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12. Define an OR model. Give examples from industry and business to explain the use of models.
13. Define OR and discuss its scope.
14. Discuss the significance and scope of Operations Research in modern management.
15. Write a detailed note on the use of models for decision-making. Your answer should specifically cover the following :
    (*i*) Need for model building
    (*ii*) Type of model appropriate to the situation
    (*iii*) Steps involved in the construction of a model
    (*iv*) Setting up criteria for evaluating different alternatives
    (*v*) Role of random numbers.

CHAPTER 2 LINEAR PROGRAMMING AND TRANSPORTATION PROBLEM

★ STRUCTURE ★

LINEAR PROGRAMMING

2.1 INTRODUCTION TO OPERATIONS RESEARCH

The roots of Operations Research (OR) can be traced many decades ago. First this term was coined by Mc Closky and Trefthen of United Kingdom in 1940 and it came in existence during world war II when the allocations of scarce resources were done to the various military operations. Since then the field has developed very rapidly. Some chronological events are listed as follows:

1952 – Operations Research Society of America (ORSA).
1957 – Operations Research Society of India (ORSI)
– International Federation of OR Societies

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1959 – First Conference of ORSI

1963 – Opsearch (the journal of OR by ORSI).

However the term 'Operations Research' has a number of different meaning. The Operational Research Society of Great Britain has adopted the following illaborate definition :

"Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific method of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies and controls. The purpose is to help management to determine its policy and actions scientifically."

Whereas ORSA has offered the following shorter definition :

"Operations Research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources."

Many individuals have described OR according to their own view. Only three are quoted below :

"OR is the art of giving bad answers to problems which otherwise have worse answers"

—**T.L. Saaty**

"OR is a scientific approach to problems solving for executive management."

—**H.M. Wagner**

"OR is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources."

—**H.A. Taha**

An abbreviated *list of applications* of OR techniques are given below :

1.	Manufacturing	:	Production scheduling
			Inventory control
			Product mix
			Replacement policies
2.	Marketing	:	Advertising budget allocation
			Supply chain management
3.	Organizational behaviour	:	Personnel planning
			Scheduling of training programs
			Recruitment policies
4.	Facility planning	:	Factory location
			Hospital planning
			Telecommunication network planning
			Warehouse location
5.	Finance	:	Investment analysis
			Portfolio analysis
6.	Construction	:	Allocation of resources to projects
			Project scheduling
7.	Military		
8.	Different fields of engineering.		

2.2 INTRODUCTION TO LINEAR PROGRAMMING PROBLEMS (LPP)

I. When a problem is identified then the attempt is to make an mathematical model. In decision-making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors. So in the formation of mathematical model the following **three phases** are carried out :

(*i*) Identify the decision variables.

(*ii*) Identify the objective using the decision variables and

(*iii*) Identify the constraints or restrictions using the decision variables.

Let there be n decision variable $x_1, x_2, \ldots, x_n$ and the general form of the mathematical model which is called as Mathematical programming problem under decision-making can be stated as follows :

Maximize/Minimize $\quad z = f(x_1, x_2, \ldots, x_n)$

Subject to, $\quad g_i(x_1, x_2, \ldots, x_n) \{\leq, \geq \text{ or } =\} b_i$

$$i = 1, 2, \ldots, m.$$

and the type of the decisions *i.e.*, $x_j \geq 0$

or, $\quad x_j \leq 0$ or x_j's are unrestricted

or $\quad$ combination types decisions.

In the above, if the functions f and g_i $(i = 1, 2, \ldots, m)$ are all linear, then the model is called "*Linear Programming Problem (LPP)*". If any one function is non-linear then the model is called "*Non-linear Programming Problem (NLPP)*".

II. We define some basic aspects of LPP in the following :

(*a*) Convex set : A set X is said to be convex if

$$x_1, x_2 \in X, \text{ then for } 0 \leq \lambda \leq 1,$$

$$x_3 = \lambda x_1 + (1 - \lambda)x_2 \in X$$

Some examples of convex sets are :

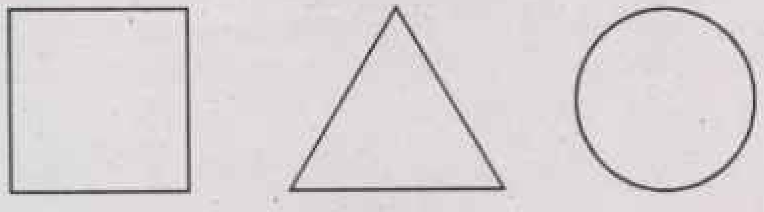


Fig. 2.1 Convex sets

Some examples of non-convex sets are :

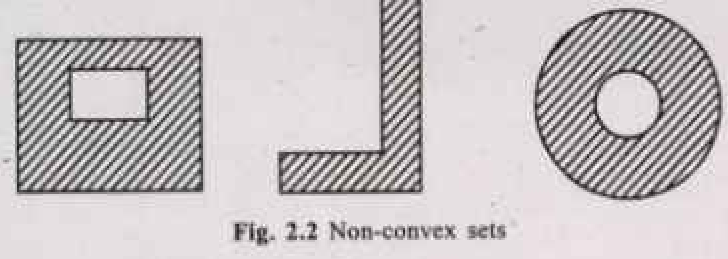


Fig. 2.2 Non-convex sets

Basically if all the points on a line segment forming by two points lies inside the set/geometric figure then it is called convex.

NOTES

(b) Extreme point or vertex or corner point of a convex set : It is a point in the convex set which cannot be expressed as $\lambda x_1 + (1 - \lambda)x_2$ where x_1 and x_2 are any two points in the convex set.

For a triangle, there are three vertices, for a rectangle there are four vertices and for a circle there are infinite number of vertices.

(c) Let $Ax = b$ be the constraints of an LPP. The set $X = \{x \mid Ax = b, x \geq 0\}$ is a convex set.

Feasible Solution : A solution which satisfies all the constraints in LPP is called feasible solution.

Basic Solution : Let m = number of constraints and n = number of variables and $m < n$. Then the solution from the system $Ax = b$ is called basic solution. In this system there are ${}^n c_m$ number of basic solutions. By setting $(n - m)$ variables to zero at a time, the basic solutions are obtained. The variables which is set to zero are known as 'non-basic' variables. Other variables are called basic variables.

Basic Feasible Solution (BFS) : A solution which is basic as well as feasible is called *basic feasible solution.*

Degenerate BFS : If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.

Optimal BFS : The BFS which optimizes the objective function is called *optimal BFS.*

2.3 GRAPHICAL METHOD

Let us consider the constraint $x_1 + x_2 = 1$. The feasible region of this constraint comprises the set of points on the straight line $x_1 + x_2 = 1$.

If the constraint is $x_1 + x_2 \geq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points above the line. Here above means away from origin.

If the constraint is $x_1 + x_2 \leq 1$, then the feasible region comprises not only the set of points on the straight line $x_1 + x_2 = 1$ but also the points below the line. Here below means towards the origin.

The above three cases depicted as follows:

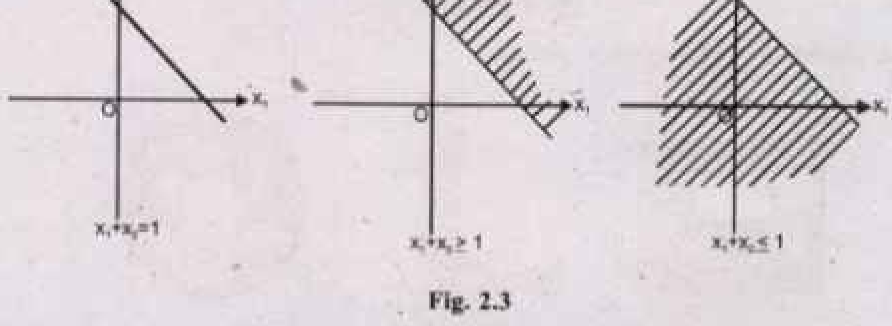


Fig. 2.3

For the constraints $x_1 \geq 1, x_1 \leq 1, x_2 \geq 1, x_2 \leq 1$ the feasible regions are depicted below :

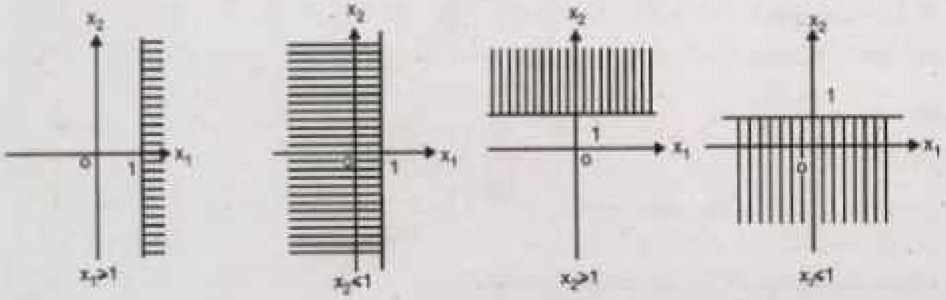


Fig. 2.4

For the constraints $x_1 - x_2 = 0, x_1 - x_2 \geq 0$ and $x_1 - x_2 \leq 0$ the feasible regions are depicted in Fig. 2.5.

The *steps of graphical method* can be stated as follows :

(*i*) Plot all the constraints and identify the individual feasible regions.

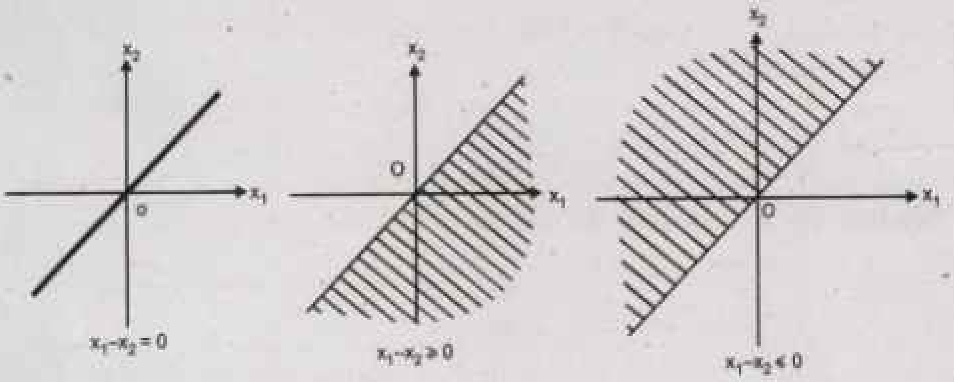


Fig. 2.5

(*ii*) Identify the common feasible region and identify the corner points *i.e.*, vertices of the common feasible region.

(*iii*) Identify the optimal solution at the corner points if exists.

Example 1. *Using graphical method solve the following LPP :*

$$\text{Maximize } z = 5x_1 + 3x_2$$

$$\begin{aligned} \text{Subject to. } 2x_1 + 5x_2 &\leq 10, \\ 5x_1 + 2x_2 &\leq 10, \\ 2x_1 + 3x_2 &\geq 6, \\ x_1 \geq 0, x_2 &\geq 0. \end{aligned}$$

Solution. Let us present all the constraints in intercept form *i.e.*,

$$\frac{x_1}{5} + \frac{x_2}{2} \leq 1 \qquad \text{...(I)}$$

$$\frac{x_1}{2} + \frac{x_2}{5} \leq 1 \qquad \text{...(II)}$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \qquad \text{...(III)}$$

The common feasible region ABC is shown in Fig. 2.6 and the individual regions are indicated by arrows. (Due to non-negativity constraints *i.e.*, $x_1 \geq 0, x_2 \geq 0$, the common feasible region is obtained in the first quadrant).

NOTES

The corner points are $A\left(\frac{18}{11}, \frac{10}{11}\right)$, $B\left(\frac{10}{7}, \frac{10}{7}\right)$ and C (0, 2). The value of the objective function at the corner points are $z_A = \frac{120}{11} = 10.91$, $z_B = \frac{80}{7} = 11.43$ and $z_C = 6$.

Here the common feasible region is bounded and the maximum has occurred at the corner point B. Hence the optimal solution is

$$x_1^* = \frac{10}{7},\ x_2^* = \frac{10}{7} \text{ and } z^* = \frac{80}{7} = 11.43.$$

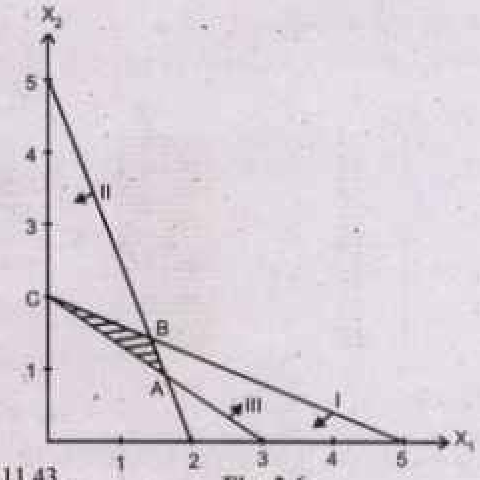


Fig. 2.6

Example 2. *Using graphical method solve the following LPP :*

$$\text{Minimize } z = 3x_1 + 10x_2$$

$$\text{Subject to, } 3x_1 + 2x_2 \geq 6,$$
$$4x_1 + x_2 \geq 4,$$
$$2x_1 + 3x_2 \geq 6,$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Solution. Let us present all the constraints in intercept form *i.e.*,

$$\frac{x_1}{2} + \frac{x_2}{3} \geq 1 \qquad ...(I)$$

$$\frac{x_1}{1} + \frac{x_2}{4} \geq 1 \qquad ...(II)$$

$$\frac{x_1}{3} + \frac{x_2}{2} \geq 1 \qquad ...(III)$$

Due to the non-negativity constraints *i.e.*, $x_1 \geq 0$ and $x_2 \geq 0$ the feasible region will be in the first quadrant.

The common feasible region is shown in Fig. 2.7, where the individual feasible regions are shown by arrows. Here the common feasible region is unbounded.

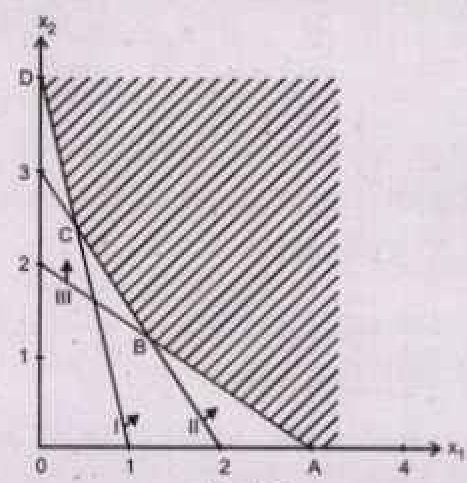


Fig. 2.7

NOTES

i.e., open with the corner points A(3, 0), $B\left(\frac{3}{5}, \frac{8}{5}\right)$, $C\left(\frac{2}{5}, \frac{12}{5}\right)$ and D (0, 4). The value of the objective function at the corner points are $z_A = 9$, $z_B = \frac{89}{5} = 17.8$, $z_C = \frac{126}{5} = 25.2$, and $z_D = 40$.

Here the minimum has occurred at A and there is no other point in the feasible region at which the objective function value is lower than 9. Hence the optimal solution is

$$x_1^* = 3,\ x_2^* = 0 \text{ and } z^* = 9$$

Example 3. *Solve the following LPP by graphical method :*

$$\text{Maximize } z = 3x_1 - 15x_2$$

$$\text{Subject to, } x_1 + x_2 \le 8,$$

$$x_1 - 4x_2 \le 8,$$

$$x_1 \ge 0,\ x_2 \text{ unrestricted in sign.}$$

Solution. Since x_2 is unrestricted in sign this means x_2 may be ≥ 0 or ≤ 0. Also $x_1 \ge 0$. Then the common feasible region will be in the first and fourth quadrant. Let us present all the constraints in intercept forms *i.e.*,

$$\frac{x_1}{8} + \frac{x_2}{8} \le 1 \qquad \text{...(I)}$$

$$\frac{x_1}{8} - \frac{x_2}{2} \le 1 \qquad \text{...(II)}$$

The common feasible region is shown in Fig. 2.8 where the individual feasible regions are shown by arrows.

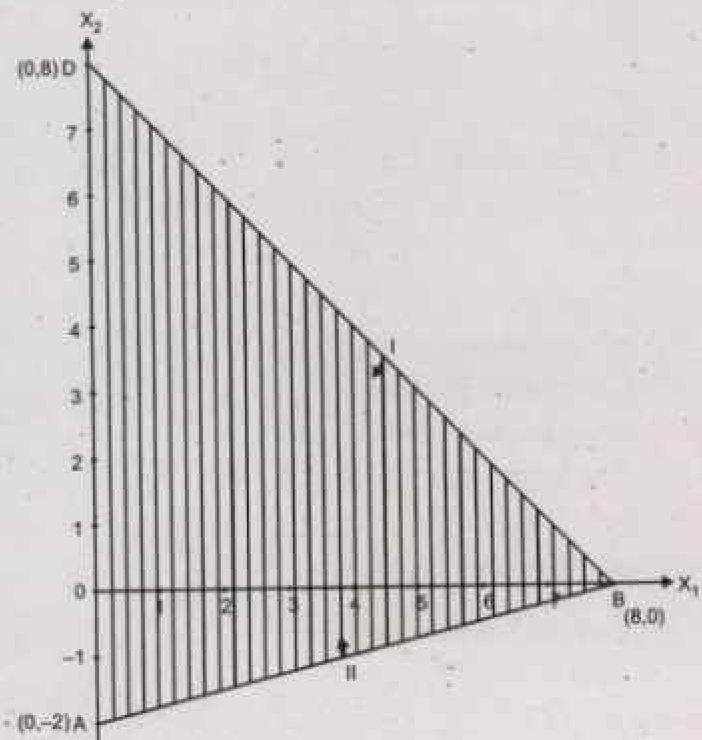


Fig. 2.8

The value of the objective function at the corner points are $z_A = 30$, $z_B = 24$ and $z_C = -120$. Since the common feasible region is bounded and the maximum has occurred at A, the optimal solution is

$$x_1^* = 0,\ x_2^* = -2 \text{ and } z^* = 30.$$

NOTES

Exceptional Cases in Graphical Method

There are three cases may arise. When the value of the objective function is maximum/minimum at more than one corner points then '*multiple optima*' solutions are obtained.

Sometimes the optimum solution is obtained at infinity, then the solution is called '*unbounded solution*'. Generally, this type of solution is obtained when the common feasible region is unbounded and the type of the objective function leads to unbounded solution.

When there does not exist any common feasible region, then there does not exist any solution. Then the given LPP is called *infeasible i.e.*, having *no solution*. For example, consider the LPP which is infeasible

$$\text{Maximize } z = 5x_1 + 10x_2$$
$$\text{Subject to, } x_1 + x_2 \le 2,$$
$$x_1 + x_2 \ge 3,$$
$$x_1, x_2 \ge 0.$$

Example 4. *Solve the following LPP using graphical method :*

Maximize $\quad z = x_1 + \frac{3}{5}x_2$

Subject to, $\quad 5x_1 + 3x_2 \le 15,$

$3x_1 + 4x_2 \le 12,$

$x_1, x_2 \ge 0.$

Solution. Let us present all the constraints in intercept forms *i.e.*,

$$\frac{x_1}{3} + \frac{x_2}{5} \le 1 \quad \text{...(I)}$$

$$\frac{x_1}{4} + \frac{x_2}{3} \le 1 \quad \text{...(II)}$$

Due to non-negativity constraints *i.e.*, $x_1 \ge 0, x_2 \ge 0$ the common feasible region is obtained in the first quadrant as shown in Fig. 2.9 and the individual feasible regions are shown by arrows.

The corner points are O(0, 0), A (3, 0), $B\left(\frac{24}{11}, \frac{15}{11}\right)$ and C(0, 3). The values of the objective function at the corner points are obtained as $z_O = 0$, $z_A = 3$, $z_B = 3$, $z_C = \frac{9}{4}$.

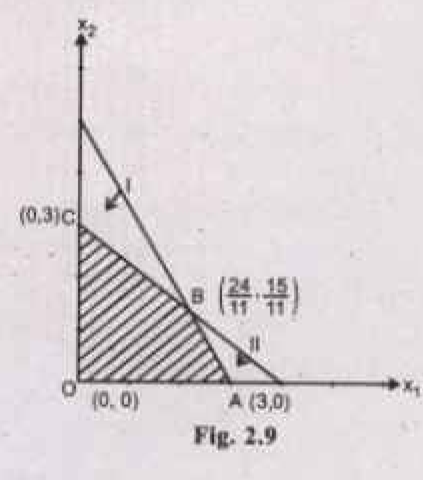


Fig. 2.9

Since the common feasible region is bounded and the maximum has occurred at two corner points *i.e.*, at A and B respectively, these solutions are called multiple optima. So the solutions are

$$x_1^* = 3,\ x_2^* = 0 \quad \text{and} \quad x_1^* = \frac{15}{11},\ x_2^* = \frac{24}{11} \text{ and } z^* = 3.$$

Example 5. *Using graphical method show that the following LPP is unbounded.*

$$\text{Maximize } z = 10x_1 + 3x_2$$

$$\text{Subject to, } -2x_1 + 3x_2 \le 6,$$

$$x_1 + 2x_2 \ge 4,$$

$$x_1,\ x_2 \ge 0.$$

Solution. Due to the non-negativity constraints *i.e.*, $x_1 \ge 0$ and $x_2 \ge 0$ the common feasible region will be obtained in the first quadrant. Let us present the constraints in the intercept forms *i.e.*,

$$\frac{x_1}{-3} + \frac{x_2}{2} \le 1 \qquad ...(I)$$

$$\frac{x_1}{4} + \frac{x_2}{2} \ge 1 \qquad ...(II)$$

The common feasible region is shown in Fig. 2.10 which is unbounded *i.e.*, open region.

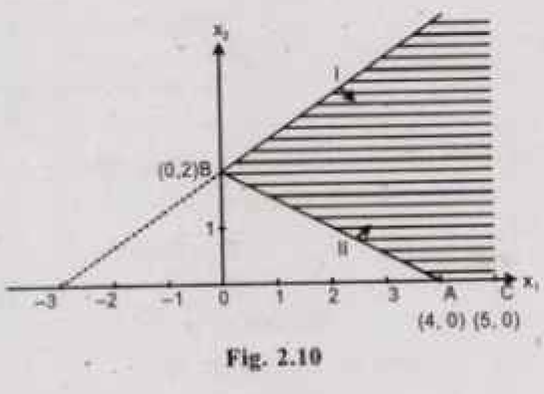


Fig. 2.10

There are two corner points A(4, 0) and B(0, 2). The objective function values are $z_A = 40$ and $z_B = 6$. Here the maximum is 40. Since the region is open, let us examine some other points.

Consider the point C(5, 0) and the value of the objective function is $z_C = 50$ which is greater than z_A. Therefore z_A is no longer optimal. If we move along x-axis, we observe that the next value is higher than the previous value and we reach to infinity for optimum value. Hence the problem is unbounded.

Note. For the same problem minimum exists which is the point B.

PROBLEMS

NOTES

Using graphical method solve the following LPP :

1. Maximize $z = 13x_1 + 117x_2$

Subject to, $x_1 + x_2 \leq 12,$

$x_1 - x_2 \geq 0,$

$4x_1 + 9x_2 \leq 36,$

$0 \leq x_1 \leq 2$ and $0 \leq x_2 \leq 10.$

2. Maximize $z = 3x_1 + 15x_2$

Subject to, $4x_1 + 5x_2 \leq 20,$

$x_2 - x_1 \leq 1,$

$0 \leq x_1 \leq 4$ and $0 \leq x_2 \leq 3.$

3. Maximize $z = 5x_1 + 7x_2$

Subject to, $3x_1 + 8x_2 \leq 12,$

$x_1 + x_2 \leq 2,$

$2x_1 \leq 3,$

$x_1, x_2 \geq 0.$

4. Minimize $z = 2x_1 + 3x_2$

Subject to, $x_2 - x_1 \geq 2,$

$5x_1 + 3x_2 \leq 15,$

$2x_1 \geq 1,$

$x_2 \leq 4,$

$x_1, x_2 \geq 0.$

5. Minimize $z = 10x_1 + 9x_2$

Subject to, $x_1 + 2x_2 \leq 10,$

$x_1 - x_2 \leq 0,$

$x_1 \leq 0, x_2 \geq 0.$

6. Minimize $z = 4x_1 + 3x_2$

Subject to, $2x_1 + 3x_2 \leq 12,$

$3x_1 - 2x_2 \leq 12,$

x_1 unrestricted in sign, $x_2 \geq 0.$

7. Maximize $z = 10x_1 + 11x_2$

Subject to, $x_1 + x_2 \geq 4,$

$0 \leq x_2 \leq 3,$

$x_1 \geq 2,$

$x_1 \geq 0.$

8. Minimize $z = -x_1 + 2x_2$

Subject to, $x_1 - x_2 \geq 1,$

$x_1 + x_2 \geq 5,$

$x_1, x_2 \geq 0.$

9. Maximize $z = 4x_1 + 5x_2$

$$\text{Subject to, } 4x_1 - 5x_2 \le 20,$$
$$x_2 - x_1 \le 1,$$
$$0 \le x_2 \le 3$$
$$0 \le x_1 \le 4.$$

10. $$\text{Maximize } z = -3x_1 + 4x_2$$
$$\text{Subject to, } -3x_1 + 4x_2 \le 12,$$
$$2x_1 - x_2 \ge -2,$$
$$x_1 \le 4,$$
$$x_1 \ge 0, x_2 \ge 0.$$

ANSWERS

1. $x_1 = 2, x_2 = 2, z^* = 260$
2. $x_1 = \frac{5}{3},\ x_2 = \frac{8}{3},\ z^* = 45$
3. $x_1 = \frac{4}{5},\ x_2 = \frac{6}{5},\ z^* = \frac{62}{5}$
4. $x_1 = \frac{1}{2},\ x_2 = \frac{5}{2},\ z^* = \frac{17}{2}$
5. $x_1 = 0, x_2 = 0, z^* = 0$
6. $x_1 = 4, x_2 = 0, z^* = 16$
7. Unbounded solution
8. Unbounded solution
9. Multiple optima :

   $x_1 = \frac{5}{3},\ x_2 = \frac{8}{3}$, and $x_1 = 4,\ x_2 = \frac{4}{5}$ and $z^* = 20$
10. Multiple optima :

    $x_1 = \frac{4}{5},\ x_2 = \frac{18}{5}$ and $x_1 = 4, x_2 = 6$ and $z^* = 12$

2.4 SIMPLEX METHOD

The algorithm is discussed below with the help of a numerical example *i.e.*, consider

$$\text{Maximize } z = 4x_1 + 8x_2 + 5x_3$$
$$\text{Subject to, } x_1 + 2x_2 + 3x_3 \le 18,$$
$$2x_1 + 6x_2 + 4x_3 \le 15,$$
$$x_1 + 4x_2 + x_3 \le 6,$$
$$x_1, x_2, x_3 \ge 0.$$

Step 1. If the problem is in minimization, then convert it to maximization as Min $z = -$ Max $(-z)$.

Step 2. All the right side constants must be positive. Multiply by -1 both sides for negative constants. All the variables must be non-negative.

Step 3. Make standard form by adding slack variables for '$\le$' type constraints, surplus variables for '$\ge$' type constraints and incorporate these variables in the objective function with zero coefficients.

NOTES

For example, Maximum $z = 4x_1 + 8x_2 + 5x_3 + 0.s_1 + 0.s_2 + 0.s_3$

$$\text{Subject to, } x_1 + 2x_2 + 3x_3 + s_1 = 18,$$
$$2x_1 + 6x_2 + 4x_3 + s_2 = 15,$$
$$x_1 + 4x_2 + x_3 + s_3 = 6,$$
$$x_1, x_2, x_3 \geq 0, s_1, s_2, s_3 \geq 0$$

Note that an unit matrix due to s_1, s_2 and s_3 variables is present in the coefficient matrix which is the key requirement for simplex method.

Step 4. Simplex method is an iterative method. Calculations are done in a table which is called simplex table. For each constraint there will be a row and for each variable there will be a column. Objective function coefficients c_j are kept on the top of the table. x_B stands for basis column in which the variables are called 'basic variables'. Solution column gives the solution, but in iteration 1, the right side constants are kept. At the bottom $z_j - c_j$ row is called 'net evaluation' row.

In each iteration one variable departs from the basis and is called departing variable and in that place one variable enter which is called entering variable to improve the value of the objective function.

Minimum ratio column determines the departing variable.

Iteration 1.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
	$z_j - c_j$								

Note. Variables which are forming the columns of the unit matrix enter into the basis column. In this table the solution is $s_1 = 18$, $s_2 = 15$, $s_3 = 6$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $z = 0$.

To test optimality we have to calculate $z_j - c_j$ for each column as follows :

$$z_j - c_j = c_B^T.[x_j] - c_j$$

For first column, $$(0, 0, 0)\begin{pmatrix}1\\2\\1\end{pmatrix} - 4 = -4$$

For second column, $$(0, 0, 0)\begin{pmatrix}2\\6\\4\end{pmatrix} - 8 = -8$$ and so on.

These are displayed in the following table :

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	18	1	2	3	1	0	0	
0	s_2	15	2	6	4	0	1	0	
0	s_3	6	1	4	1	0	0	1	
	$z_j - c_j$		-4	-8	-5	0	0	0	

↑

NOTES

Decisions : If all $z_j - c_j \geq 0$. Then the current solution is optimal and stop. Else, Select the negative most value from $z_j - c_j$ and the variable corresponding to this value will be the entering variable and that column is called '*key column*'. Indicate this column with an upward arrow symbol.

In the given problem '-8' is the most negative and variable x_2 is the entering variable. If there is a tie in the most negative, break it arbitrarily.

To determine the *departing variable*, we have to use minimum ratio. Each ratio is calculated as $\frac{[\text{soln.}]}{[\text{key column}]}$, componentwise division only for positive elements (*i.e.*, > 0) of the key column. In this example,

$$\text{min.}\left\{\frac{18}{2}, \frac{15}{6}, \frac{6}{4}\right\} = \text{min.}\{9, 2.5, 1.5\} = 1.5$$

The element corresponding to the min. ratio *i.e.*, here s_3 will be the departing variable and the corresponding row is called 'key row' and indicate this row by an outward arrow symbol. The intersection element of the key row and key column is called key element. In the present example, 4 is the key element which is highlighted. This is the end of this iteration, the final table is displayed as follow:

Iteration 1.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	18	1	2	3	1	0	0	$\frac{18}{2} = 9$
0	s_2	15	2	6	4	0	1	0	$\frac{15}{6} = 2.5$ →
0	s_3	6	1	$\boxed{4}$	1	0	0	1	$\frac{6}{4} = 1.5$
	$z_j - c_j$		-4	-8	-5	0	0	0	

↑

Step 5. For the construction of the next iteration (new) table the following calculations are to be made :

(a) Update the x_B column and the c_B column.

(b) Divide the key row by the key element.

(c) Other elements are obtained by the following formula :

NOTES

$$\begin{pmatrix}\text{new}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{old}\\ \text{element}\end{pmatrix} - \frac{\begin{pmatrix}\text{element}\\ \text{corresponding to}\\ \text{key row}\end{pmatrix} \cdot \begin{pmatrix}\text{element}\\ \text{corresponding to}\\ \text{key column}\end{pmatrix}}{\text{key element}}$$

(d) Then go to step 4.

Iteration 2.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	15	$\frac{1}{2}$	0	$\frac{5}{2}$	1	0	$-\frac{1}{2}$	$15\times\frac{3}{5}=6$
0	s_2	6	$\frac{1}{2}$	0	$\boxed{\frac{5}{2}}$	0	1	$-\frac{3}{2}$	$6\times\frac{2}{5}=2.4$ →
8	x_2	$\frac{3}{2}$	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{3}{4}\times 4=3$
	$z_j - c_j$		−2	0	−3	0	0	2	
					↑				

Iteration 3.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	9	0	0	0	1	−1	1	-
5	x_3	$\frac{12}{5}$	$\frac{1}{5}$	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{12}{5}\times\frac{5}{1}=12$
8	x_2	$\frac{9}{10}$	$\boxed{\frac{1}{5}}$	1	0	0	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{9}{10}\times\frac{5}{1}=4.5$ →
	$z_j - c_j$		$-\frac{7}{5}$	0	0	0	$\frac{6}{5}$	$\frac{1}{5}$	
			↑						

Iteration 4.

		c_j	4	8	5	0	0	0	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	s_3	ratio
0	s_1	9	0	0	0	1	−1	1	

5	x_3	$\frac{3}{2}$	0	− 1	1	0	$\frac{1}{2}$	− 1	
4	x_1	$\frac{9}{2}$	1	5	0	0	$-\frac{1}{2}$	2	
	$z_j - c_j$		0	7	0	0	$\frac{1}{2}$	3	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \qquad x_1^* = \frac{9}{2},\ x_2^* = 0,\ x_3^* = \frac{3}{2},\ z^* = \frac{51}{2}.$$

Note (exceptional cases).

(*a*) If in the key column, all the elements are non-positive *i.e.*, zero or negative, then min. ratio cannot be calculated and the problem is said to be unbounded.

(*b*) In the net evaluation of the optimal table all the basic variables will give the value zero. If any non-basic variable give zero net evaluation then it indicates that there is an alternative optimal solution. To obtain that solution, consider the corresponding column as key column and apply one simplex iteration.

(*c*) For negative variables, $x \leq 0$, set $x = -x'$, $x' \geq 0$.

For unrestricted variables set $x = x' - x''$ where $x', x'' \geq 0$.

Example 6. *Solve the following by simplex method :*

$$\textit{Maximize } z = x_1 + 3x_2$$

$$\textit{Subject to, } -x_1 + 2x_2 \leq 2,\ x_1 - 2x_2 \leq 2,\ x_1, x_2 \geq 0.$$

Solution. Standard form of the given LPP can be written as follows :

$$\text{Maximum } z = x_1 + 3x_2 + 0.s_1 + 0.s_2$$

$$\text{Subject to, } -x_1 + 2x_2 + s_1 = 2,\ x_1 - 2x_2 + s_2 = 2,$$

$$x_1, x_2 \geq 0,\ s_1, s_2 \text{ slacks} \geq 0.$$

Iteration 1.

		c_j	1	3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_1	2	− 1	$\boxed{2}$	1	0	$\frac{2}{2} = 1$ →
0	s_2	2	1	− 2	0	1	–
	$z_j - c_j$		− 1	− 3 ↑	0	0	

Iteration 2.

NOTES

		c_j	1	3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
3	x_2	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	
0	s_2	4	0	0	1	1	
	$z_j - c_j$		$-\frac{5}{2}$	0	$\frac{3}{2}$	0	
			↑				

Since all the elements in the key column are non-positive, we cannot calculate min. ratio. Hence the given LPP is said to be unbounded.

PROBLEMS

Solve the following LPP by simplex method:

1. Maximize $z = 3x_1 + 2x_2$
   S/t, $5x_1 + x_2 \le 10$, $4x_1 + 5x_2 \le 60$; $x_1, x_2 \ge 0$
2. Maximize $z = 5x_1 + 4x_2 + x_3$
   S/t, $6x_1 + x_2 + 2x_3 \le 12$, $8x_1 + 2x_2 + x_3 \le 30$,
   $4x_1 + x_2 - 2x_3 \le 16$, $x_1, x_2, x_3 \ge 0$
3. Maximize $z = 3x_1 + 2x_2$
   S/t, $3x_1 + 4x_2 \le 12$, $2x_1 + 5x_2 \le 10$, $x_1, x_2 \ge 0$
4. Maximize $z = 3x_1 + 2x_2 + x_3$
   S/t, $3x_1 + x_2 + 2x_3 \le 20$, $x_1 + 3x_2 + 4x_3 \le 16$, $x_1, x_2, x_3 \ge 0$
5. Maximize $z = 4x_1 - 2x_2 - x_3$
   S/t, $x_1 + x_2 + x_3 \le 3$, $2x_1 + 2x_2 + x_3 \le 4$, $x_1 - x_2 \le 0$, x_1, $x_2, x_3 \ge 0$
6. Maximize $z = 5x_1 + 3x_2 + 3x_3$
   S/t, $4x_1 + 4x_2 + 3x_3 \le 12000$, $0.4x_1 + 0.5x_2 + 0.3x_3 \le 1800$,
   $0.2x_1 + 0.2x_2 + 0.1x_3 \le 960$, $x_1, x_2, x_3 \ge 0$
7. Maximize $z = 3x_1 + 2x_2 + 2x_3$
   S/t, $2x_1 - x_2 + 3x_3 \le 18$, $x_1 + x_2 + 2x_3 \le 12$, $x_1, x_2, x_3 \ge 0$
8. Maximize $z = 3x_1 + x_2 + x_3 + x_4$
   S/t, $-2x_1 + 2x_2 + x_3 = 4$, $3x_1 + x_2 + x_4 = 6$, $x_i \ge 0$ for all i
9. Maximize $z = x_1 + x_2$
   S/t, $x_1 - 2x_2 \le 2$, $-x_1 + 2x_2 \le 2$, $x_1, x_2 \ge 0$
10. Find all the optimal BFS to the following :
    Maximize $z = x_1 + x_2 + x_3 + x_4$
    S/t, $x_1 + x_2 \le 2$, $x_3 + x_4 \le 5$, $x_1, x_2, x_3, x_4 \ge 0$

ANSWERS

1. $x_1 = 0, x_2 = 10, z^* = 20$ (It 3)
2. $x_1 = 0, x_2 = 12, x_3 = 0, z^* = 48$ (It 3)
3. $x_1 = 4, x_2 = 0, z^* = 12$ (It 2)
4. $x_1 = \frac{11}{2}, x_2 = \frac{7}{2}, x_3 = 0, z^* = \frac{47}{2}$ (It 3)
5. $x_1 = 1, x_2 = 1, x_3 = 0, z^* = 2$ (It 3)
6. $x_1 = 3000, x_2 = 0, x_3 = 0, z^* = 15000$ (It 2)
7. $x_1 = 10, x_2 = 2, x_3 = 0, z^* = 34$ (It 3)
8. Solution 1 : $x_1 = 1, x_2 = 3, x_3 = x_4 = 0$ (It 2)
   Solution 2 : $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 4, z^* = 6$
9. Unbounded solution (It 2)
10. (2, 0, 5, 0), (0, 2, 5, 0), (0, 2, 0, 5), (2, 0, 0, 5).

NOTES

2.5 BIG M METHOD

The method is also known as 'penalty method' due to Charnes. If there is '≥' type constraint, we add surplus variable and if there is '=' type, then the constraint is in equilibrium. Generally, in these cases there may not be any unit matrix in the standard form of the coefficient matrix.

To bring unit matrix we take help of another type of variable, known as 'artificial variable'. The addition of artificial variable creates infeasibility in the system which was already in equilibrium. To overcome this, we give a very large number denoted as M to the coefficient of the artificial variable in the objective function. For maximization problem, we add "– M. (artificial variable)" in the objective function so that the profit comes down. For minimization problem we add "M.(artificial variable)" in the objective function so that the cost goes up. Therefore the simplex method tries to reduce the artificial variable to the zero level so that the feasibility is restored and the objective function is optimized.

The only **drawback** of the big M method is that the value of M is not known but it is a very large number. Therefore, we cannot develop computer program for this method.

Note. (*a*) Once the artificial variable departs from the basis, it will never again enter in the subsequent iterations due to big M. Due to this we drop the artificial variable column in the subsequent iterations once the variable departs from the basis.

(*b*) If in the optimal table, the artificial variable remains with non-zero value, then the problem is said to be '*infeasible*'.

If the artificial variable remains in the optimal table with zero value, then the solution is said to be 'pseudo optimal'.

(*c*) The rule for 'multiple solution' and 'unbounded solution' are same as given by simplex method. The big-M method is a simple variation of simplex method.

Example 7. *Using Big-M method solve the following LPP :*

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$\text{S/t. } x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4;\ x_1, x_2 \geq 0$$

NOTES

Solution. Standard form of the given LPP is

$$\text{Min. } z = -\text{ Max. } (-2 \quad = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t, } x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1, x_2 \geq 0,\ s_1, s_2 \text{ surplus} \geq 0,\ a_1, a_2 \text{ artificial} \geq 0$$

Iteration 1.

		c_j	-10	-3	0	0	$-M$	$-M$	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
$-M$	a_1	3	1	2	-1	0	1	0	$\frac{3}{2} = 1.5$
$-M$	a_2	4	1	$\boxed{4}$	0	-1	0	1	$\frac{4}{4} = 1$ →
	$z_j - c_j$	8	$-2M + 10$	$-6M + 3$	M	M	0	0	
				↑					

Iteration 2.

		c_j	-10	-3	0	0	$-M$	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	ratio
$-M$	a_1	1	$\boxed{\frac{1}{2}}$	0	-1	$\frac{1}{2}$	1	$\frac{1}{1/2} = 2$ →
-3	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{1/4} = 4$
	$z_j - c_j$	-	$-\frac{M}{2} + \frac{37}{4}$	0	M	$-\frac{M}{2} + \frac{3}{4}$	0	
			↑					

Iteration 3.

		c_j	-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
-10	x_1	2	1	0	-2	$\boxed{1}$	$\frac{2}{1} = 2$ →
-3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	–
	$z_j - c_j$		0	0	$\frac{37}{2}$	$-\frac{17}{2}$	
						↑	

NOTES

Iteration 4. (Optimal)

	c_j		-10	-3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_2	2	1	0	-2	1	
-3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
	$z_j - c_j$		$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \quad x_1^* = 0,\ x_2^* = \frac{3}{2},\ z^* = \frac{9}{2}.$$

Example 8. *Solve the following LPP by Big-M method :*

$$\text{Minimize } z = 2x_1 + x_2 + 3x_3$$

$$S/t,\ -3x_1 + x_2 - 2x_3 \geq 1,\ x_1 - 2x_2 + x_3 \geq 2;\ x_1, x_2, x_3 \geq 0.$$

Solution. The standard form of the given problem can be written as follows :

$$\text{Min. } z = -\text{Max. } (-z = -2x_1 - x_2 - 3x_3 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$S/t,\ -3x_1 + x_2 - 2x_3 - s_1 + a_1 = 1,$$

$$x_1 - 2x_2 + x_3 - s_2 + a_2 = 2,$$

$x_1, x_2, x_3 \geq 0$, s_1, s_2 surplus $s \geq 0$, a_1, a_2 artificial variables ≥ 0.

Iteration 1.

		c_j	-2	-1	-3	0	0	$-M$	$-M$	Min.
c_B	x_B	soln.	x_1	x_2	x_3	s_1	s_2	a_1	a_2	ratio
$-M$	a_1	1	-3	1	-2	-1	0	1	0	
$-M$	a_2	2	1	-2	1	0	-1	0	1	
	$z_j - c_j$		$2M + 2$	$M + 1$	$M + 3$	M	M	0	0	

Since all $z_j - c_j \geq 0$, the first iteration itself give optimal solution. But in solution *i.e.*, $a_1 = 1$, $a_2 = 2$ present with non-zero value. Hence the given problem does not possess any feasible solution.

2.6 TWO PHASE METHOD

To overcome the drawback of Big-M method, two phase method has been framed. In the first phase an auxiliary LP Problem is formulated as follows :

Minimize T = Sum of artificial variables

S/t, original constraints

which is solved by simplex method. Here artificial variables act as decision variables. So Big-M is not required in the objective function. If $T^* = 0$, then go to phase two calculations, else ($T^* \neq 0$) write the problem is infeasible. In phase two, the optimal table of phase one is considered with the following modifications :

NOTES

Delete the artificial variable's columns and incorporate the original objective function and also update the c_B values. Calculate $z_j - c_j$ values. If all $z_j - c_j \geq 0$, the current solution is optimal else go to the next iteration.

Note. (*a*) Multiple solutions, if it exists, can be detected from the optimal table of phase two.

(*b*) In phase I, the problem is always minimization type irrespective of the type of the original given objective function.

Example 9. *Using two phase method solve the following LPP :*

$$\text{Minimize } z = 10x_1 + 3x_2$$

$$\text{S/t, } x_1 + 2x_2 \geq 3,\ x_1 + 4x_2 \geq 4;\ x_1, x_2 \geq 0$$

Solution. Standard form of the given LPP is

$$\text{Min.}z = -\text{ Max. } (-z = -10x_1 - 3x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$

$$\text{S/t, } x_1 + 2x_2 - s_1 + a_1 = 3,$$

$$x_1 + 4x_2 - s_2 + a_2 = 4,$$

$$x_1, x_2 \geq s_1, s_2, \text{ surplus } > a_1, a_2 \text{ artificial} \geq 0$$

Phase I Min $T = a_1 + a_2 = -$ Max

$$(-T = 0.\ x_1 + 0.x_2 + 0.s_1 + 0.s_2 - a_1 - a_2)$$

$$\text{S/t., } x_1 + 2x_2 - s_1 + a_1 = 3;\ x_1 + 4x_2 - s_2 + a_2 = 4,$$

$$x_1, x_2, s_1, s_2, a_1, a_2 > 0$$

Iteration 1.

		c_j	0	0	0	0	−1	−1	Min.	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio	
−1	a_1	3	1	2	−1	0	1	0	$\frac{3}{2} = 1.5$	
−1	a_2	4	1	$\boxed{4}$	0	−1	0	1	$\frac{4}{4} = 1$	→
	$z_j - c_j$		−2	−6	1	1	0	0		
				↑						

Iteration 2.

		c_j	0	0	0	0	−1	−1	Min.	
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio	
−1	a_1	1	$\boxed{\frac{1}{2}}$	0	−1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{1/2} = 2$	→

0	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{1/4}=4$
$z_j - c_j$			$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	
			↑						

Iteration 3.

		c_j	0	0	0	0	−1	−1	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	a_1	a_2	ratio
0	x_1	2	1	0	−2	1	2	−1	
0	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
$z_j - c_j$			0	0	0	0	1	1	

Since all $z_j - c_j \geq 0$, the solution is optimal $a_1^* = 0$, $a_2^* = 0$ and $T^* = 0$. Therefore we go to phase II calculations.

Phase II

Iteration 1.

		c_j	−10	−3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
−10	x_1	2	1	0	−2	$\boxed{1}$	$\frac{2}{1}=2$ →
−3	x_2	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
$z_j - c_j$			0	0	$\frac{37}{2}$	$-\frac{17}{2}$	

Iteration 2.

		c_j	−10	−3	0	0	Min.
c_B	x_B	soln.	x_1	x_2	s_1	s_2	ratio
0	s_2	2	1	0	−2	1	
−3	x_2	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	
$z_j - c_j$			$\frac{17}{2}$	0	$\frac{3}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal.

$$\therefore \qquad x_1^* = 0, x_2^* = \frac{3}{2}, \; z^* = \frac{9}{2}.$$

Example 10. *Solve the following LPP by two phase method.*

NOTES

$$\textit{Maximize } z = 2x_1 + x_2 - 3x_3$$
$$S/t.\ x_1 + 2x_2 + 2x_3 \geq 12,\ 3x_1 - 2x_2 + 4x_3 \leq 10$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0.$$

Solution. Set $x_2 = -x'_2, x'_2 \geq 0$.

The standard form of the given LPP is

$$\text{Maximize } z = 2x_1 - x'_2 - 3x_3 + 0.s_1 + 0.s_2 - Ma_1$$
$$S/t,\ x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12,$$
$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10,$$
$$x_1, x'_2\ x_3 \geq 0,\ s_1 \text{ (surplus)} \geq 0, s_2 \text{ (slack)} \geq 0 \text{ and } a_1 \text{ (artificial)} \geq 0.$$

Phase I.

$$\text{Minimize T} = a_1 = -\text{Max. } (-\text{T} = 0.x_1 + 0.x'_2 + 0.x_3 + 0.s_1 + 0.s_2 - a_1)$$
$$\text{S/t, } x_1 - 2x'_2 + 2x_3 - s_1 + a_1 = 12$$
$$3x_1 + 2x'_2 + 4x_3 + s_2 = 10$$
$$x_1, x'_2, x_3, s_1, s_2, a_1 \geq 0.$$

Iteration 1.

		c_j	0	0	0	0	0	− 1	Min.
c_B	x_B	soln.	x_1	x'_2	x_3	s_1	s_2	a_1	ratio
− 1	a_1	12	1	− 2	2	− 1	0	1	$\frac{12}{2} = 6$
0	s_1	10	3	2	$\boxed{4}$	0	1	0	$\frac{10}{4} = 2.5$ →
$z_j - c_j$			− 1	2	− 2	1	0	0	
					↑				

Iteration 2.

		c_j	0	0	0	0	0	− 1	Min.
c_B	x_B	soln.	x_1	x'_2	x_3	s_1	s_2	a_1	ratio
− 1	a_1	7	$-\frac{1}{2}$	− 3	0	− 1	$-\frac{1}{2}$	1	
0	x_3	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	1	0	$\frac{1}{4}$	0	
$z_j - c_j$			$\frac{1}{2}$	3	0	1	$\frac{1}{2}$	0	

Since all $z_j - c_j \geq 0$, the current solution is optimal and $T^* = 7 \neq 0$.

This implies that there does not exist any feasible solution to the given LPP.

Note. In simplex, Big-M and two phase methods, if there is a tie in min. ratio or in negative most value of net evaluation, the optimal feasible solution will lead to degenerate solution.

NOTES

PROBLEMS

Solve the following LPP using *Big-M* method and *Two phase* method :

1. Minimize $z = 2x_1 + 3x_2$

   S/t, $2x_1 + x_2 \geq 1$, $x_1 + 2x_2 \geq 1$; $x_1, x_2 \geq 0$.

2. Maximize $z = 5x_1 + 3x_2$

   S/t, $2x_1 - 4x_2 \leq 16$, $3x_1 + 4x_2 \geq 12$; $x_1, x_2 \geq 0$.

3. Maximize $z = 2x_1 + 3x_2 + 2x_3$

   S/t, $3x_1 + 2x_2 + 2x_3 = 16$, $2x_1 + 4x_2 + x_3 = 20$,

   $x_1 \geq 0$, x_2 unrestricted in sign, $x_3 \geq 0$.

4. Maximize $z = 2x_1 + 3x_2 + x_3$

   S/t, $3x_1 + 2x_2 + x_3 = 15$, $x_1 + 4x_2 = 10$,

   x_1 unrestricted in sign, $x_2, x_3 \geq 0$.

5. Maximize $z = 2x_1 + 2x_2 + 3x_3$

   S/t, $x_1 - 2x_2 + x_3 \leq 8$, $3x_1 + 4x_2 + 2x_3 \geq 2$,

   $x_1 \geq 0$, $x_2 \leq 0$, $x_3 \geq 0$.

6. Maximize $z = 3x_1 + 2x_2 + x_3 - x_4$

   S/t, $x_1 + 2x_2 + 3x_3 = 16$, $3x_1 + x_2 + 2x_3 = 20$,

   $2x_1 + x_2 + x_3 + x_4 = 12$; $x_1, x_2, x_3, x_4 \geq 0$.

7. Minimize $z = x_1 + 2x_2$

   S/t, $2x_1 + x_2 = 4$, $3x_1 + 4x_2 \geq 5$, $x_1 + x_2 \leq 4$,

   $x_1, x_2 \geq 0$.

8. Minimize $z = x_1 + 3x_2 + 5x_3$

   S/t, $2x_1 + 5x_2 + x_3 \geq 12$, $x_1 + 2x_2 + 3x_3 \geq 10$,

   $x_1, x_2, x_3 \geq 0$.

9. Minimize $z = 5x_1 + x_2$

   S/t, $2x_1 + x_2 \leq 2$, $3x_1 + 4x_2 \geq 12$; $x_1, x_2 \geq 0$.

10. Maximize $z = x_1 - 3x_2$

    S/t, $-x_1 + 2x_2 \leq 15$, $x_1 + 3x_2 = 10$; $x_1 \geq 0$, $x_2 \leq 0$.

11. Find a BFS of the following system :

    $x_1 + x_2 \geq 1$, $-2x_1 + x_2 \geq 2$, $2x_1 + 3x_2 \leq 6$; $x_1, x_2 \geq 0$.

ANSWERS

1. $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $z^* = \frac{5}{3}$ (Big-M 3 It)
2. Unbounded solution. (Big-M 4 It)

NOTES

3. $x_1 = 0, x_2 = 4, x_3 = 4, z^* = 20$ (Big-M 4 It)
4. Unbounded solution (Big-M 2 It)
5. $x_1 = 0, x_2 = 0, x_3 = 8, z^* = 24$. (Big-M 5 It)
6. $x_1 = 4, x_2 = 0, x_3 = 4, x_4 = 0, z^* = 16$. (Big-M 4 It)
7. $x_1 = 2, x_2 = 0, z^* = 2$. (Big-M 3 It)
8. $x_1 = 10, x_2 = 0, x_3 = 0, z^* = 10$. (Big-M 5 It)
9. Infeasible solution.
10. Unbounded solution.
11. $x_1 = 0, x_2 = 2$ (use Phase-I) (3 It).

2.7 FORMULATION PROBLEMS

Example 11. *A manufacturer produces two types of machines M_1 and M_2. Each M_1 requires 4 hrs. of grinding and 2 hrs. of polishing whereas each M_2 model requires 2 hrs. of grinding and 4 hrs. of polishing. Manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hrs a week and each polisher works 40 hrs a week. Profit on an M_1 model is ₹ 3 and on an M_2 model is ₹ 4. Whatever is produced is sold in the market. How should the manufacturer allocate his production capacity to two types of models so that he may make the maximum profit in a week. Formulate the LPP and solve graphically.*

Solution. Let x_1 = Number of M_1 machines, and

x_2 = Number of M_2 machines to be produced in a week.

The above data is summarized as follows :

	M_1	M_2
Grinding	4 hrs.	2 hrs.
Polishing	2 hrs.	4 hrs.
Profit	₹ 3	₹ 4

	Time available per week
2 Grinding	80 hrs.
3 Polishing	120 hrs.

Therefore the LPP can be formulated as follows :

$$\text{Maximize profit} = 3x_1 + 4x_2$$
$$S/t,\ 4x_1 + 2x_2 \le 80 \quad \text{(grinding)}$$
$$2x_1 + 4x_2 \le 120 \quad \text{(polishing)}$$
$$x_1, x_2 \ge 0$$

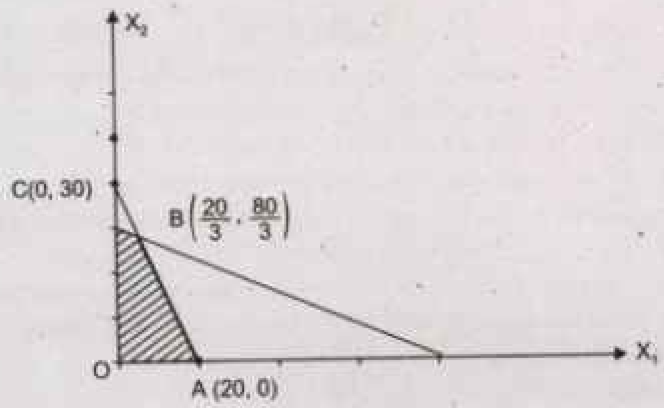
The graphical region is shown below.

Profit at A = 60

Profit at B = 126.67

Profit at C = 120

NOTES



$\therefore$ The optimal solution is $x_1 = \frac{20}{3}$, $x_2 = \frac{80}{3}$, and max. profit = ₹ 126.67.

Example 12. *A firm can produce three types of woolen clothes, say, A, B and C using three kinds of wool, say red wool, green wool and blue wool. One unit of length of type A cloth needs 2 yards of red wool and 3 yards of blue wool; one unit length of type B cloth needs 3 yards of red wool, 2 yards of green wool and 2 yards of blue wool; and one unit length of type C cloth needs 5 yards of green wool and 4 yards of blue wool. The firm has only a stock of 8 yards of red wool, 10 yards of green wool and 15 yards of blue wool. It is assumed that income obtained from one unit length of type A cloth is ₹ 3, of type B cloth is ₹ 5 and that of type C cloth is ₹ 4. Formulate the above problem as a LP problem.*

Solution. The given data is summarized below :

Wool	*Garment type*			*Stock*
	A	*B*	*C*	*available*
Red	2	3	-	8
Green	-	2	5	10
Blue	3	2	4	15
Income (₹)	3	5	4	.

Suppose that he produces x_1, x_2 and x_3 unit lengths of A, B and C clothes respectively. Then the LPP is :

$$\text{Maximize income} = 3x_1 + 5x_2 + 4x_3$$

$$\text{S/t, } 2x_1 + 3x_2 \le 8 \text{ (Red wool)}$$

$$2x_2 + 5x_3 \le 10 \qquad \text{(Green wool)}$$

$$3x_1 + 2x_2 + 5x_3 \le 15 \qquad \text{(Blue wool)}$$

$$x_1, x_2, x_3 \ge 0$$

The solution is obtained as $x_1^* = 1.67$, $x_2^* = 1.56$, $x_3^* = 1.38$, $z^* = 18.29$ (It 4, Simplex).

NOTES

PROBLEMS

1. A manufacturer of furniture makes only chair and tables. A chair requires two hours on m/c A and six hours on m/c B. A table requires five hours on m/c A and two hours on m/c B. 16 hours are available on m/c A and 22 hours on m/c B per day. Profits for a chair and table be ₹ 1 and ₹ 5 respectively. Formulate the LPP of finding daily production of these items for maximum profit and solve graphically.
2. A tailor has 80 sq. m of cotton material and 120 sq. m of woolen material. A suit requires 1 sq. m of cotton and 3 sq. m of woolen material and a dress requires 2 sq. m of each. A suit sells for ₹ 500 and a dress for ₹ 400. Pose a LPP in terms of maximizing the income.
3. A company owns two mines : mine A produces 1 tonne of high grade ore, 3 tonnes of medium grade ore and 5 tonnes of low grade ore each day; and mine B produces 2 tonnes of each of the three grades of ore each day. The company needs 80 tonnes of high grade ore, 160 tonnes of medium grade ore and 200 tonnes of low grade ore. If it costs ₹ 200 per day to work each mine, find the number of days each mine has to be operated for producing the required output with minimum total cost.
4. A company manufactures two products A and B. The profit per unit sale of A and B is ₹ 10 and ₹ 15 respectively. The company can manufacture at most 40 units of A and 20 units of B in a month. The total sale must not be below ₹ 400 per month. If the market demand of the two items be 40 units in all, write the problem of finding the optimum number of items to be manufactured for maximum profit, as a problem of LP. Solve the problem graphically or otherwise.
5. A company is considering two types of buses—ordinary and semideluxe for transportation. Ordinary bus can carry 40 passengers and requires 2 mechanics for servicing. Semideluxe bus can carry 60 passengers and requires 3 mechanics for servicing. The company can transport at least 300 persons daily and not more than 12 mechanics can be employed. The cost of purchasing buses is to be minimized, given that the ordinary bus costs ₹ 1,20,000 and semideluxe bus costs ₹ 1,50,000. Formulate this problem as a LPP.
6. A pharmaceutical company has 100 kg. of ingredient A, 180 kg. of ingredient B and 120 kg. of ingredient C available per month. They can use these ingredients to make three basic pharmaceutical products namely 5-10-5; 5-5-10 and 20-5-10; where the numbers in each case represent the percentage by weight of A, B and C respectively in each of the products. The cost of these ingredients are given below :

Ingredient	*Cost per kg. (₹)*
A	80
B	20
C	50
Inert ingredients	20

Selling price of these products are ₹ 40.5, ₹ 43 and ₹ 45 per kg. respectively. There is a capacity restriction of the company for the product 5-10-5, so they cannot produce more than 30 kg per month. Determine how much of each of the products they should produce in order to maximize their monthly profit.

7. A fruit squash manufacturing company manufactures three types of squashes. The basic formula are :

5 litre lemonade : 2 oz. lemons, 2 kg of sugar, 2 oz. citric acid and water.

5 litre grape fruits : 11/2 kg of grape fruit, 11/2 kg of sugar, 11/2 oz. citric acid and water.

5 litre orangeade : 11/2 dozen oranges, 11/2 kg of sugar, 1 oz. citric acid and water.

The squashes sell at

Lemonade : ₹ 37.50 per 5 litre;

Grape fruit : ₹ 40.00 per 5 litre;

Orangeade : ₹ 42.50 per 5 litre.

In the last week of the season they have in stock 2500 dozen lemons, 2000 kg grape fruit, 750 dozen oranges, 5000 kg of sugar and 3000 ozs. citric acid. What should be their manufacturing quantities in the week to maximize the turnover ?

NOTES

8. A farmer is raising cows in his farm. He wishes to determine the qualities of the available types of feed that should be given to each cow to meet certain nutritional requirements at a minimum cost. The numbers of each type of basic nutritional ingredient contained within a kg of each feed type is given in the following table, along with the daily nutritional requirements and feed costs.

Nutritional ingredient	*kg of corn*	*kg of tankage*	*kg of green grass*	*Min. daily requirement*
Carbohydrates	9	2	4	20
Proteins	3	8	6	18
Vitamins	1	2	6	15
Cost	7	6	5	

Formulate a linear programming model for this problem so as to determine the optimal mix of feeds.

ANSWERS

1. No chairs and 3.2 tables to be produced for max. profit of ₹ 16.
2. Max. sells = $500x_1 + 400x_2$
   S/t. $x_1 + 2x_2 \le 80$, $3x_1 + 2x_2 \le 120$, x_1 = no. of suits ≥ 0 and x_2 = no. of dresses ≥ 0.
3. Mine A to be operated for 40 days and mine B to be operated for 20 days and min. cost = ₹ 12000.
4. Max. profit = $10x_1 + 15x_2$
   S/t. $x_1 \le 40$, $x_2 \le 20$, $x_1 + x_2 \ge 40$, $10x_1 + 15x_2 \ge 400$, $x_1, x_2 \ge 0$ and $x_1^* = 40$, $x_2^* = 20$, max. profit = ₹ 700.
5. Min. cost = $1{,}20{,}000x_1 + 1{,}50{,}000x_2$
   S/t. $40x_1 + 60x_2 \ge 300$, $2x_1 + 3x_2 \le 12$, $x_1, x_2 \ge 0$.
6. Let x_1, x_2, x_3 be three products in kg to be manufactured.
   Max. profit = $16x_1 + 17x_2 + 10x_3$
   S/t. $x_1 + x_2 + 4x_3 \le 2000$, $2x_1 + x_2 + x_3 \le 3600$,
   $x_1 + 2x_2 + 2x_3 \ge 2400$, $x_1 \le 30$, $x_1 + x_2 + x_3 \ge 0$
   Solution. $x_1 = 30$, $x_2 = 1185$, $x_3 = 0$, Profit = ₹ 20,625. (Simplex 3It)
7. Let $5x_1$, $5x_2$, $5x_3$ litre be the lemonade, grape fruit and orangeade to be manufactured per week.

Max. profit = $37.5x_1 + 40x_2 + 42.5x_3$
S/t, $2x_1 \le 2500$, $3x_2 \le 4000$, $3x_3 \le 1500$,
$4x_1 + 3x_2 + 3x_3 \le 10{,}000$, $4x_1 + 3x_2 + 2x_3 \le 6000$, $x_1, x_2, x_3 \ge 0$.

8. Min. cost = $7x_1 + 6x_2 + 5x_3$
S/t, $9x_1 + 2x_2 + 4x_3 \geq 20$,
$3x_1 + 8x_2 + 6x_3 \geq 18$
$x_1 + 2x_2 + 6x_3 \geq 15$
$x_1, x_2, x_3 \geq 0$.

NOTES

2.8 REVISED SIMPLEX METHOD (RSM)

I. Algorithm

Step 1. Write the standard form of the given LPP and convert it into maximization type if it is in minimization type *i.e.*,

$$\text{Max. } z = cx$$
$$\text{S/t, } Ax = b,\ x \ge 0.$$

Use the following notations :

$c^T = [c_1, c_2, \ldots, c_n]$ Profit coefficients.

Columns of A as $A_1, A_2, \ldots, A_m$.

$\pi = (\pi_1, \pi_2, \ldots)$ Simplex multipliers
x_B = Basis vector
c_B^T = Profit coefficient in the basis
B = Basis matrix, B^{-1} = Basis inverse
$\bar{c}_j$ = Net evaluations,
j = Index of non-basic variables
$\bar{b}$ = Current BFS

Step 2. For iteration 1

$$B = I,\ B^{-1} = I$$

else for other iterations

Find $$B = [x_{B_i}] = [A_{x_{B_i}}] \text{ and hence find } B^{-1}.$$

Step 3. Calculate

$$\pi = c_B^T . B^{-1} \text{ and } \bar{b} = B^{-1}.b \text{ (current solution)}$$
$$\bar{c}_j = \pi A_j - c_j$$

Decisions : If all $\bar{c}_j \ge 0$ then the current BFS is optimal, else select the negative most of $\bar{c}_j$, say $\bar{c}_k$. Then x_k will be the '*Entering Variable*' and $\bar{A}_k$ = key column = $B^{-1}.A_k$.

Step 4. Produce the following revised simplex table :

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*

Encircle the key element obtained from the min. ratio $\{[\bar{b}]/[\text{Key column}]\}$.

Element corresponding to the key element will depart from $[x_B]$.

Step 5. Go to step 2.

Repeat the procedure until optimal BFS is obtained.

Note. (*a*) If, in step 4, all the elements in the key column are non-positive, then the given problem is unbounded.

(*b*) If, in the optimal BFS, artificial variables (if any) take zero value then the solution is degenerate else, for non-zero value, the given problem is said to be infeasible.

II. Advantages

In computational point of view, the Revised Simplex Method is superior than ordinary simplex method. Due to selected column calculations in revised simplex method, less memory is required in computer. Whereas the ordinary simplex method requires more memory space in computer.

Example 13. *Using revised simplex method solve the following LPP :*

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

$$\text{S/t. } x_1 + 2x_2 + 2x_3 \leq 8$$

$$3x_1 + 4x_2 + x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0.$$

Solution. Standard form of the given LPP is

$$\text{Max. } z = 5x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2$$

$$\text{S/t, } x_1 + 2x_2 + 2x_3 + s_1 = 8$$

$$3x_1 + 4x_2 + x_3 + s_2 = 7$$

$$x_1, x_2, x_3 \geq 0, \ s_1, s_2 \text{ are slacks and } \geq 0$$

Then, $$A_1 = \begin{bmatrix}1\\3\end{bmatrix}, A_2 = \begin{bmatrix}2\\4\end{bmatrix}, A_3 = \begin{bmatrix}2\\1\end{bmatrix}, A_4 = \begin{bmatrix}1\\0\end{bmatrix}, A_5 = \begin{bmatrix}0\\1\end{bmatrix}, b = \begin{bmatrix}8\\7\end{bmatrix}$$

Let us consider the index of the variables x_1 be 1, x_2 be 2, x_3 be 3, s_1 be 4, s_2 be 5.

Iteration 1.

$$x_B = (s_1, s_2), \ B = [A_4, A_5] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I, \ B^{-1} = I.$$

$$c_B^T = (0, 0), \ \bar{b} = B^{-1}.b = b, \ J = (1, 2, 3).$$

$$\pi = c_B^T.B^{-1} = (0, 0)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = (0, 0) = (\pi_1, \pi_2).$$

Net evaluations :

$\bar{c}_1 = \pi A_1 - c_1 = -5 \leftarrow$ negative most and entering variable is x_1

$\bar{c}_2 = \pi A_2 - c_2 = -2$

$\bar{c}_3 = \pi A_3 - c_3 = -3.$

NOTES

Key column : $B^{-1}.A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Table 1

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
s_1	1 0	8		1
s_2	0 1	7	x_1	3

This indicates the departing variable as s_2.

Iteration 2.

$$x_B = (s_1, x_1),\ B = [A_4, A_1] = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix},\ B^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\bar{b} = B^{-1}.b = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 17/3 \\ 7/3 \end{bmatrix},\ J = (2, 3, 5).$$

$$\pi = c_B^T.B^{-1} = [0, 5]\begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = \left(0, \frac{5}{3}\right).$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{14}{3}.$$

$$\bar{c}_3 = \pi A_3 - c_3 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3 = -\frac{4}{3} \leftarrow \text{Entering variable } x_3$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(0, \frac{5}{3}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{5}{3}$$

Key column : $B^{-1}.A_3 = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix}$.

Table 2

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
s_1	1 −1/3	17/3	x_3	5/3
x_2	0 1/3	7/3		1/3

(This indicates the departing variable as s_1).

Iteration 3.

$$x_B = (x_3, x_1),\ B = [A_3, A_1] = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$$

$J = 2, 4, 5$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 17/5 \\ 6/5 \end{pmatrix}$$

$$\pi = c_B^T.B^{-1} = (3, 5)\begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} = \left(\frac{4}{5}, \frac{7}{5}\right).$$

Net evaluations :

$$\bar{c}_2 = \pi A_2 - c_2 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 2 \\ 4 \end{pmatrix} - 2 = \frac{26}{5}.$$

$$\bar{c}_4 = \pi A_4 - c_4 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = \frac{4}{5}$$

$$\bar{c}_5 = \pi A_5 - c_5 = \left(\frac{4}{5}, \frac{7}{5}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = \frac{7}{5}$$

As all $\bar{c}_j > 0 \Rightarrow$ the current $\bar{b}$ is optimal.

$\therefore\ x_1^* = \frac{6}{5},\ x_2^* = 0,\ x_3^* = \frac{17}{5}$ and $z^* = \frac{81}{5}.$

Example 14. *Solve by revised simplex method.*

$$\textit{Minimize } z = 12x_1 + 20x_2$$
$$\textit{S/t, } 6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0.$$

Solution. Standard form :

$$\text{Min. } z = -\text{Max. } (-z = -12x_1 - 20x_2 + 0.s_1 + 0.s_2 - Ma_1 - Ma_2)$$
$$\text{S/t, } 6x_1 + 8x_2 - s_1 + a_1 = 100$$
$$7x_1 + 12x_2 - s_2 + a_2 = 120$$

$x_1, x_2 \geq 0$, s_1, s_2 surplus and ≥ 0, a_1, a_2 artificial and ≥ 0.

Let $A_1 = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$, $A_2 = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $A_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_6 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$

Let the index of the variables $x_1, x_2, s_1, s_2, a_1, a_2$ be 1, 2, 3, 4, 5 and 6 respectively.

Iteration 1.

$$x_B = (a_1, a_2),\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1},\ \bar{b} = b,\ c_B^T = [-M, -M]$$

$$\pi = c_B^T B^{-1} = [-M, -M],\ J = (1, 2, 3, 4)$$

Net evaluations :

$\bar{c}_1 = \pi A_1 - c_1 = -13\text{ M} + 12$

$\bar{c}_2 = \pi A_2 - c_2 = -20\text{ M} + 20 \leftarrow$ Most negative and x_2 as entering variable

$$\bar{c}_3 = \pi A_3 - c_3 = M$$
$$\bar{c}_4 = \pi A_4 - c_4 = M$$

$$\text{Key column} = B^{-1}.A_2 = \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

NOTES

Table 1

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
a_1	1 0	100	x_2	8
a_2	0 1	120		12

(This table indicates a_2 as departing variable).

Iteration 2.

$$x_B = (a_1, x_2),\ B = [A_3, A_2] = \begin{bmatrix} 1 & 8 \\ 0 & 12 \end{bmatrix},\ B^{-1} = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix}$$

$$J = (1, 3, 4, 6)$$

$$\pi = c_B^T.B^{-1} = (-M, -20)\begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix} = \left(-M, \frac{2}{3}M - \frac{5}{3}\right)$$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 1 & -2/3 \\ 0 & 1/12 \end{pmatrix}\begin{pmatrix} 100 \\ 120 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$$

Net evaluations :

$$\bar{c}_1 = \pi A_1 - c_1 = -\frac{4}{3}M + \frac{1}{3}$$
$$\bar{c}_3 = \pi A_3 - c_3 = M$$
$$\bar{c}_4 = \pi A_4 - c_4 = -\frac{2}{3}M + \frac{5}{3}$$
$$\bar{c}_6 = \pi A_6 - c_6 = \frac{5}{3}M - \frac{5}{3}$$

$$\text{Key column} = B^{-1}.A_1 = \begin{bmatrix} 4/3 \\ 7/12 \end{bmatrix}$$

Table 2

x_B	B^{-1}	$\bar{b}$	*Entering variable*	*Key column*
a_1	1 −2/3	20	x_1	4/3
x_2	0 1/12	10		7/12

(This table indicates a_1 as departing variable).

Iteration 3.

$$x_B = (x_1, x_2),\ B = (A_1, A_2) = \begin{pmatrix} 6 & 8 \\ 7 & 12 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix}$$

$$J = (3, 4, 5, 6)$$

$$\pi = c_B^T . B^{-1} = (-12, -20)\begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix} = \left(-\frac{1}{4}, -\frac{3}{2}\right)$$

$$\bar{b} = B^{-1}.b = \begin{pmatrix} 3/4 & -1/2 \\ -7/16 & 3/8 \end{pmatrix}\begin{pmatrix} 100 \\ 120 \end{pmatrix} = \begin{pmatrix} 15 \\ 5/4 \end{pmatrix}$$

Net evaluations :

$$\bar{c}_3 = \pi A_3 - c_3 = \frac{1}{4}$$

$$\bar{c}_4 = \pi A_4 - c_4 = \frac{3}{2}$$

$$\bar{c}_5 = \pi A_5 - c_5 = M - \frac{1}{4}$$

$$\bar{c}_6 = \pi A_6 - c_6 = M - \frac{3}{2}$$

Since all $\bar{c}_j > 0 \Rightarrow$ the current $\bar{b}$ is optimal.

$\therefore\ x_1^* = 15,\ x_2^* = \frac{5}{4}$ and $z^* = 205.$

PROBLEMS

Using revised simplex method solve the following LPP :

1. Maximize $z = x_1 + x_2 + 3x_3$
   S/t, $3x_1 + 2x_2 + x_3 \le 3,\ 2x_1 + x_2 + 2x_3 \le 2;\ x_1, x_2, x_3 \ge 0.$
2. Maximize $z = 3x_1 + 4x_2$
   S/t, $x_1 - x_2 \ge 0,\ -x_1 + 3x_2 \le 3;\ x_1, x_2 \ge 0.$
3. Minimize $z = x_1 + x_2$
   S/t, $2x_1 + x_2 \ge 4,\ x_1 + 7x_2 \ge 7;\ x_1, x_2 \ge 0.$
4. Minimize $z = 2x_1 - x_2 + 2x_3$
   S/t, $-x_1 + x_2 + x_3 = 4,\ -x_1 + x_2 - x_3 \le 6,$
   $x_1 \le 0,\ x_2 \ge 0,\ x_3$ unrestricted in sign.
5. Maximize $z = -x_1 + 2x_2 + 3x_3$
   S/t, $-2x_1 + x_2 + 3x_3 = 2,\ 2x_1 + 3x_2 + 4x_3 = 1;\ x_1, x_2, x_3 \ge 0.$
6. Maximize $z = 2x_1 + x_2 + 3x_3$
   S/t, $x_1 + x_2 + 2x_3 \le 5,\ 2x_1 + 3x_2 + 4x_3 = 12;\ x_1, x_2, x_3 \ge 0.$
7. Maximize $z = 5x_1 + 2x_2 + 3x_3$
   S/t, $x_1 + 2x_2 + 2x_3 \le 8,\ 3x_1 + 4x_2 + x_3 \le 7;\ x_1, x_2, x_3 \ge 0.$

ANSWERS

NOTES

1. $x_1 = 0, x_2 = 0, x_3 = 1, z^* = 3.$
2. Unbounded solution.
3. $x_1 = \frac{21}{13},\ x_2 = \frac{10}{13},\ z^* = \frac{31}{13}.$
4. $x_1 = -5, x_2 = 0, x_3 = -1, z^* = -12$ (Iteration 3)
5. Infeasible solution.
6. $x_1 = 3, x_2 = 2, x_3 = 0, z^* = 8.$
7. $x_1 = \frac{6}{5},\ x_2 = 0,\ x_3 = \frac{17}{5},\ z^* = \frac{81}{5}$ (Iteration 3)

2.9 INTRODUCTION AND FORMULATION

For every LP Problem we can construct another LP problem using the same data. These two problems try to achieve two different objectives within the same data. The original problem is called *Primal* problem and the constructed problem is called *Dual*. This is illustrated through the following example :

A company makes three products X, Y, Z using three raw materials A, B and C. The raw material requirement is given below : (for 1 unit of product).

	X	*Y*	*Z*	*Availability*
A	1	2	1	36 units
B	2	1	4	60 units
C	2	5	1	45 units
Profit	₹ 40	₹ 25	₹ 50	

Let the company decide to produce x_1, x_2 and x_3 units of the products X, Y and Z respectively in order to maximize the profit. We obtain the following LP problems :

$$\text{Maximize profit} = 40x_1 + 25x_2 + 50x_3$$

$$\begin{aligned}\text{Subject to, } x_1 + 2x_2 + x_3 &\le 36,\\ 2x_1 + x_2 + 4x_3 &\le 60,\\ 2x_1 + 5x_2 + x_3 &\le 45,\\ x_1, x_2, x_3 &\ge 0.\end{aligned}$$

Adding slack variables s_1, s_2 and s_3 to the constraints, we solve the problem by simplex method. The optimal solution is

$$x_1 = 20,\ x_2 = 0,\ \ x_3 = 5 \text{ and optimal profit} = ₹\ 1050.$$

Suppose the company wishes to sell the three raw materials A, B and C instead of using them for production of the products X, Y and Z. Let the selling prices be ₹ y_1, ₹ y_2 and ₹ y_3 per unit of raw materials A, B and C respectively.

The cost of the purchaser due to all raw materials is

$$36y_1 + 60y_2 + 45y_3.$$

Then the purchaser forms the following LP problem :

$$\text{Minimize } T = 36y_1 + 60y_2 + 45y_3$$

$$\begin{aligned} \text{Subject to, } y_1 + 2y_2 + 2y_3 &\geq 40, \\ 2y_1 + y_2 + 5y_3 &\geq 25, \\ y_1 + 4y_2 + y_3 &\geq 50, \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

The solution is obtained as :

$$y_1 = 0,\ y_2 = 10,\ y_3 = 10,\ \text{Optimal cost} = ₹\ 1050.$$

In the above, the company's problem is called primal problem and purchaser's problem is called dual problem. Also we can use these two terms interchangeably. In the primal problem, the company achieve a profit of ₹ 1050 by producing 20 units of X and 5 units of Z. Instead, if the company sells the raw material B with ₹ 10 per unit and C with ₹ 10 per unit then also the company achieve a sale of ₹ 1050.

(*a*) Formulation

In the above, both the problems are called **symmetric** problem since the objective function is maximization (minimization), all the constraints are '≤' type (≥ type) and non-negative decision variables.

The decision variables in the primal are called primal variables and the decision variables in the dual are called dual variables.

Let us consider the following table for formulation of the dual.

Primal (Maximization)	*Dual (Minimization)*
Right hand side constants	Cost vector
Cost vector	Right hand side constants.
Coefficient matrix	Transpose of coefficient matrix
'≤'	'≥'
Max. $z = cx$ S/t, $Ax \leq b$ $x \geq 0$	Min. $T = b^T y$ S/t $A^T y \geq c^T$ $y \geq 0$

(*b*) Asymmetric Primal-Dual Problems

Primal (Maximization)	*Dual (Minimization)*
a. Coefficient matrix	A^T
b. Right hand side constants	Cost vector

NOTES

c. Cost vector	Right hand side constants
i-th constraint	i-th dual variable
$\leq$ type	$y_i \geq 0$
$\geq$ type	$y_i \leq 0$
= type	y_i unrestricted in sign
j-th primal variable	j-th dual constraint
x_j unrestricted in sign	= type
$x_j \leq 0$	$\leq$ type
$x_j \geq 0$	$\geq$ type

Also in (*a*) and (*b*),

Number of primal constraints = Number of dual variables.

Number of primal variables = Number of dual constraints.

Note. The dual of the dual is the primal.

Example 15. *Obtain the dual of*

$$\textit{Minimize } z = 8x_1 + 3x_2 + 15x_3$$

$$\textit{Subject to, } 2x_1 + 4x_2 + 3x_3 \geq 28,$$

$$3x_1 + 5x_2 + 6x_3 \geq 30,$$

$$x_1, x_2, x_3 \geq 0.$$

Solution. Let y_1 and y_2 be the variables corresponding to the first and second constraints respectively. Objective function, maximize $T = 28y_1 + 30y_2$. There will be three dual constraints due to three primal variables. In primal

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 5 & 6 \end{bmatrix}, c = [8, 3, 15]$$

$\therefore$ In dual
$$A^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq c^T$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 3 \\ 15 \end{pmatrix}$$

$$\Rightarrow \quad 2y_1 + 3y_2 \leq 8 \text{ (due to } x_1)$$

$$4y_1 + 5y_2 \leq 3 \text{ (due to } x_2)$$

$$3y_1 + 6y_2 \leq 15 \text{ (due to } x_3)$$

Hence the dual problem is

$$\text{Maximize } T = 28y_1 + 30y_2$$

$$\text{Subject to, } 2y_1 + 3y_2 \leq 8$$

$$4y_1 + 5y_2 \leq 3$$

$$3y_1 + 6y_2 \le 15$$
$$y_1, y_2 \ge 0.$$

Example 16. *Find the dual of*

NOTES

$$\text{Maximize } z = 2x_1 + x_2 + 5x_3$$
$$\text{Subject to, } x_1 + x_2 + x_3 = 10,$$
$$4x_1 - x_2 + 2x_3 \ge 12,$$
$$3x_1 + 2x_2 - 3x_3 \le 6,$$
$$x_1, x_2, x_3 \ge 0.$$

Solution. First we have to express all the constraints in '≤' form due to maximization problem:

The first constraint : $x_1 + x_2 + x_3 \le 10$

and $x_1 + x_2 + x_3 \ge 10$

$\Rightarrow$ $-x_1 - x_2 - x_3 \le -10$

The second constraint : $-4x_1 + x_2 - 2x_3 \le -12$

Let y_1, y_2, y_3 and y_4 be four dual variables corresponding to the newly converted constraints respectively.

Objective function : Minimize $T = 10y_1 - 10y_2 - 12y_3 + 6y_4$

Again,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -4 & 1 & -2 \\ 3 & 2 & -3 \end{bmatrix}, c = [2, 1, 5]$$

$\therefore$ Constraints in dual :
$$A^T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \ge c^T.$$

Thus the dual problem is

$$\text{Minimize } T = 10y_1 - 10y_2 - 12y_3 + 6y_4.$$
$$\text{Subject to, } y_1 - y_2 - 4y_3 + 3y_4 \ge 2 \text{ (due to } x_1)$$
$$y_1 - y_2 + y_3 + 2y_4 \ge 1 \text{ (due to } x_2)$$
$$y_1 - y_2 - 2y_3 - 3y_4 \ge 5 \text{ (due to } x_3)$$
$$y_1, y_2, y_3, y_4 \ge 0.$$

Set $w_1 = y_1 - y_2$, $w_2 = -y_3$, $w_3 = y_4 \Rightarrow w_1$ unrestricted in sign, $w_2 \le 0$, $w_3 \ge 0$. This conversion leads to

$$\text{Minimize } T = 10w_1 + 12w_2 + 6w_3$$

$$\text{Subject to, } w_1 + 4w_2 + 3w_3 \geq 2,$$
$$w_1 - w_2 + 2w_3 \geq 1,$$
$$w_1 + 2w_2 - 3w_3 \geq 5,$$

NOTES

w_1 unrestricted in sign, $\qquad w_2 \leq 0,\ w_3 \geq 0.$

Example 17. *Find the dual of*

$$\textit{Maximize } z = 5x_1 + 4x_2 - 3x_3$$
$$\textit{Subject to, } 2x_1 + 4x_2 - x_3 \leq 14,$$
$$x_1 - 2x_2 + x_3 = 10,$$

$x_1 \geq 0$, x_2 *unrestricted in sign,* $\qquad x_3 \leq 0.$

Solution. First we have introduce non-negative variables.

$\therefore$ Set $\qquad x_2 = x'_2 - x''_2;\ x'_2, x''_2 \geq 0$ and $x_3 = -x'_3,\ x'_3 \geq 0.$

The given problem reduces to

$$\text{Maximize } z = 5x_1 + 4x'_2 - 4x''_2 + 3x'_3$$
$$\text{Subject to, } 2x_1 + 4x'_2 - 4x''_2 + x'_3 < 14$$
$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$
$$x_1, x'_2, x''_2\ x'_3 \geq 0$$

The second constraint is expressed as

$$x_1 - 2x'_2 + 2x''_2 - x'_3 \leq 10$$

and
$$-x_1 + 2x'_2 - 2x''_2 + x'_3 \leq -10$$

Let y_1, y_2, y_3 be the three dual variables corresponding to the three constraints respectively. Then the symmetric dual is

$$\text{Minimize T} = 14y_1 + 10y_2 - 10y_3$$
$$\text{Subject to, } 2y_1 + y_2 - y_3 \geq 5 \text{ (due to } x_1)$$
$$4y_1 - 2y_2 + 2y_3 \geq 4 \text{ (due to } x'_1)$$
$$-4y_1 + 2y_2 - 2y_3 \geq -4 \text{ (due to } x''_2)$$
$$y_1 - y_2 + y_3 \geq -3 \text{ (due to } x'_3)$$
$$y_1, y_2, y_3 \geq 0$$

Set $\qquad w_1 = y_1,\ w_2 = y_2 - y_3 \Rightarrow w_1 \geq 0$ and w_2 unrestricted.

Also the second and third constraint reduces to

$$4y_1 - 2y_2 + 2y_3 = 4$$

Therefore the dual is

$$\text{Minimize T} = 14w_1 + 10w_2$$
$$\text{Subject to, } 2w_1 + w_2 \geq 5,$$
$$4w_1 - 2w_2 = 4,$$
$$-w_1 + w_2 \leq 3$$

$w_1 \geq 0$ and w_2 unrestricted in sign.

2.10 DUALITY THEOREMS

Theorem 1. (Weak Duality)

Consider the symmetric primal (max. type) and Dual (min. type). The value of the objective function of the (dual) minimum problem for any feasible solution is always greater than or equal to that of the maximum problem (primal) for any feasible solution.

Proof. Let x^0 be a feasible solution to the primal.

Then $Ax^0 \le b$, $x^0 \ge 0$ and $z = cx^0$.

Let y^0 be a feasible solution to the dual.

Then $A^T y^0 \ge c^T$, $y^0 \ge 0$ and $T = b^T y^0$.

Taking transpose on both sides, we have

$$c \le (y^0)^T.A$$

$$\Rightarrow \quad cx^0 \le (y^0)^T.Ax^0$$

$$\Rightarrow \quad cx^0 \le (y^0)^T.b$$

$$\Rightarrow \quad cx^0 \le b^T.y^0 \qquad (\because (y^0)^T b = b^T y^0)$$

Hence proved.

Theorem 2.

Let x^0 and y^0 be the feasible solutions to the corresponding primal and dual problem such that $cx^0 = b^T y^0$, then x^0 and y^0 are optimal solutions to the respective problems.

Proof. Let x^* be any other feasible solution to the primal problem.

Then by Theorem 1, $cx^* \le b^T y^0$

$$\Rightarrow \quad cx^* \le .cx^0$$

Hence x^0 is an optimal solution to the primal problem because the primal problem is a maximization problem.

Similarly, we can prove that y^0 is an optimal solution for the dual problem.

Theorem 3. (Fundamental Theorem of Duality)

If both the primal and dual problems are feasible and both have optimal solutions then the optimal values of the objective functions of both the problems are equal.

Theorem 4. (Complementary Slackness Conditions (CSC))

Let x^0 and y^0 be the feasible solutions for the primal and dual problems respectively. Let u be the slack variables of the primal and v be the surplus variables of the dual. Then x^0 and y^0 are optimal solutions to the respective primal and dual problems respectively iff

$$(x^0)^T.v = 0 \text{ and } (y^0)^T.u = 0$$

Results on Feasibility

NOTES

		Primal (Max. z)	
		Feasible Solution	*Infeasible solution*
Dual (Min. T)	Feasible solution	Max. z = Min T	Dual unbounded (Min. T → – ∞)
	Infeasible solution	Primal unbounded (Max. z → ∞)	May occur.

Let the primal as : Minimize $z = -x_1 - x_2$

$$S/t,\ x_1 - x_2 = 3,$$
$$x_1 - x_2 = -3,$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Then the dual can be written as

$$\text{Maximize } T = 3y_1 - 3y_2$$
$$S/t,\ y_1 + y_2 \leq -1,$$
$$-y_1 - y_2 \leq -1,$$
$$y_1, y_2 \text{ unrestricted in sign.}$$

Here both the primal and the dual are inconsistent and hence no feasible solutions.

Example 18. *Using the C.S.C. find the optimal solution of the following primal.*

$$\text{Minimize } z = 2x_1 + 3x_2 + 5x_3 + 3x_4 + 2x_5$$
$$S/t,\ x_1 + x_2 + 2x_3 + 3x_4 + x_5 \geq 4,$$
$$2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3,$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

Solution. The dual is

$$\text{Maximize } T = 4y_1 + 3y_2$$
$$S/t,\ y_1 + 2y_2 \leq 2$$
$$y_1 - 2y_2 \leq 3$$
$$2y_1 + 3y_2 \leq 5$$
$$3y_1 + y_2 \leq 3$$
$$y_1 + y_2 \leq 2$$
$$y_1, y_2 \geq 0.$$

The solution of this dual, by graphically is $y_1^* = \frac{4}{5}$, $y_2^* = \frac{3}{5}$, $T^* = 5$. Let u_1, u_2, u_3, u_4 and u_5 be the slack variables of the dual and v_1, v_2 be the surplus variables of the primal. Then by C.S.C., we have

$$x_1u_1 = 0,\ x_2u_2 = 0,\ x_3u_3 = 0,$$
$$x_4u_4 = 0,\ x_5u_5 = 0,\ y_1v_1 = 0,\ y_2v_2 = 0.$$

Since y_1^* and y_2^* are non-zero $\Rightarrow$ $v_1 = v_2 = 0$.

It is also seen that at optimality, the two constraints $y_1 + 2y_2 \le 2$ and $3y_1 + y_2 \le 3$ are satisfying in equality sense which mean $u_1^* = 0$ and $u_4^* = 0$.

For the remaining constraints, u_2^*, u_3^* and u_5^* are non-zero *i.e.*, by C.S.C., $x_2^* = 0$, $x_3^* = 0$ and $x_5^* = 0$.

Then the primal constraints reduces to

$$x_1^* + 3x_4^* = 4$$
$$2x_1^* + x_4^* = 3.$$

Solving we get $x_1^* = 1$ and $x_4^* = 1$.

Hence the optimal solution of the primal is

$$x_1^* = 1, x_2^* = 0, x_3^* = 0, x_4^* = 1, x_5^* = 0 \text{ and } z^* = 5.$$

NOTES

2.11 DUALITY OF SIMPLEX METHOD

The fundamental theorem of duality helps to obtain the optimal solution of the dual from optimal table of the primal and vice-versa. Using C.S.C., the correspondence between the primal (dual) variables and slack and/or surplus variables of the dual (primal) to be identified. Then the optimal solution of the dual (primal) can be read off from the net evaluation row of the primal (dual) of the simplex table.

For example, if the primal variable corresponds to a slack variable of the dual, then the net evaluation of the slack variable in the optimal table will give the optimal solution of the primal variable.

Example 19. *Using the principle of duality solve the following problem :*

$$\text{Minimize } z = 4x_1 + 14x_2 + 3x_3$$
$$\text{S/t. } -x_1 + 3x_2 + x_3 \ge 3,$$
$$2x_1 + 2x_2 - x_3 \ge 2,$$
$$x_1, x_2, x_3 \ge 0.$$

Solution. The dual problem is

$$\text{Maximize T} = 3y_1 + 2y_2$$
$$\text{S/t. } -y_1 + 2y_2 \le 4$$
$$3y_1 + 2y_2 \le 14$$
$$y_1 - y_2 \le 3$$
$$y_1, y_2 \ge 0$$

Standard form :

$$\text{Maximize T} = 3y_1 + 2y_2 + 0.u_1 + 0.u_2 + 0.u_3$$
$$\text{S/t. } -y_1 + 2y_2 + u_1 = 4$$
$$3y_1 + 2y_2 + u_2 = 14$$
$$y_1 - y_2 + u_3 = 3$$

$y_1, y_2 \ge 0$, u_1, u_2, u_3 are slacks and ≥ 0.

Let the surplus variables of the dual v_1 and v_2.

Then by C.S.C., $y_1v_1 = 0$, $y_2v_2 = 0$,

$$x_1u_1 = 0,\ x_2u_2 = 0,\ x_3u_3 = 0.$$

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Let us solve the dual by simplex method and the optimal table is given below (Iteration 3) :

		c_j	3	2	0	0	0
c_B	x_B	Soln.	y_1	y_2	u_1	u_2	u_3
0	u_1	6	0	0	1	$-\frac{1}{5}$	$\frac{3}{5}$
2	y_2	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$
3	y_1	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$
$z_j - c_j$			0	0	0	1	0

The optimal solution of the dual is $y_1^* = 4, y_2^* = 1, T^* = 14$.

The optimal solution of the primal can be read off from the $(z_j - c_j)$-row. Since x_1, x_2, x_3 corresponds to u_1, u_2, u_3 respectively, then

$$x_1^* = 0, x_2^* = 1, x_3^* = 0, \text{and } z^* = 14.$$

2.12 THE DUAL SIMPLEX METHOD

Step 1. Convert the minimization LP problem into an symmetric maximization LP problem (*i.e.*, all constraints are $\leq$ type) if it is in the minimization form.

Step 2. Introduce the slack variables and obtain the first iteration dual simplex table.

		c_j	
c_B	x_B	Soln.	(x)
	$[x_{B_i}]$		
	$z_j - c_j$		
	Max. ratio		

Step 3.(*a*) If all $z_j - c_j$ and x_{B_i} are non-negative, then an optimal basic feasible solution has been attained.

(*b*) If all $z_j - c_j \geq 0$ and at least one of x_{B_i} is negative then go to step 4.

(*c*) If at least one $(z_j - c_j)$ is negative, the method is not applicable.

Step 4. Select the most negative of x_{B_i}'s and that basic variable will leave the basis and the corresponding row is called 'key-row'.

Step 5.(*a*) If all the elements of the key row is positive, then the problem is infeasible.

(*b*) If at least one element is negative then calculate the maximum ratios as follows :

NOTES

$$\text{Max}\left\{\frac{(z_j - c_j)\text{ value}}{\text{Negative element of the key row}}\right\}$$

The maximum ratio column is called 'key column' and the intersection element of key row and key column is called 'key element'.

Step 6. Obtain the next table which is the same procedure as of simplex method.

Step 7. Go to step 3.

Note. 1. Difference between simplex method and dual-simplex method : In simplex method, we move from a feasible non-optimal solution to feasible optimal solution. Whereas in dual simplex method, we move from an infeasible optimal solution to feasible optimal solution.

2. The term 'dual' is used in dual simplex method because the rules for leaving and entering variables are derived from the dual problem but are used in the primal problem.

Example 20. *Using dual simplex method solve the following LP problem.*

$$\textit{Minimize } z = 4x_1 + 2x_2$$

$$S/t,\ x_1 + 2x_2 \geq 2,\ 3x_1 + x_2 \geq 3,$$

$$4x_1 + 3x_2 \geq 6,\ x_1, x_2 \geq 0.$$

Solution. $\quad$ Min. $z = -$ Max. $(-z) = -$ Max. $(-z = -4x_1 - 2x_2)$.

Multiply -1 to all the $\geq$ constraints to make $\leq$ type.

Then the standard form is obtained as follows :

$$\text{Max } -z = -4x_1 - 2x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$S/t,\ -x_1 - 2x_2 + s_1 = -2$$

$$-3x_1 - x_2 + s_2 = -3$$

$$-4x_1 - 3x_2 + s_3 = -6$$

$x_1, x_2 \geq 0$, s_1, s_2, s_3 are slacks and ≥ 0.

Iteration 1.

		c_j	-4	-2	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	-2	-1	-2	1	0	0	
0	s_2	-3	-3	-1	0	1	0	
0	s_3	-6	-4	-3	0	0	1	→ Key row
$z_j - c_j$			4	2	0	0	0	
Max. ratio			$\frac{4}{-4}$	$\frac{2}{-3}$				
				↑ Key column				

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Iteration 2.

		c_j	-4	-2	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	2	5/3	0	1	0	$-2/3$	
0	s_2	-1	$-5/3$	0	0	1	$-1/3$	$\rightarrow$ Key row
-2	x_2	2	4/3	1	0	0	$-1/3$	
$z_j - c_j$			4/3	0	0	0	2/3	
Max. ratio			$-\frac{4}{5}$	–	–	–	-2	
			$\uparrow$ Key column					

Iteration 3.

		c_j	-4	-2	0	0	0
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3
0	s_1	1	0	0	1	1	-1
-4	x_1	3/5	1	0	0	$-3/5$	1/5
-2	x_2	6/5	0	1	0	4/5	$-3/5$
$z_j - c_j$			0	0	0	4/5	2/5

Hence optimal feasible solution is $x_1^* = \frac{3}{5}$, $x_2^* = \frac{6}{5}$ and $z^* = \frac{24}{5}$.

Example 21. *Use dual simplex method to solve the following LP problem.*

$$\textit{Maximize } z = -4x_1 - 3x_2$$

$$\textit{Subject to, } x_1 + x_2 \le 1,\ x_2 \ge 1,\ -x_1 + 2x_2 \le 1,\ x_1, x_2 \ge 0.$$

Solution. The constraint $x_2 \ge 1$ is rewritten as $-x_2 \le -1$. Adding slack variables, the standard form is

$$\text{Maximize } z = -4x_1 - 3x_2 + 0.s_1 + 0.s_2 + 0.s_3$$

$$\text{S/t, } x_1 + x_2 + s_1 = 1,\ -x_2 + s_2 = -1,\ -x_1 + 2x_2 + s_3 = 1$$

$x_1, x_2 \ge 0$, s_1, s_2, s_3 are slacks and ≥ 0.

NOTES

Iteration 1.

		c_j	−4	−3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	1	1	1	1	0	0	
0	s_2	−1	0	$\boxed{-1}$	0	1	0	→ Key row
0	s_3	1	−1	2	0	0	1	
$z_j - c_j$			4	3	0	0	0	
Max. ratio			–	−3	–	–	–	
				↑ Key column				

Iteration 2.

		c_j	−4	−3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	0	1	0	1	1	0	
−3	x_2	1	0	1	0	−1	0	
0	s_3	−1	$\boxed{-1}$	0	0	2	1	→ Key row
$z_j - c_j$			4	0	0	3	0	
Max. ratio			−4	–	–	–	–	
			↑ Key column					

Iteration 3.

		c_j	−4	−3	0	0	0	
c_B	x_B	Soln.	x_1	x_2	s_1	s_2	s_3	
0	s_1	−1	0	0	1	3	1	→ Key row
−3	x_2	1	0	1	0	−1	0	
−4	x_1	1	1	0	0	−2	−1	
$z_j - c_j$			0	0	0	11	4	
Max. ratio			–	–	–	–	–	

Since all the elements in the key row are positive, the given problem is infeasible.

2.13 ECONOMIC INTERPRETATION OF DUAL VARIABLE

NOTES

Let x^* and y^* be the optimal solutions of the respective primal and dual problems respectively and objective function values are same *i.e.*, $z^* = T^*$. In the primal, the small change in the resources (*i.e.*, right hand side constants) gives the small change in z^*. Consequently, the y^* value for each primal constraint gives the net change in the optimal value of the objective function for unit increase in right hand side constants. Hence the dual variables are called '**shadow prices**'.

TRANSPORTATION PROBLEM

2.14 INTRODUCTION AND MATHEMATICAL FORMULATION

Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation. If different modes of transportation considered then the problem is called 'solid T.P'. In this chapter, we shall deal with simple T.P.

Suppose there are m factories where a certain product is produced and n markets where it is needed. Let the supply from the factories be $a_1, a_2, ..., a_m$ units and demands at the markets be $b_1, b_2, ..., b_n$ units.

Also consider

c_{ij} = Unit of cost of shipping from factory i to market j.

x_{ij} = Quantity shipped from factory i to market j.

Then the LP formulation can be started as follows :

$$\text{Minimize } z = \text{Total cost of transportation}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

$$\text{Subject to, } \sum_{j=1}^{n} x_{ij} < a_i,\ i = 1, 2, ..., m.$$

(Total amount shipped from any factory does not exceed its capacity)

$$\sum_{i=1}^{m} x_{ij} \geq b_j,\ j = 1, 2, ..., n.$$

(Total amount shipped to a market meets the demand of the market)

$x_{ij} \geq 0$ for all i and j.

Here the market demand can be met if

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j.$$

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ *i.e.*, total supply = total demand, the problem is said to be "Balanced T.P." and all the constraints are replaced by equality sign.

$$\text{Minimize } z = \sum\sum c_{ij}\, x_{ij}$$

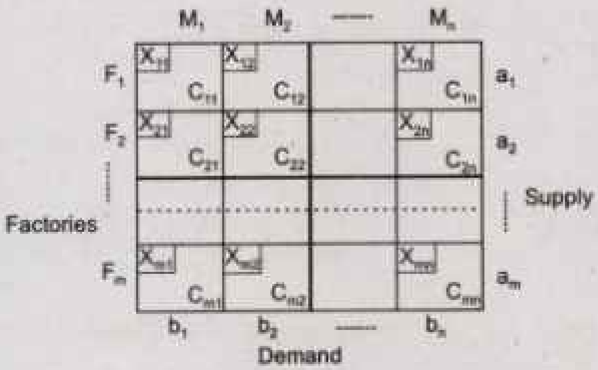
$$\text{Subject to, } \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n.$$

$x_{ij} \geq 0$ for all i and j.

(Total $m + n$ constraints and mn variables)

The T.P. can be represented by *table form* as given below :



In the above, each cell consists of decision variable x_{ij} and per unit transportation cost c_{ij}.

Theorem 5. A necessary and sufficient condition for the existence of a feasible solution to a T.P. is that the T.P. is balanced.

Proof. (Necessary part)

Total supply from an origin $\quad \sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, \ldots, m.$

Overall supply, $\quad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

Total demand met of a destination

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j = 1, 2, \ldots, n.$$

Overall demand, $\quad \sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j\,.$

Since overall supply exactly met the overall demand.

$$\sum_{i}\sum_{j} x_{ij} = \sum_{j}\sum_{i} x_{ij}$$

$$\Rightarrow \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

NOTES

(Sufficient part) Let $\sum_i a_i = \sum_j b_j = l$ and $x_{ij} = a_i b_j / l$ for all i and j.

$$\text{Then } \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} (a_i b_j)/l = a_i \left(\sum_{j=1}^{n} b_j \right) \Big/ l = a_i = 1, 2, \ldots, m.$$

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} (a_i b_j)/l = b_j \left(\sum_{i=1}^{m} a_i \right) \Big/ l = b_j, j = 1, 2, \ldots, n.$$

$x_{ij} \geq 0$ since a_i and b_j are non-negative.

Therefore x_{ij} satisfies all the constraints and hence x_{ij} is a feasible solution.

Theorem 6. The number of basic variables in the basic feasible solution of an $m \times n$ T.P. is $m + n - 1$.

Proof. This is due to the fact that the one of the constraints is redundant in balanced T.P.

We have overall supply, $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$

and overall demand $\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$

Since $\sum_i a_i = \sum_j b_j$, the above two equations are identical and we have only $m + n - 1$ independent constraints. Hence the theorem is proved.

Note. 1. If any basic variable takes the value zero then the basic feasible solution (BFS) is said to be degenerate. Like LPP, all non-basic variables take the value zero.

2. If a basic variable takes either positive value or zero, then the corresponding cell is called 'Basic cell' or 'Occupied cell'. For non-basic variable the corresponding cell is called 'Non-basic cell' or 'Non-occupied cell' or 'Non-allocated cell'.

Loop. This means a closed circuit in a transportation table connecting the occupied (or allocated) cells satisfying the following :

(*i*) It consists of vertical and horizontal lines connecting the occupied (or allocated) cells.

(*ii*) Each line connects only two occupied (or allocated) cells.

(*iii*) Number of connected cells is even.

(*iv*) Lines can skip the middle cell of three adjacent cells to satisfy the condition (*ii*).

The following are the examples of loops.

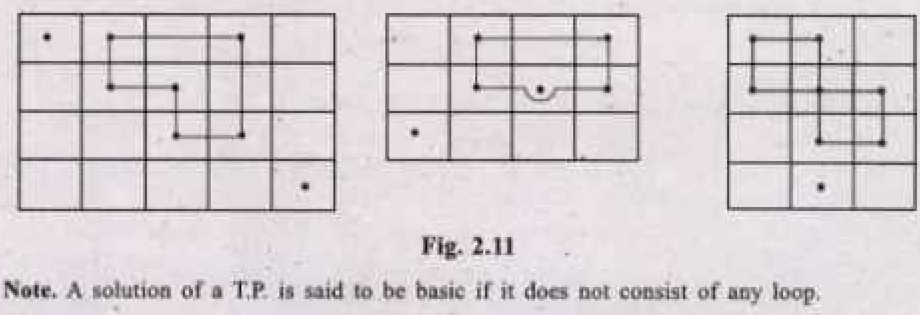


Fig. 2.11

Note. A solution of a T.P. is said to be basic if it does not consist of any loop.

NOTES

2.15 FINDING INITIAL BASIC FEASIBLE SOLUTION

In this section three methods are to be discussed to find initial BFS of a T.P. In advance, it can be noted that the above three methods may give different initial BFS to the same T.P. Also allocation = minimum (supply, demand).

(*a*) North-West Corner Rule (NWC)

(*i*) Select the north west corner cell of the transportation table.

(*ii*) Allocate the min (supply, demand) in that cell as the value of the variable.

If supply happens to be minimum, cross-off the row for further consideration and adjust the demand.

If demand happens to be minimum, cross-off the column for further consideration and adjust the supply.

(*iii*) The table is reduced and go to step (*i*) and continue the allocation until all the supplies are exhausted and the demands are met.

Example 22. *Find the initial BFS of the following T.P. using NWC rule.*

To

From		M_1	M_2	M_3	M_4	Supply
	F_1	3	2	4	1	20
	F_2	2	4	5	3	15
	F_3	3	5	2	6	25
	F_4	4	3	1	4	40
	Demand	30	20	25	25	

Solution. Here, total supply = 100 = total demand. So the problem is balanced T.P. The north-west corner cell is (1, 1) cell. So allocate min. (20, 30) = 20 in that cell. Supply exhausted. So cross-off the first row and demand is reduced to 10. The reduced table is

	M_1	M_2	M_3	M_4	
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	10	20	25	25	

Here the north-west corner cell is (2, 1) cell. So allocate min. (15, 10) = 10 in that cell. Demand met. So cross-off the first column and supply is reduced to 5. The reduced table is

	M_2	M_3	M_4	
F_2	4	5	3	5
F_3	5	2	6	25
F_4	3	1	4	40
	20	25	25	

NOTES

Here the north-west corner cell is (2, 2) cell. So allocate min. (5, 20) = 5 in that cell. Supply exhausted. So cross-off the second row (due to F2) and demand is reduced to 15. The reduced table is

	M_2	M_3	M_4	
F_3	5	2	6	25
F_4	3	1	4	40
	15	25	25	

Here the north-west corner cell is (3, 2) cell. So allocate min. (25, 15) = 15 in that cell. Demand met. So cross-off the second column (due to M_2) and supply is reduced to 10. The reduced table is

	M_3	M_4	
F_3	2	6	10
F_4	1	4	40
	25	25	

Here the north-west corner cell is (3, 3) cell. So allocate min. (10, 25) = 10 in that cell. Supply exhausted. So cross-off the third row (due to F_3) and demand is reduced to 15. The reduced table is

	M_3	M_4	
F_4	1	4	40
	25	25	

continuing we obtain the allocation 15 to (4, 3) cell and 25 to (4, 4) cell so that supply exhausted and demand met. The **complete allocation** is shown below:

	M_1	M_2	M_3	M_4
F_1	[20] 3	2	4	1
F_2	[10] 2	[5] 4	5	3
F_3	3	[15] 5	[10] 2	6
F_4	4	3	[15] 1	[25] 4

Thus, the initial BFS is

$$x_{11} = 20,\ x_{21} = 10,\ x_{22} = 5,\ x_{32} = 15,\ x_{33} = 10,\ x_{43} = 15,\ x_{44} = 25.$$

The transportation cost

$$= 20 \times 3 + 10 \times 2 + 5 \times 4 + 15 \times 5 + 10 \times 2 + 5 \times 1 + 25 \times 4$$

$$= ₹\ 310.$$

NOTES

(b) Least Cost Entry Method (LCM) (or Matrix Minimum Method)

(*i*) Find the least cost from transportation table. If the least value is unique, then go for allocation.

If the least value is not unique then select the cell for allocation for which the contributed cost is minimum.

(*ii*) If the supply is exhausted cross-off the row and adjust the demand.

If the demand is met cross-off the column and adjust the supply.

Thus the matrix is reduced.

(*iii*) Go to step (*i*) and continue until all the supplies are exhausted and all the demands are met.

Example 23. *Find the initial BFS of Example 1 using least cost entry method:*

	M_1	M_2	M_3	M_4	
F_1	3	2	4	1	20
F_2	2	4	5	3	15
F_3	3	5	2	6	25
F_4	4	3	1	4	40
	30	20	25	25	

Solution. Here the least value is 1 and occurs in two cells (1, 4) and (4, 3). But the contributed cost due to cell (1, 4) is 1 × min (20, 25) *i.e.*, 20 and due to cell (4, 3) is 1 × min. (40, 25) *i.e.*, 25. So we selected the cell (1, 4) and allocate 20. Cross-off the first row since supply exhausted and adjust the demand to 5. The reduced table is given below :

2	4	5	3	15
3	5	2	6	25
4	3	1	4	40
30	20	25	5	

The least value is 1 and unique. So allocate min. (40, 25) = 25 in that cell. Cross-off the third column (due to M_3) since the demand is met and adjust the supply to 15. The reduced table is given below :

2	4	3	15
3	5	6	25
4	3	4	15
30	20	5	

The least value is 2 and unique. So allocate min. (15, 30) = 15 in that cell. Cross-off the second row (due to F_2) since the supply exhausted and adjust the demand to 15. The reduced table is given below :

3	5	6	25
4	3	4	15
15	20	5	

NOTES

The least value is 3 and occurs in two cells (3, 1) and (4, 2). The contributed cost due to cell (3, 1) is 3 × min. (25, 15) = 45 and due to cell (4, 2) is 3 × min. (15, 20) = 45. Let us select the (3, 1) cell for allocation and allocate 15. Cross-off the first column (due to M_1) since demand is met and adjust the supply to 10. The reduced table is given below :

5	6	10
3	4	15
20	5	

Continuing the above method and we obtain the allocations in the cell (4, 2) as 15, in the cell (3, 2) as 5 and in the cell (3, 4) as 5. The complete allocation is shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[15] 2	4	5	3
F_3	[15] 3	[5] 5	2	[5] 6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20, x_{21} = 15, x_{31} = 15, x_{32} = 5, x_{34} = 5, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 20 \times 1 + 15 \times 2 + 15 \times 3 + 5 \times 5 + 5 \times 6 + 15 \times 3 + 25 \times 1$$
$$= ₹\ 220.$$

Note. If the least cost is only selected columnwise then it is called 'column minima' method. If the least cost is only selected rowwise then it is called 'row minima' method.

(c) Vogel's Approximation Method (VAM)

(*i*) Calculate the row penalties and column penalties by taking the difference between the lowest and the next lowest costs of every row and of every column respectively.

(*ii*) Select the largest penalty by encircling it. For tie cases, it can be broken arbitrarily or by analyzing the contributed costs.

(*iii*) Allocate in the least cost cell of the row/column due to largest penalty.

(*iv*) If the demand is met, cross off the corresponding column and adjust the supply.

NOTES

If the supply is exhausted, cross-off the corresponding row and adjust the demand.

Thus the transportation table is reduced.

(v) Go to Step (*i*) and continue until all the supplies exhausted and all the demands are met.

Example 24. *Find the initial BFS of example 1 using Vogel's approximation method.*

Solution.

	M_1	M_2	M_3	M_4	Row penalties
F_1	3	2	4	[20] 1	20 (1)
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
	30	20	25	25	
Column penalties	(1)	(1)	(1)	(2)	

Since there is a tie in penalties, let us break the tie by considering the contributed costs. Due to M_4, the contributed cost is $1 \times \min. (20, 25) = 20$. While due to F_4, the contributed cost is $1 \times \min. (40, 25) = 25$. So select the column due to M_4 for allocation and we allocate min. (20, 25) *i.e.*, 20 in (1, 4) cell. Then cross-off the first row as supply is exhausted and adjust the corresponding demand as 5. The reduced table is

	M_1	M_2	M_3	M_4	Row penalties
F_2	2	4	5	3	15 (1)
F_3	3	5	2	6	25 (1)
F_4	4	3	1	4	40 (2)
	30	20	25	5	
Column penalties	(1)	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_4. Allocate in (4, 3) cell as min. $(40, 25) = 25$. Cross-off the third column due to M_3, since demand is met and adjust the corresponding supply to 15. The reduced table is

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_3	3	5	6	25 (2)
F_4	4	3	4	15 (1)
	30	20	5	
Column penalties	(1)	(1)	(1)	

Here the largest penalty is 2 which is due to F_3. Allocate in (3, 1) cell as min. (25, 30) = 25. Cross-off the third row due to F_3 since supply is exhausted and adjust the corresponding demand to 5. The reduced table is

	M_1	M_2	M_4	Row penalties
F_2	2	4	3	15 (1)
F_4	4	3	4	15 (1)
	5	20	5	
Column penalties	(2)	(1)	(1)	

Here the largest penalty is 2 which is due to M_1. Allocate in (2, 1) cell as min. (15, 5) = 5. Corss-off the first column due to M_1 since demand is met and adjust the supply to 10. The reduced table is

	M_2	M_4	Row penalties
F_2	4	3	10 (1)
F_4	3	4	15 (1)
	20	5	
Column penalties	(1)	(1)	

Here tie has occurred. The contributed cost is minimum due to (2, 4) cell which is 3 × min. (10, 5) = 15. So allocate min. (10, 5) = 5 in (2, 4) cell. Cross-off the fourth column which is due to M_4 since demand is met and adjust the corresponding supply to 5. On continuation we obtain the allocation of 5 in (2, 2) cell and 15 in (4, 2) cell. The complete allocation in shown below :

	M_1	M_2	M_3	M_4
F_1	3	2	4	[20] 1
F_2	[5] 2	[5] 4	5	[5] 3
F_3	[25] 3	5	2	6
F_4	4	[15] 3	[25] 1	4

The initial BFS is

$$x_{14} = 20, x_{21} = 5, x_{22} = 5, x_{24} = 5, x_{31} = 25, x_{42} = 15, x_{43} = 25.$$

The transportation cost

$$= 1 \times 20 + 2 \times 5 + 4 \times 5 + 3 \times 5 + 3 \times 25 + 3 \times 15 + 1 \times 25$$

$$= ₹\ 210.$$

2.16 UV-METHOD/MODI METHOD

Taking the initial BFS by any method discussed above, this method find the optimal solution to the transportation problem. The steps are given below :

NOTES

(*i*) For each row consider a variable u_i and for each column consider another variable v_j.

Find u_i and v_j such that

$u_i + v_j = c_{ij}$ for every basic cells.

(*ii*) For every non-basic cells, calculate the net evaluations as follows :

$\bar{c}_{ij} = u_i + v_j - c_{ij}$

If all $\bar{c}_{ij}$ are non-positive, the current solution is optimal.

If at least one $\bar{c}_{ij} > 0$, select the variable having the largest positive net evaluation to enter the basis.

(*iii*) Let the variable x_{rc} enter the basis. Allocate an unknown quantity θ to the cell (r, c).

Identify a loop that starts and ends in the cell (r, c).

Subtract and add θ to the corner points of the loop clockwise/anticlockwise.

(*iv*) Assign a minimum value of θ in such a way that one basic variable becomes zero and other basic variables remain non-negative. The basic cell which reduces to zero leaves the basis and the cell with θ enters into the basis.

If more than one basic variables become zero due to the minimum value of θ, then only one basic cell leaves the basis and the solution is called degenerate.

(*v*) Go to step (*i*) until an optimal BFS has been obtained.

Note. In step (*ii*), if all $\bar{c}_{ij} < 0$, then the optimal solution is unique. If at least one $\bar{c}_{ij} < 0$, then we can obtain alternative solution. Assign θ in that cell and repeat one iteration (from step (*iii*)).

Example 25. *Consider the initial BFS by LCM of Example 2, find the optimal solution of the T.P.*

Solution. Iteration 1.

	M_1	M_2	M_3	M_4	
F_1	3	2	4	$\boxed{20}$ 1	$u_1 = -5$
F_2	$\boxed{15}$ 2	4	5	3	$u_2 = -1$
F_3	$\boxed{15}$ 3	$\boxed{5}$ 5	2	$\boxed{5}$ 6	$u_3 = 0$ (Let)
F_4	4	$\boxed{15}$ 3	$\boxed{25}$ 1	4	$u_4 = -2$
	$V_1 = 3$	$V_2 = 5$	$V_3 = 3$	$V_4 = 6$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -5,\ \bar{c}_{12} = -2,\ \bar{c}_{13} = -6,\ \bar{c}_{22} = 0,\ \bar{c}_{23} = -3,\ \bar{c}_{24} = 2,\ \bar{c}_{33} = 1,\ \bar{c}_{41} = -3,\ \bar{c}_{44} = 0.$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

NOTES

Select the cell (2, 4) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is as follow:

3	2	4	[20] 1
[$15-\theta$] 2	4	5	3
[$15+\theta$] 3	[5] 5	2	[$5-\theta$] 6
4	[15] 3	[25] 1	4

Select θ = min. (5, 15) = 5. The cell (3, 4) leaves the basis and the cell (2, 4) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	[5] 5	2	6	$u_3 = 1$
4	[15] 3	[25] 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3,\ \bar{c}_{12} = 0,\ \bar{c}_{13} = -4,\ \bar{c}_{22} = 0,\ \bar{c}_{23} = -3,\ \bar{c}_{33} = 1,\ \bar{c}_{34} = -2,\ \bar{c}_{41} = -3,\ \bar{c}_{44} = -2.$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	[20] 1
[10] 2	4	5	[5] 3
[20] 3	[$5-\theta$] 5	[θ] 2	6
4	[$15+\theta$] 3	[$25-\theta$] 1	4

Select θ = min. (5, 25) = 5. The cell (3, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

NOTES

Iteration 3.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	5	[5] 2	6	$u_3 = 1$
4	[20] 3	[20] 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus, the optimal solution is

$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$

The optimal transportation cost

$= 1 \times 20 + 2 \times 10 + 3 \times 5 + 3 \times 20 + 2 \times 5 + 3 \times 20 + 1 \times 20 =$ ₹ 205.

Example 26. *Consider the initial BFS by VAM of Example 3, find the optimal solution of the T.P.*

Solution. Iteration 1.

3	2	4	[20] 1	$u_1 = -2$
[5] 2	[5] 4	5	[5] 3	$u_2 = 0$ (Let)
[25] 3	5	2	6	$u_3 = 1$
4	[15] 3	[25] 1	4	$u_4 = -1$
$V_1 = 2$	$V_2 = 4$	$V_3 = 2$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3, \bar{c}_{12} = 0, \bar{c}_{13} = -4, \bar{c}_{23} = -3, \bar{c}_{32} = 0, \bar{c}_{33} = 1, \bar{c}_{34} = -2, \bar{c}_{41} = -3, \bar{c}_{44} = -2.$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal.

Select the cell (3, 3) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown below :

3	2	4	[20] 1
[5]+θ 2	[5]$-\theta$ 4	5	[5] 3
[25]$-\theta$ 3	5	θ 2	6
4	[15]+θ 3	[25]$-\theta$ 1	4

NOTES

Select θ = min. (5, 25, 25) = 5. The cell (2, 2) leaves the basis and the cell (3, 3) enters into the basis. Thus the current solution is updated.

Iteration 2.

3	2	4	[20] 1	$u_1 = -2$
[10] 2	4	5	[5] 3	$u_2 = 0$ (Let)
[20] 3	5	[5] 2	6	$u_3 = 1$
4	[20] 3	[20] 1	4	$u_4 = 0$
$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	$V_4 = 3$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{11} = -3, \bar{c}_{12} = -1, \bar{c}_{13} = -5, \bar{c}_{22} = -1, \bar{c}_{23} = -5, \bar{c}_{32} = -1, \bar{c}_{34} = -2, \bar{c}_{41} = -2, \bar{c}_{44} = -1.$

Since all $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the optimal solution is

$$x_{14} = 20, x_{21} = 10, x_{24} = 5, x_{31} = 20, x_{33} = 5, x_{42} = 20, x_{43} = 20.$$

The optimal transportation cost = ₹ 205.

Note. To find optimal solution to a T.P., the number of iterations by uv-method is always more if we consider the initial BFS by NWC.

2.17 DEGENERACY IN T.P.

A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of *uv*-method. An initial BFS could become degenerate when the supply and demand in the intermediate stages of any one method (NWC/LCM/VAM) are equal corresponding to a selected cell for allocation. In *uv*-method it is identified only when more than one corner points in a loop vanishes due to minimum value of θ.

For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.

For the degeneracy in *uv*-method, arbitrarily we can make one corner as non-basic cell and put zero in the other corner.

Example 27. *Find the optimal solution to the following T.P.:*

Source	Destination 1	2	3	Available
1	50	30	190	10
2	80	45	150	30
3	220	180	50	40
Requirement	40	20	20	80

NOTES

Solution. Let us find the initial BFS using VAM :

	1	2	3	Row penalties
1	50	30	190	10 (20)
2	80	45	150	30 (35)
3	220	180	50	40 (130)
	40	20	20	
Column penalties	(30)	(15)	(100)	

Select (3, 3) cell for allocation and allocate min (40, 20) = 20 in that cell. Cross-off the third column as the requirement is met and adjust the availability to 20. The reduced table is given below :

	1	2	Row penalties
1	50	30	10 (20)
2	80	45	30 (35)
3	220	180	20 (40)
	40	20	
Column penalties	(30)	(15)	

Select (3, 2) cell for allocation. Now there is a tie in allocation. Let us allocate 20 in (3, 2) cell and cross-off the second column and adjust the availability to zero. The reduced table is given below :

	1	
1	50	10
2	80	30
3	220	0
	40	

On continuation we obtain the remaining allocations as 0 in (3, 1) cell, 30 in (2, 1) cell and 10 in (1, 1) cell. The complete initial BFS is given below and let us apply the first iteration of *uv*-method :

Iteration 1.

[10] 50	30	190	$u_1 = -170$
[30] 80	45	150	$u_2 = -140$
[0] 220	[20] 180	[20] 50	$u_3 = 0$ (Let)
$V_1 = 220$	$V_2 = 180$	$V_3 = 50$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{12} = -20,\ \bar{c}_{13} = -310,\ \bar{c}_{22} = -5,\ \bar{c}_{23} = -240.$$

NOTES

Since all $\bar{c}_{ij} < 0$, the current solution is optimal. Hence, the optimal solution is

$$x_{11} = 10, x_{21} = 30, x_{31} = 0, x_{32} = 20, x_{33} = 20.$$

The transportation cost

$$= 50 \times 10 + 80 \times 30 + 0 + 180 \times 20 + 50 \times 20$$
$$= ₹\ 7500.$$

2.18 MAX-TYPE T.P.

Instead of unit cost in transportation table, unit profit is considered then the objective of the T.P. changes to maximize the total profits subject to supply and demand restrictions. Then this problem is called 'max-type' T.P.

To obtain optimal solution, we consider

$$\text{Loss} = -\ \text{Profit}$$

and convert the max type transportation matrix to a loss matrix. Then all the methods described in the previous sections can be applied. Thus the optimal BFS obtained for the loss matrix will be the optimal BFS for the max-type T.P.

Example 28. *A company has three plants at locations A, B and C, which supply to four markets D, E, F and G. Monthly plant capacities are 500, 800 and 900 units respectively. Monthly demands of the markets are 600, 700, 400 and 500 units respectively. Unit profits (in rupees) due to transportation are given below :*

	D	E	F	G
A	8	5	3	6
B	7	4	5	2
C	6	8	4	2

Determine an optimal distribution for the company in order to maximize the total transportation profits.

Solution. The given problem is balanced max type T.P. All profits are converted to losses by multiplying – 1.

	D	E	F	G	
A	–8	–5	–3	–6	500
B	–7	–4	–5	–2	800
C	–6	–8	–4	–2	900
	600	700	400	500	2200

The initial BFS by LCM is given below:

NOTES

$\boxed{500}$ -8	-5	-3	-6
$\boxed{100}$ -7	-4	$\boxed{400}$ -5	$\boxed{300}$ -2
-6	$\boxed{700}$ -8	-4	$\boxed{200}$ -2

To find optimal solution let us apply *uv*-method.

Iteration 1.

$\boxed{500}$ $-\theta$ -8	-5	-3	θ -6	$u_1 = -1$
$\boxed{100}$ $+\theta$ -7	-4	$\boxed{400}$ -5	$\boxed{300}$ $-\theta$ -2	$u_2 = 0$
-6	$\boxed{700}$ -8	-4	$\boxed{200}$ -2	$u_3 = 0$ (Let)
$V_1 = -7$	$V_2 = -8$	$V_3 = -5$	$V_4 = -2$	

For non-basic cells : $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$$\bar{c}_{12} = -4,\ \bar{c}_{13} = -3,\ \bar{c}_{14} = 3,\ \bar{c}_{22} = -4,\ \bar{c}_{31} = -1,\ \bar{c}_{33} = -1.$$

Since all $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal. Select the cell (1, 4) due to largest positive value and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (500, 300) = 300. The cell (2, 4) leaves the basis and the cell (1, 4) enters into the basis. Thus the current solution is updated.

Iteration 2.

$\boxed{200}$ $-\theta$ -8	-5	-3	$\boxed{300}$ $+\theta$ -6	$u_1 = -4$
$\boxed{400}$ -7	-4	$\boxed{400}$ -5	-2	$u_2 = -3$
θ -6	$\boxed{700}$ -8	-4	$\boxed{200}$ $-\theta$ -2	$u_3 = 0$ (Let)
$V_1 = -4$	$V_2 = -8$	$V_3 = -2$	$V_4 = -2$	

For non-basic cells,

$$\bar{c}_{12} = -7,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -7,\ \bar{c}_{24} = -3,\ \bar{c}_{31} = 2,\ \bar{c}_{33} = 2.$$

Since all the $\bar{c}_{ij}$ are not non-positive, the current solution is not optimal. There is a tie in largest positive values. Let us select the cell (3, 1) and assign an unknown quantity θ in that cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (200, 200) = 200. Since only one cell will leave the basis, let the cell (3, 3) leaves the basis and assign a zero in the cell (1, 1). The cell (3, 1) enters into the basis. Thus the current solution is updated.

Iteration 3.

[0] −8	−5	−3	[500] −6	$u_1 = -2$
[400] −7	−4	[400] −5	−2	$u_2 = -1$
[200] −6	[700] −8	−4	−2	$u_3 = 0$ (Let)
$V_1 = -6$	$V_2 = -8$	$V_3 = -4$	$V_4 = -4$	

NOTES

For non-basic cells,

$$\bar{c}_{12} = -5,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -5,\ \bar{c}_{24} = -3,\ \bar{c}_{33} = 0,\ \bar{c}_{34} = -4.$$

Since all the $\bar{c}_{ij}$ are non-positive, the current solution is optimal.

Thus the optimal solution, which is degenerate, is

$$x_{11} = 0,\ x_{14} = 500,\ x_{21} = 400,\ x_{23} = 400,\ x_{31} = 200,\ x_{32} = 700.$$

The maximum transportation profit

$$= 0 + 3000 + 2800 + 2000 + 1200 + 5600 = ₹\ 14600.$$

Since $\bar{c}_{33} = 0$, this indicates that there exists an alternative optimal solution. Assign an unknown quantity θ in the cell (3, 3). Identify a loop and subtract and add θ to the corner points of the loop which is shown below:

[0] −8	−5	−3	[500] −6
[400] +θ −7	−4	[400] −θ −5	−2
[200] −θ −6	[700] −8	θ −4	−2

Select θ = min. (200, 400) = 200. The cell (3, 1) leaves the basis and the cell (3, 3) enters into the basis.

Iteration 4.

[0] −8	−5	−3	[500] −6	$u_1 = -2$
[600] −7	−4	[200] −5	−2	$u_2 = -1$
−6	[700] −8	[200] −4	−2	$u_3 = 0$ (Let)
$V_1 = -6$	$V_2 = -8$	$V_3 = -4$	$V_4 = -4$	

For non-basic cells,

$$\bar{c}_{12} = -5,\ \bar{c}_{13} = -3,\ \bar{c}_{22} = -5,\ \bar{c}_{24} = -3,\ \bar{c}_{31} = 0,\ \bar{c}_{34} = -2.$$

Since all the $\bar{c}_{ij}$ are non-positive, the current solution is optimal. Thus the alternative optimal solution is

$$x_{11} = 0,\ x_{14} = 500,\ x_{21} = 600,\ x_{23} = 200,\ x_{32} = 700,\ x_{33} = 200.$$

and the maximum transportation profit is ₹ 14,600.

NOTES

2.19 UNBALANCED T.P.

If total supply ≠ total demand, the problem is called unbalanced T.P. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required. Suppose, (supply =) $\sum a_i > \sum b_j$ (= demand). Then add one dummy destination with demand $= \left(\sum a_i - \sum b_j\right)$ with either zero transportation costs or some penalties, if they are given. Suppose (supply =) $\sum a_i < \sum b_j$ (= demand). Then add one dummy source with supply = $= \left(\sum b_j - \sum a_i\right)$ with either zero transportation costs or some penalties, if they are given.

After making it balanced the mathematical formulation is similar to the balanced T.P.

Example 29. *A company wants to supply materials from three plants to three new projects. Project I requires 50 truck loads, project II requires 40 truck loads and project III requires 60 truck loads. Supply capacities for the plants P_1, P_2 and P_3 are 30, 55 and 45 truck loads. The table of transportation costs are given as follows:*

	I	II	III
P_1	7	10	12
P_2	8	12	7
P_3	4	9	10

Determine the optimal distribution.

Solution. Here total supplies = 130 and total requirements = 150. The given problem is unbalanced T.P. To make it balanced consider a dummy plants with supply capacity of 20 truck loads and zero transportation costs to the three projects. Then the balanced T.P. is

		To I	To II	To III	
	P_1	7	10	12	30
From	P_2	8	12	7	55
	P_3	4	9	10	45
	P_4 (Dummy)	0	0	0	20
		50	40	60	

NOTES

Using VAM, we obtain the initial BFS as given below :

[5] 7	[20] 10	[5] 12
8	12	[55] 7
[45] 4	9	10
0	[20] 0	0

To find optimal solution let us apply *uv*-method.

Iteration 1.

[5] 7	[20]+θ 10	[5]−θ 12	$u_1 = 0$ (Let)
8	12	[55] 7	$u_2 = -5$
[45] 4	9	10	$u_3 = -3$
0	[20] −θ 0	θ 0	$u_4 = -10$
$V_1 = 7$	$V_2 = 10$	$V_3 = 12$	

For non-basic cells, $\bar{c}_{ij} = u_i + v_j - c_{ij}$

$\bar{c}_{21} = -6,\ \bar{c}_{22} = -7,\ \bar{c}_{32} = -2,\ \bar{c}_{33} = -1,\ \bar{c}_{41} = -3,\ \bar{c}_{43} = 2,$

Since $\bar{c}_{43}$ is only positive value assign an unknown quantity θ in (4, 3) cell. Identify a loop and subtract and add θ to the corner points of the loop which is shown above.

Select θ = min. (5, 20) = 5 so that the cell (1, 3) leaves the basis and the cell (4, 3) enters into the basis.

Iteration 2.

[5] 7	[25] 10	12	$u_1 = 0$ (Let)
8	12	[55] 7	$u_2 = -3$
[45] 4	9	10	$u_3 = -3$
0	[15] 0	[5] 0	$u_4 = -10$
$V_1 = 7$	$V_2 = 10$	$V_3 = 10$	

For non-basic cells, we obtain

$\bar{c}_{13} = -2,\ \bar{c}_{21} = -4,\ \bar{c}_{22} = -5,\ \bar{c}_{32} = -2,\ \bar{c}_{33} = -3,\ \bar{c}_{41} = -3$

Since $\bar{c}_{ij} < 0$, the current solution is optimal. Thus the optimal solution is

Supply 15 truck loads from P_1 to I, 25 truck loads from P_1 to II, 55 truck loads from P_2 to III, 45 truck loads from P_3 to I. Demands of 15 truck loads for II and 5 truck loads for III will remain unsatisfied.

2.20 SUMMARY

- "Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence.
- In decision-making all the decisions are taken through some variables which are known as decision variables. In engineering design, these variables are known as design vectors.
- A solution which satisfies all the constraints in LPP is called feasible solution.
- A solution which is basic as well as feasible is called basic feasible solution.
- If a basic variable takes the value zero in a BFS, then the solution is said to be degenerate.
- The BFS which optimizes the objective function is called optimal BFS.
- Transportation problem (T.P.) is generally concerned with the distribution of a certain commodity/product from several origins/sources to several destinations with minimum total cost through single mode of transportation.
- A BFS of a T.P. is said to be degenerate if one or more basic variables assume a zero value. This degeneracy may occur in initial BFS or in the subsequent iterations of *uv*-method.
- For the degeneracy in initial BFS, arbitrarily we can delete the row due to supply adjusting the demand to zero or delete the column due to demand adjusting the supply to zero whenever there is a tie in demand and supply.
- If total supply ≠ total demand, the problem is called unbalanced T.P.. To obtain feasible solution, the unbalanced problem should be converted to balanced problem by introducing dummy source or dummy destination, whichever is required.

2.21 REVIEW QUESTIONS

1. Obtain the dual of the following LP problems :

(*a*)
$$\text{Maximize } z = 4x_1 + 2x_2 + x_3 + 6x_4$$
$$\text{Subject to, } 6x_1 - 3x_2 + x_3 + 5x_4 \le 15,$$
$$x_1 - x_2 + 6x_3 + 2x_4 \ge 8,$$
$$x_1, x_2, x_3, x_4 \ge 0$$

(*b*)
$$\text{Maximize } z = 2x_1 + x_2$$
$$\text{Subject to, } 2x_1 + 3x_2 \ge 4,$$
$$3x_1 + 4x_2 \le 10,$$
$$x_1 + 5x_2 = 9,$$
$$x_1 \ge 0, x_2 \ge 0.$$

NOTES

(c) $$\text{Minimize } z = 3x_1 + 4x_2 - x_3$$
$$\text{Subject to, } 2x_1 + 3x_2 + 5x_3 \geq 10,$$
$$3x_1 + 10x_3 \leq 14,$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \geq 0.$$

(d) $$\text{Minimize } z = 10x_1 + 15x_2$$
$$\text{Subject to, } 3x_1 + 2x_2 = 15,$$
$$5x_1 + 4x_2 = 20,$$
$$x_1, x_2 \text{ unrestricted in sign.}$$

(e) $$\text{Maximize } z = x_1 - 2x_2 + 3x_3$$
$$\text{Subject to, } 2x_1 + 5x_3 \leq 16,$$
$$5x_2 + 4x_3 \geq 8,$$
$$x_1 + x_2 + x_3 = 10,$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted in sign.}$$

2. Use principle of duality to solve the following LP problems :

(a) $$\text{Minimize } z = 4x_1 + 3x_2$$
$$\text{S/t, } 2x_1 + x_2 \geq 40,\ x_1 + 2x_2 \geq 50,\ x_1 + x_2 \geq 35$$
$$x_1, x_2 \geq 0$$

(b) $$\text{Maximize } z = 2x_1 + x_2$$
$$\text{S/t, } x_1 + 2x_2 \leq 10,\ x_1 + x_2 \leq 6,\ x_1 - x_2 \leq 2,\ x_1 - 2x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

(c) $$\text{Minimize } z = 6x_1 + x_2$$
$$\text{S/t, } 2x_1 + x_2 \geq 3,\ x_1 - x_2 \geq 0,\ x_1, x_2 \geq 0$$

(d) Minimize $z = 30x_1 + 30x_2 + 10x_3$

S/t, $2x_1 + x_2 + x_3 > 6,\ x_1 + x_2 + 2x_3 \leq 8,\ x_1, x_2, x_3 \geq 0$

(e) $$\text{Maximize } z = 5x_1 + 2x_2$$
S/t, $x_1 - x_2 \leq 1,\ x_1 + x_2 \geq 4,\ x_1 - 3x_2 \leq 3,\ x_1, x_2 \geq 0$

3. Using the complementary slackness condition solve the following LP problem:
$$\text{Maximize } z = 2x_1 + 3x_2 + 6x_3$$
S/t, $x_1 + 3x_2 + 4x_3 \leq 4,\ 2x_1 + x_2 + 3x_3 \leq 2,\ x_1, x_2, x_3 \geq 0.$

4. With the help of the following example, verify that the dual of the dual is the primal.
$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$
$$\text{S/t, } 4x_1 + 3x_2 - x_3 \leq 20,\ 3x_1 + 2x_2 + 5x_3 = 18,$$
$$0 \leq x_1 \leq 4,\ x_2 \geq 0,\ x_3 \leq 0.$$

5. Verify the fundamental theorem of duality using the following LP problems:

(a) $$\text{Maximize } z = 2x_1 + 10x_2$$
$$\text{S/t, } 2x_1 + 5x_2 \leq 16,\ 6x_1 \leq 30,\ x_1, x_2 \geq 0.$$

(b) $$\text{Minimize } z = 2x_1 - x_2$$

$$\text{S/t, } x_1 + x_2 \le 5,\ x_1 + 2x_2 \ge 8,\ x_1, x_2 \ge 0.$$

6. Use dual-simplex method to solve the following LP problems :

(a) $$\text{Minimize } z = x_1 + 3x_2$$

$$\text{S/t, } 2x_1 + x_2 \ge 4,\ 3x_1 + 2x_2 \ge 5,\ x_1, x_2 \ge 0.$$

(b) $$\text{Minimize } z = 2x_1 + x_2$$

$$\text{S/t, } x_1 + x_2 \ge 2,\ 3x_1 + 2x_2 \ge 4,\ x_1, x_2 \ge 0.$$

(c) $$\text{Minimize } z = 2x_1 + 3x_2 + 10x_3$$

$$\begin{aligned} \text{S/t, } 2x_1 - 5x_2 + 4x_3 &\ge 30, \\ 3x_1 + 2x_2 - 5x_3 &\ge 25, \\ x_1 + 3x_2 + x_3 &\le 30, \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

(d) $$\text{Maximize } z = -2x_1 - x_2 - 3x_3$$

$$\text{S/t, } -3x_1 + x_2 - 2x_3 - x_4 = 1,\ x_1 - 2x_2 + x_3 - x_5 = 2,\ x_i \ge 0\ \forall i$$

(e) $$\text{Minimize } z = 2x_1 + 3x_2 + 4x_3$$

$$\text{S/t, } 3x_1 + 10x_2 + 5x_3 \ge 3,\ 3x_1 - 10x_2 + 9x_3 \le 30,$$
$$x_1 + 2x_2 + x_3 \ge 4,\ x_1, x_2, x_3 \ge 0.$$

(f) $$\text{Minimize } z = 6x_1 + 2x_2 + 5x_3 + 3x_4$$

$$\text{S/t, } 3x_1 + 2x_2 - 3x_3 + 5x_4 \ge 10,\ 4x_2 + 3x_3 - 5x_4 \ge 12,$$
$$5x_1 - 4x_2 + x_3 + x_4 \ge 10,\ x_1, x_2, x_3, x_4 \ge 0.$$

(g) $$\text{Maximize } z = -x_1 - 2x_2 - 3x_3$$

$$\text{S/t, } 2x_1 - x_2 - x_3 \ge 4,\ x_1 - x_2 + 2x_3 \le 8,\ x_1, x_2, x_3 \ge 0.$$

7. One unit of product A requires 3 units of raw material and 2 hours of labour and contributes the profit of ₹ 7. One unit of product B requires one unit of raw material and one hour of labour and contributes the profit of ₹ 5. There are 48 units of raw material and 40 hours of labour available. The objective is to maximize the profit. Calculate the shadow prices of the raw material and labour.

8. There are three sources which store a given product. The sources supply these products to four dealers. The capacities of the sources and the demands of the dealers are given. Capacities $S_1 = 150$, $S_2 = 40$, $S_3 = 80$, Demands $D_1 = 90$, $D_2 = 70$, $D_3 = 50$, $D_4 = 60$. The cost matrix is given as follows:

		To			
		D_1	D_2	D_3	D_4
From	S_1	27	23	31	69
	S_2	10	45	40	32
	S_3	30	54	35	57

Find the minimum cost of T.P.

NOTES

9. There are three factories F_1, F_2, F_3 situated in different areas with supply capacities as 200, 400 and 350 units respectively. The items are shipped to five markets M_1, M_2, M_3, M_4 and M_5 with demands as 150, 120, 230, 200, 250 units respectively. The cost matrix is given as follows :

	M_1	M_2	M_3	M_4	M_5
F_1	2	5	6	4	7
F_2	4	3	5	8	8
F_3	4	6	2	1	5

Determine the optimal shipping cost and shipping patterns.

10. Find the initial basic feasible solution to the following T.P. using (*a*) NWC, (*b*) LCM, and (*c*) VAM :

(*i*)

To

From	D	E	F	G	H	
A	11	7	5	8	9	50
B	10	11	8	4	5	90
C	9	6	12	5	5	60
	20	40	20	40	80	

(*ii*)

To

From	A	B	C	D	E	
I	9	10	0	8	9	90
II	11	12	5	8	3	20
III	4	9	1	2	0	50
IV	8	0	3	5	6	50
	80	60	20	40	10	

11. Solve the following transportation problem :

To

From	D_1	D_2	D_3	D_4	D_5	
S_1	3	5	2	1	3	45
S_2	2	1	–	4	6	55
S_3	5	4	3	1	2	65
S_4	–	4	6	5	7	50
	27	42	51	62	33	

(Supply from S_2 to D_3 and S_4 to D_1 are restricted)

NOTES

12. A transportation problem for which the costs, origin and availabilities, destinations and requirements are given below :

	D_1	D_2	D_3	
O_1	2	1	2	40
O_2	9	4	7	60
O_3	1	2	9	10
	40	50	20	

Check whether the following basic feasible solution $x_1 = 20$, $x_{13} = 20$, $x_{21} = 10$, $x_{22} = 50$, and $x_{31} = 10$ is optimal. If not, find an optimal solution.

13. Goods have to be transported from sources S_1, S_2 and S_3 to destinations D_1, D_2 and D_3. The T.P. cost per unit capacities of the sources and requirements of the destinations are given in the following table :

	D_1	D_2	D_3	Capacity
S_1	8	5	6	120
S_2	15	10	12	80
S_3	3	9	10	80
Requirement	150	80	50	

Determine a T.P. schedule so that the cost is minimized.

14. Four products are produced in four machines and their profit margins are given by the table as follows :

	P_1	P_2	P_3	P_4	Capacity
M_1	10	7	8	6	40
M_2	5	9	6	4	55
M_3	7	4	11	5	60
M_4	4	10	7	8	45
Requirement	35	42	68	55	

Find a suitable production plan of products in machines so that the profit is maximized while the capacities and requirements are met.

15. Identical products are produced in four factories and sent to four warehouses for delivery to the customers. The costs of transportation, capacities and demands are given as below :

Warehouses

Factories	W_1	W_2	W_3	W_4	Capacities
F_1	9	6	11	5	200
F_2	4	5	8	5	150
F_3	7	8	4	6	350
F_4	3	3	10	10	250
Demands	260	100	340	200	

NOTES

Find the optimal schedule of delivery for minimization of cost of transportation. Is there any alternative solution ? If yes, then find it.

16. Starting with LCM initial BFS, find the optimal solution to the following T.P. problem :

To

From						Supply
	5	1	2	4	3	60
	1	4	2	3	6	55
	4	2	3	5	2	40
	3	5	6	3	7	50
Demands	42	33	41	52	27	

17. A company manufacturing air coolers has two plants located at Mumbai and Kolkata with a weekly capacity of 200 units and 100 units respectively. The company supplies air coolers to its 4 show-rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units respectively. The cost per unit (in ₹) is shown in the following table:

	Ranchi	Delhi	Lucknow	Kanpur
Mumbai	90	90	100	100
Kolkata	50	70	130	85

Plan the production programmes so as to minimize the total cost of transportation.

ANSWERS

1. (a) Minimize $T = 15y_1 - 8y_2$

Subject to, $6y_1 - y_2 \geq 4$

$-3y_1 + y_2 \geq 2$

$y_1 - 6y_2 \geq 1$

$5y_1 - 2y_2 \geq 6$

$y_1, y_2 \geq 0.$

and then set $w_1 = y_1$ and $w_2 = -y_2$.

(b) Minimize $T = 4w_1 + 10w_2 + 9w_3$

S/t, $2w_1 + 3w_2 + w_3 \geq 2$

$3w_1 + 4w_2 + 5w_3 \geq 1$

$w_1 \leq 0,\ w_2 \geq 0,\ w_3$ unrestricted.

(c) Maximize $T = -10w_1 - 14w_2$

S/t, $-2w_1 - 3w_2 \leq 3$

$-w_1 \leq 4$

$5w_1 + 10w_2 \geq 1$

$w_1 \leq 0,\ w_2 \geq 0.$

(*d*) Maximize $T = 15w_1 + 20w_2$

$S/t,\ 3w_1 + 5w_2 = 10,\ 2w_1 + 4w_2 = 15,$

w_1, w_2 unrestricted.

(*e*) Minimize $T = 16w_1 + 8w_2 + 10w_3$

$S/t,\ 2w_1 + w_3 \geq 1,\ 5w_2 + w_3 \leq -2,\ 5w_1 + 4w_2 + w_3 = 3$

$w_1 \geq 0,\ w_2 \leq 0,\ w_3$ unrestricted.

2. (*a*) $x_1 = 5, x_2 = 30, z^* = 110.$

(*b*) $x_1 = 4, x_2 = 2, z^* = 10.$

(*c*) $x_1 = 1, x_2 = 1, z^* = 7.$

(*d*) $x_1 = \frac{4}{3}, x_2 = 0,\ x_3 = \frac{10}{3}, z^* = \frac{220}{3}.$

(*e*) Unbounded solution.

3. $x_1 = 0,\ x_2 = \frac{4}{5},\ x_3 = \frac{2}{5},\ z^* = 4.8.$

5. (*a*) Max. $z = 32 =$ Min. T.

(*b*) Min. $z = -5 =$ Max. T.

6. (*a*) $x_1 = 2, x_2 = 0, z^* = 2$ (It – 3)

(*b*) $x_1 = 0, x_2 = 2, z^* = 2$ (It – 2)

(*c*) $x_1 = 15, x_2 = 0, x_3 = 0, z^* = 30$ (It – 2)

(*d*) Infeasible solution (It – 3)

(*e*) $x_1 = 0, x_2 = 2, x_3 = 0, z^* = 6$ (It – 2)

(*f*) $x_1 = 0,\ x_2 = \frac{11}{3},\ x_3 = 0,\ x_4 = \frac{8}{15},\ z^* = \frac{134}{15}$ (It – 3)

(*g*) $x_1 = 2, x_2 = 0, x_3 = 0, z^* = -2$ (It – 2)

7. Shadow prices for raw material is zero and for labour is five.

8. $x_{11} = 30, x_{12} = 70, x_{13} = 50, x_{24} = 40, x_{31} = 60, x_{34} = 20.$

Minimum T.P. cost = ₹ 8190.

9. Solution 1:

$x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 80, x_{25} = 200, x_{33} = 150, x_{34} = 200.$

Solution 2 :

$x_{11} = 150, x_{15} = 50, x_{22} = 120, x_{23} = 230, x_{25} = 50, x_{34} = 200, x_{35} = 150.$

Minimum shipping cost = ₹ 3510.

10. (*i*) (*a*) $x_{11} = 20, x_{12} = 30, x_{22} = 10, x_{23} = 20, x_{24} = 40, x_{25} = 20, x_{35} = 60.$

T.P. cost = ₹ 1260.

NOTES

(b) $x_{11} = 20, x_{12} = 10, x_{13} = 20, x_{24} = 40, x_{25} = 50, x_{32} = 30, x_{35} = 30.$
T.P. cost = ₹ 1130.

(c) $x_{12} = 40, x_{13} = 10, x_{21} = 20, x_{23} = 10, x_{24} = 40, x_{25} = 20, x_{35} = 60.$
T.P. cost = ₹ 1170.

(ii) (a) $x_{11} = 80, x_{12} = 10, x_{22} = 20, x_{32} = 30, x_{33} = 20, x_{34} = 0, x_{44} = 40,$
$x_{45} = 10.$
T.P. cost = ₹ 1610.

(b) $x_{11} = 70, x_{13} = 20, x_{21} = 10, x_{22} = 10, x_{31} = 0, x_{34} = 40, x_{35} = 10,$
$x_{42} = 50.$
T.P. cost = ₹ 940.

(c) $x_{11} = 60, x_{12} = 10, x_{13} = 20, x_{21} = 10, x_{25} = 10, x_{31} = 10, x_{34} = 40,$
$x_{42} = 50.$
T.P. cost = ₹ 900.

11. Solution 1 :

$x_{13} = 15, x_{14} = 30, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14,$
$x_{43} = 36.$

Solution 2 :

$x_{13} = 45, x_{21} = 27, x_{22} = 28, x_{34} = 32, x_{35} = 33, x_{42} = 14, x_{43} = 6,$
$x_{44} = 30.$

Minimum T.P. cost = ₹ 512.

12. The new optimal solution is $x_{11} = 30, x_{13} = 10, x_{22} = 50, x_{23} = 10,$
$x_{31} = 10.$

Minimum T.P. cost = ₹ 360.

13. $x_{11} = 70, x_{12} = 0, x_{13} = 50, x_{22} = 80, x_{31} = 80.$

Minimum T.P. cost = ₹ 1900.

14. $x_{11} = 35, x_{14} = 5, x_{22} = 42, x_{23} = 8, x_{24} = 5, x_{33} = 60, x_{44} = 45.$

Total Profit = ₹ 1846.

15. Solution 1 :

$x_{14} = 160, x_{21} = 110, x_{24} = 40, x_{33} = 340, x_{41} = 150, x_{42} = 100.$

Solution 2 :

$x_{14} = 200, x_{21} = 110, x_{33} = 340, x_{41} = 150, x_{42} = 100.$

Minimum T.P. cost = ₹ 3550.

16. Solution 1 :

$x_{12} = 33, x_{13} = 27, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{33} = 3, x_{35} = 27, x_{44} = 50.$

Solution 2 :

$x_{12} = 30, x_{13} = 30, x_{21} = 42, x_{23} = 11, x_{24} = 2, x_{32} = 3, x_{35} = 27, x_{44} = 50.$

Minimum T.P. cost = ₹ 370.

17. $x_{12} = 75, x_{13} = 95, x_{14} = 30, x_{21} = 75, x_{22} = 25.$

Minimum T.P. cost = ₹ 24750.

CHAPTER 3 ASSIGNMENT MODEL AND GAME THEORY

NOTES

★ STRUCTURE ★

ASSIGNMENT MODEL

3.1 INTRODUCTION AND MATHEMATICAL FORMULATION

Consider n machines $M_1, M_2, ..., M_n$ and n different jobs $J_1, J_2, ..., J_n$. These jobs to be processed by the machines one to one basis *i.e.*, each machine will process exactly one job and each job will be assigned to only one machine. For each job the processing cost depends on the machine to which it is assigned. Now, we have to determine the assignment of the jobs to the machines one to one basis such that the total processing cost is minimum. This is called an *assignment problem.*

If the number of machines is equal to the number of jobs then the above problem is called *balanced* or *standard* assignment problem. Otherwise, the problem is called *unbalanced* or *non-standard* assignment problem. Let us consider a balanced assignment problem.

For linear programming problem formulation, let us define the decision variables as

$$x_{ij} = \begin{cases} 1, \text{if job } j \text{ is assigned to machine } i \\ 0, \text{otherwise} \end{cases}$$

and the cost of processing job j on machine i as c_{ij}. Then, we can formulate the assignment problem as follows :

NOTES

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \quad ...(1)$$

subject to, $\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, ..., n$

(Each machine is assigned exactly to one job)

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, ..., n.$$

(Each job is assigned exactly to one machine)

$$x_{ij} = 0 \text{ or } 1, \text{ for all } i \text{ and } j$$

In **matrix form**, Minimize $z = Cx$

subject to, $Ax = 1$,

$$x_{ij} = 0 \text{ or } 1, i, j = 1, 2, ..., n.$$

where A is a $2n \times n^2$ matrix and total unimodular *i.e.*, the determinant of every subsquare matrix formed from it has value 0 or 1. This property permits us to replace the constraint $x_{ij} = 0$ or 1 by the constraint $x_{ij} \geq 0$. Thus we obtain

$$\text{Minimize } z = Cx$$

$$\text{subject to, } Ax = 1, x \geq 0$$

The **dual** of (1) with the non-negativity restrictions replacing the 0-1 constraints can be written as follows :

$$\text{Maximize W} = \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

subject to, $u_i + v_j \leq c_{ij}$, $\quad i, j = 1, 2, ..., n.$

u_i, v_j unrestricted in signs $\quad i, j = 1, 2, ..., n.$

Example 1. *A company is facing the problem of assigning four operators to four machines. The assignment cost in rupees in given below :*

$$\begin{array}{cc} & \text{Machine} \\ & \begin{array}{ccccc} \quad & M_1 & M_2 & M_3 & M_4 \end{array} \\ \text{Operator} & \left[\begin{array}{ccccc} \text{I} & 5 & 7 & - & 4 \\ \text{II} & 7 & 5 & 3 & 2 \\ \text{III} & 9 & 4 & 6 & - \\ \text{IV} & 7 & 2 & 7 & 6 \end{array}\right] \end{array}$$

In the above, operators I and III cannot be assigned to the machines M_3 and M_4 respectively. Formulate the above problem as a LP model.

Solution. Let $x_{ij} = \begin{cases} 1, \text{if the } i\text{th operator is assigned to } j\text{th machine} \\ 0, \text{otherwise} \end{cases}$

$$i, j = 1, 2, 3, 4.$$

be the decision variables.

By the problem, $x_{13} = 0$ and $x_{34} = 0$.

The LP model is given below :

$$\text{Minimize } z = 5x_{11} + 7x_{12} + 4x_{14} + 7x_{21} + 5x_{22} + 3x_{23} + 2x_{24} + 9x_{31} + 4x_{32} + 6x_{33} + 7x_{41} + 2x_{42} + 7x_{43} + 6x_{44}$$

subject to,

(Operator assignment constraints)

$$x_{11} + x_{12} + x_{14} = 1$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} = 1$$
$$x_{41} + x_{42} + x_{43} + x_{44} = 1$$

(Machine assignment constraints)

$$x_{11} + x_{21} + x_{31} + x_{41} = 1$$
$$x_{12} + x_{22} + x_{32} + x_{42} = 1$$
$$x_{23} + x_{33} + x_{43} = 1$$
$$x_{14} + x_{24} + x_{44} = 1$$
$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

3.2 HUNGARIAN ALGORITHM

This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix. The **algorithm** can be started as follows :

(*a*) Bring at least one zero to each row and column of the cost matrix by subtracting the minimum of the row and column respectively.

(*b*) Cover all the zeros in cost matrix by ***minimum*** number of horizontal and vertical lines.

(*c*) If number of lines = order of the matrix, then select the zeros as many as the order of the matrix in such a way that they cover all the rows and columns.

(Here $A_{n \times n}$ means *n*th order matrix)

(*d*) If number of lines ≠ order of the matrix, then perform the following and create a new matrix :

(*i*) Select the minimum element from the uncovered elements of the cost matrix by the lines.

(*ii*) Subtract the uncovered elements from the minimum element.

(*iii*) Add the minimum element to the junction (*i.e.*, crossing of the lines) elements.

(*iv*) Other elements on the lines remain unaltered.

(*v*) Go to Step (*b*).

NOTES

Example 2. *A construction company has four engineers for designing. The general manager is facing the problem of assigning four designing projects to these engineers. It is also found that Engineer 2 is not competent to design project 4. Given the time estimate required by each engineer to design a given project, find an assignment which minimizes the total time.*

		Projects			
		P1	P2	P3	P4
	E1	6	5	13	2
Engineers	E2	8	10	4	–
	E3	10	3	7	3
	E4	9	8	6	2

Solution. Let us first bring zeros rowwise by subtracting the respective minima from all the row elements respectively.

4	3	11	0
4	6	0	–
7	0	4	0
7	6	4	0

Let us bring zero columnwise by subtracting the respective minima from all the column elements respectively. Here the above operations is to be performed only on first column, since at least one zero has appeared in the remaining columns.

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

(This completes Step-*a*)

Now (Step-*b*) all the zeros are to be covered by minimum number of horizontal and vertical lines which is shown below. It is also to be noted that this covering is not unique.

0	3	11	0
0	6	0	–
3	0	4	0
3	6	4	0

It is seen that number of lines = 4 = order of the matrix. Therefore by Step-*c*, we can go for assignment *i.e.*, we have to select 4 zeros such that they cover all the rows and columns which is shown below:

$\boxed{0}$	3	11	0
0	6	$\boxed{0}$	–
3	$\boxed{0}$	4	0
3	6	4	$\boxed{0}$

Therefore the optimal assignment is

$E1 \rightarrow P1$, $E2 \rightarrow P3$, $E3 \rightarrow P2$, $E4 \rightarrow P4$

and the minimum total time required = 6 + 4 + 3 + 2 = 15 units.

Example 3. *Solve the following job machine assignment problem. Cost data are given below :*

		Machines					
		1	2	3	4	5	6
	A	21	35	20	20	32	28
	B	30	31	22	25	28	30
	C	28	29	25	27	27	21
Jobs	D	30	30	26	26	31	28
	E	21	31	25	20	27	30
	F	25	29	22	25	30	21

Solution. Let us first bring zeros first rowwise and then columnwise by subtracting the respective minima elements from each row and each column respectively and the cost matrix, thus obtained, is as follows :

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

By Step-*b*, all the zeros are covered by minimum number of horizontal and vertical lines which is shown below :

NOTES

0	11	0	0	7	8
7	5	0	3	1	8
6	4	4	6	1	0
3	0	0	0	0	2
0	7	5	0	2	10
3	4	1	4	4	0

Here number of lines ≠ order of the matrix. Hence, we have to apply Step-*d*. The minimum uncovered element is 1. By applying Step-*d* we obtain the following matrix :

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, by Step-*b*, we cover all the zeros by minimum number of horizontal and vertical straight lines.

0	11	1	0	7	9
6	4	0	2	0	8
5	3	4	5	0	0
3	0	1	0	0	3
0	7	6	0	2	11
2	3	1	3	3	0

Now, the number of lines = order of the matrix. So, we can go for assignment by Step-*c*. The assignment is shown below :

NOTES

$\boxed{0}$	11	1	0	7	9
6	4	$\boxed{0}$	2	0	8
5	3	4	5	$\boxed{0}$	0
3	$\boxed{0}$	1	0	0	3
0	7	6	$\boxed{0}$	2	11
2	3	1	3	3	$\boxed{0}$

The optimal assignment is A→1, B→3, C→5, D→2, E→4, F→6. An alternative assignment is also obtained as A→4, B→3, C→5, D→2, E→1, F→6. For both the assignments, the minimum cost is 21 + 22 + 27 + 30 + 20 + 21 *i.e.,* ₹ 141.

3.3 UNBALANCED ASSIGNMENTS

For unbalanced or non-standard assignment problem number of rows ≠ number of columns in the assignment cost matrix *i.e.,* we deal with a rectangular cost matrix. To find an assignment for this type of problem, we have to first convert this unbalanced problem into a balanced problem by adding dummy rows or columns with zero costs so that the defective function will be unaltered. For machine-job problem, if number of machines (say, m) > number of jobs (say, n), then create m-n dummy jobs and the processing cost of dummy jobs as zero. When a dummy job gets assigned to a machine, that machine stays idle. Similarly the other case *i.e.,* $n > m$, is handled.

Example 4. *Find an optimal solution to an assignment problem with the following cost matrix :*

	M1	M2	M3	M4	M5
J1	13	5	20	5	6
J2	15	10	16	10	15
J3	6	12	14	10	13
J4	13	11	15	11	15
J5	15	6	16	10	6
J6	6	15	14	5	12

Solution. The above problem is unbalanced. We have to create a dummy machine M6 with zero processing time to make the problem as balanced assignment problem. Therefore, we obtain the following :

NOTES

	M1	M2	M3	M4	M5	M6 (dummy)
J1	13	5	20	5	6	0
J2	15	10	16	10	15	0
J3	6	12	14	10	13	0
J4	13	11	15	11	15	0
J5	15	6	16	10	6	0
J6	6	15	14	5	12	0

Let us bring zeros columnwise by subtracting the respective minima elements from each column respectively and the cost matrix, thus obtained, is as follows:

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Let us cover all the zeros by minimum number of horizontal and vertical lines and is given below :

7	0	6	0	0	0
9	5	2	5	9	0
0	7	0	5	7	0
7	6	1	6	9	0
9	1	2	5	0	0
0	10	0	0	6	0

Now, the number of lines ≠ order of the matrix. The minimum uncovered element by the lines is 1. Using Step-*d* of the Hungarian algorithm and covering all the zeros by minimum number of lines we obtain as follows :

7	0	6	0	1	1
8	4	1	4	9	0
0	7	0	5	8	1
6	5	0	5	9	0
8	0	1	4	0	0
0	10	0	0	7	1

Now, the number of lines = order of the matrix and we have to select 6 zeros such that they cover all the rows and columns. This is done in the following :

7	$\boxed{0}$	6	0	1	1
8	4	1	4	9	$\boxed{0}$
$\boxed{0}$	7	0	5	8	1
6	5	$\boxed{0}$	5	9	0
8	0	1	4	$\boxed{0}$	0
0	10	0	$\boxed{0}$	7	1

Therefore, the optimal assignment is

J1→M2, J2→M6, J3→M1, J4→M3, J5→M5, J6→M4 and the minimum cost = ₹ (5 + 0 + 6 + 15 + 6 + 5) = ₹ 37.

In the above, the job J2 will not get processed since the machine M6 is dummy.

3.4 MAX-TYPE ASSIGNMENT PROBLEMS

When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix. Then the minimization of the loss matrix is the same as the maximization of the profit matrix.

Example 5. *A company is faced with the problem of assigning 4 jobs to 5 persons. The expected profit in rupees for each person on each job are as follows :*

	Job			
Persons	J1	J2	J3	J4
I	86	78	62	81
II	55	79	65	60
III	72	65	63	80
IV	86	70	65	71
V	72	70	71	60

Find the assignment of persons to jobs that will result in a maximum profit.

Solution. The above problem is unbalanced max-type assignment problem. The maximum element is 86. By subtracting all the elements from it obtain the following opportunity loss matrix.

NOTES

0	8	24	5
31	7	21	26
14	21	23	6
0	16	21	15
14	16	15	26

Now, a dummy job J5 is added with zero losses. Then bring zeros in each column by subtracting the respective minimum element from each column we obtain the following matrix.

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Let us cover all the zeros by minimum number of lines and is as follows:

0	1	9	0	0
31	0	6	21	0
14	14	8	1	0
0	9	6	10	0
14	9	0	21	0

Since, the number of lines = order of the matrix, we have to select 5 zeros such that they cover all the rows and columns. This is done in the following :

0	1	9	$\boxed{0}$	0
31	$\boxed{0}$	6	21	0
14	14	8	1	$\boxed{0}$
$\boxed{0}$	9	6	10	0
14	9	$\boxed{0}$	21	0

The optimal assignment is

I→J4, II→J2, III→J5, IV→J1, V→J3 and maximum profit = ₹ (81 + 79 + 86 + 71) = ₹ 317. Here person III is idle.

Note. The max type assignment problem can also be converted to a minimization problem by multiplying all the elements of the profit matrix by – 1. Then the Hungarian method can be applied directly.

PROBLEMS

NOTES

1. Solve the following assignment problems:

(*a*)

	A	B	C	D	E
I	12	20	20	18	17
II	20	12	5	11	8
III	20	5	12	5	9
IV	18	11	5	12	10
V	17	8	9	10	12

(*b*)

Jobs

		J1	J2	J3	J4	J5	J6
	A	18	10	25	10	11	22
	B	20	15	21	–	20	18
	C	11	17	19	15	18	17
Persons	D	18	16	20	16	20	21
	E	20	–	21	15	11	17
	F	11	15	19	12	15	20

2. A machine tool decides to make six sub-assemblies through six contractors A, B, C, D, E and F. Each contractor is to receive only one sub-assembly from A1, A2, A3, A4, A5 and A6. But the contractors C and E are not competent for the A4 and A2 assembly respectively. The cost of each subassembly by the bids submitted by each contractor is shown below (in hundred rupees) :

	A1	A2	A3	A4	A5	A6
A	15	10	11	18	13	22
B	9	12	18	10	14	11
C	9	15	11	–	22	11
D	14	13	9	12	15	10
E	10	–	11	22	13	18
F	10	14	15	12	13	14

Find the optimal assignments of the assemblies to contractors so as to minimize the total cost.

3. Five programmers, in a computer centre, write five programmes which run successfully but with different times. Assign the programmers to the programmes in a such a way that the total time taken by them is minimum taking the following time matrix:

Programmes

		P1	P2	P3	P4	P5
	A	80	66	65	65	73
	B	76	75	70	70	75
Programmers	C	74	73	72	70	66
	D	75	75	71	71	73
	E	76	66	66	70	75

NOTES

4. Consider the problem of assigning seven jobs to seven persons. The assignment costs are given as follows :

		Jobs						
		I	II	III	IV	V	VI	VII
Persons	A	9	6	12	11	13	15	11
	B	14	13	14	14	10	20	15
	C	18	6	17	11	15	13	11
	D	10	11	12	15	15	14	13
	E	15	6	18	15	10	14	12
	F	9	18	15	20	14	13	11
	G	14	15	12	13	11	17	20

Determine the optimal assignment schedule.

5. Solve the following unbalanced assignment problems :

(*a*)

		Machines				
		M1	M2	M3	M4	M5
Jobs	J1	12	15	15	13	14
	J2	8	14	11	15	8
	J3	15	13	11	11	13
	J4	8	12	8	11	10

(*b*)

		Machines			
		M1	M2	M3	M4
Jobs	J1	16	17	14	19
	J2	15	16	14	9
	J3	10	15	9	9
	J4	10	13	10	13
	J5	15	9	14	18

6. There are five operators and six machines in a machine shop. The assignment costs are given in the table below :

		Machine					
		M1	M2	M3	M4	M5	M6
Operator	A	5	–	22	6	8	6
	B	14	9	15	9	14	15
	C	8	12	12	10	8	5
	D	11	13	11	6	9	14
	E	8	9	11	13	–	12

Operator A cannot operate machine M2 and operator E cannot operate machine M5. Find the optimal assignment schedule.

7. A batch of 4 jobs can be assigned to 5 different machines. The setup time for each job on various machines is given below :

		Machine				
		1	2	3	4	5
Jobs	J1	3	9	6	5	6
	J2	4	5	5	7	4
	J3	5	5	3	4	4
	J4	6	8	4	5	5

Find an optimal assignment of jobs to machines which will minimize the total setup time.

8. A construction company has to move six large cranes from old construction sites to new construction sites. The distances (in miles) between the old and the new sites are given below :

		New sites				
		A	B	C	D	E
	I	12	9	8	11	7
	II	11	10	8	12	7
Old	III	9	12	7	6	9
sites	IV	9	8	11	10	10
	V	10	9	9	6	11
	VI	11	11	7	8	9

Determine a plan for moving the cranes such that the total distance involved in the move will be minimum.

9. A company wants to assign five salesperson to five different regions to promote a product. The expected sales (in thousand) are given below :

		Regions				
		I	II	III	IV	V
	S1	27	54	37	100	85
	S2	55	66	45	80	32
Salesperson	S3	72	58	74	80	85
	S4	39	88	74	59	72
	S5	72	66	45	69	85

Solve the above assignment problem to find the maximum total expected sale.

10. A company makes profit (₹) while processing different jobs on different machines (one machine to one job only). Now, the company is facing problem of assigning 4 machines to 5 jobs. The profits are estimated as given below :

		Job				
		J1	J2	J3	J4	J5
	A	21	16	35	42	16
	B	15	20	30	35	15
Machine	C	20	16	30	27	18
	D	15	18	32	27	15

Determine the optimal assignment for maximum total profits.

ANSWERS

1. (*a*) I→A, II→E, III→D, IV→C, V→B, Min. cost = 38.

   (*b*) A→J6, B→J2, C→J1, D→J3, E→J5, F→J4, Min. cost = 91.

2. A→A2, B→A4, C→A1, D→A6, E→A3, F→A5

   A→A2, B→A4, C→A6, D→A3, E→A1, F→A5

A→A2, B→A4, C→A6, D→A3, F→A5, E→A1

For each assignment, min. cost = ₹ 6300.

NOTES

3. A→P3, B→P4, C→P5, D→P1, E→P2

   A→P4, B→P3, C→P5, D→P1, E→P2

   Min. total time = 342 units.

4. A→VII, B→V, C→IV, D→I, E→II, F→VI, G→III

   A→VII, B→V, C→IV, D→VI, E→II, F→I, G→III

   A→I, B→V, C→IV, D→VI, E→II, F→VII, G→III

   Min. total cost = 73.

5. (*a*) J1→M1, J2→M5, J3→M4, J4→M3, M2 is idle.

   Min. total cost = 39.

   (*b*) J2→M4, J3→M3, J4→M1, J5→M2, J1 is not processed.

   Min. total cost = 37.

6. A→M5, B→M2, C→M6, D→M4, E→M1, M3 is idle.

   Min. total cost = 36.

7. J1→1, J2→5, J3→3, J4→4

   J1→1, J2→5, J3→4, J4→3

   Min. total time = 15, Machine 2 is idle.

8. I→E, III→A, IV→B, V→D, VI→C

   Min. total distance = 37 miles

   Crane II is not moved.

9. S1→IV, S2→I, S3→III, S4→II, S5→5, Max. total profit = ₹ 40200.

10. A→J4, B→J2, C→J1, D→J3, Job J5 is idle.

    Max. total profit = ₹ 114.

3.5 ROUTING PROBLEMS

There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)'. Suppose there is a network of n cities and a salesman wants to make a tour *i.e.*, starting from a city 1 he will visit each of the other $(n - 1)$ cities once and will return to city 1. In this tour the objective is to minimize either the total distance travelled or the cost of travelling by the salesman.

(a) Mathematical Formulation

Let the cities be numbered as 1, 2, ..., n and the distance matrix as follows:

$$D = \begin{array}{c} \text{From} \backslash \text{To} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 \cdots & n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} \cdots & d_{1n} \\ d_{21} & d_{22} \cdots & d_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} \cdots & d_{nn} \end{bmatrix} \end{array}$$

Generally an infinity symbol is placed in the principal diagonal elements where there is no travelling. So d_{ij} represents the distance from city i to city j $(i \neq j)$. If the cost of travelling is considered then D is referred as cost matrix. It is also to be noted that D may be symmetric in which case the problem is called 'Symmetric TSP' or asymmetric in which case the problem is called 'Asymmetric TSP'.

Let us define the decision variables as follows :

$$x_{ij} = \begin{cases} 1, \text{ if he travels from city } i \text{ to city } j \\ 0, \text{ otherwise} \end{cases}$$

where $i, j = 1, 2, ..., n$

Then the linear programming formulation can be stated as follows :

Minimize $$z = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}\, x_{ij}$$

Subject to, $$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, ..., n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, ..., n$$

$$x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j = 1, 2, ..., n$$

and $$x = (x_{ij}) \text{ is a tour.}$$

The above problem has been solved with various approaches *e.g.*, Graph Theoretic Approach, Dynamic Programming, Genetic algorithm etc.

The above problem looks like a special type of Assignment problem. Consider a 4×4 assignment problem and a solution as $1 - 4, 2 - 3, 3 - 1, 4 - 2$ which can also be viewed as a tour *i.e.*, $1 - 4 - 2 - 3 - 1$. If the solution is $1 - 4, 2 - 3, 3 - 2, 4 - 1$ then this consists of two sub-tours $1 - 4 - 1, 2 - 3 - 2$.

Here one algorithm known as 'Branch and Bound' algorithm is described below :

(b) Branch and Bound Algorithm for TSP

(*i*) Ignoring tour, solve [D] using Hungarian Algorithm. The transformed matrix is denoted as $[D_0]$. If there is a tour, stop, else goto next step while storing the solution in a node denoted by TSP.

(*ii*) Calculate the **evaluation** for the variables in $[D_0]$ whose values are zero *i.e.*, $x_{ij} = 0$ where evaluation means the sum of smallest elements of the i-th row and the j-th column excluding the (i, j)th entry.

(*iii*) Select the variable with highest evaluation, say x_{ij}. If there is a tie, break it arbitrarily. The variable x_{ij} is called the branching variable.

(*iv*) Create a left branch (TSP1) with $x_{ij} = 0$. To implement this put $d_{ij} = \infty$ in $[D_0]$ *i.e.*, travelling from city i to city j is restricted.

Set [D] = transformed $[D_0]$ and goto step (*i*).

(*v*) Create a right branch (TSP2) with $x_{ij} = 1$. This means the salesman must visit city j from city i. To implement this take $[D_0]$ of the parent node.

Delete the i-th row and j-th column and put $d_{ji} = \infty$ (to prevent a subtour).

Set [D] = transformed [D_0] and goto step (i).

NOTES

Note.

(a) There may be a situation arises in step (i) where further solution is not possible then we shall stop that branch.

(b) There may be multiple tours. We shall select the tour with minimum distance or travelling cost.

(c) Calculate total distance (TD) from the given [D] which increases with the level of the tree.

Example 6. *Solve the following travelling salesman problem using branch and bound algorithm.*

$$D = \begin{array}{c|c} \text{From} \backslash \text{To} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \infty & 3 & 6 & 5 \\ 3 & \infty & 5 & 8 \\ 6 & 5 & \infty & 2 \\ 5 & 8 & 2 & \infty \end{bmatrix} \end{array}$$

Solution. Let us apply the Hungarian Algorithm on [D] and obtain the following matrix :

D_0 =

	1	2	3	4
1	∞	0	3	2
2	0	∞	2	5
3	4	3	∞	0
4	3	6	0	∞

The solution is 1 – 2, 2 – 1, 3 – 4, 4 – 3. *i.e.*, there exists two subtours 1 – 2 – 1, 3 – 4 –3. The total distance (TD) = 3 + 3 + 2 + 2 = 10 units.

Then we have to calculate the evaluations for the variables having the value zero in [D_0].

Variable	Evaluation
x_{12}	2 + 3 = 5
x_{21}	2 + 3 = 5
x_{34}	3 + 2 = 5
x_{43}	3 + 2 = 5

Since there are ties in the values, let us select x_{12} as branching variable.

Subproblem TSP1

Let $x_{12} = 0 \Rightarrow$ Put $d_{12} = \infty$ in [D_0] and obtain

NOTES

$$D_0 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & \infty & 3 & 2 \\ 2 & 0 & \infty & 2 & 5 \\ 3 & 4 & 3 & \infty & 0 \\ 4 & 3 & 6 & 0 & \infty \\ \hline \end{array}$$

$$\Downarrow$$

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & \infty & \infty & 1 & 0 \\ 2 & 0 & \infty & 2 & 5 \\ 3 & 4 & 0 & \infty & 0 \\ 4 & 3 & 1 & 0 & \infty \\ \hline \end{array} \quad \text{(Apply Hungarian Algorithm)}$$

The solution is 1 – 4, 2 – 1, 3 – 2, 4 – 3 *i.e.*, 1 – 4 – 3 – 2 – 1 which is a tour and TD = 5 + 3 + 5 + 2 = 15 units from [D].

Subproblem TSP2

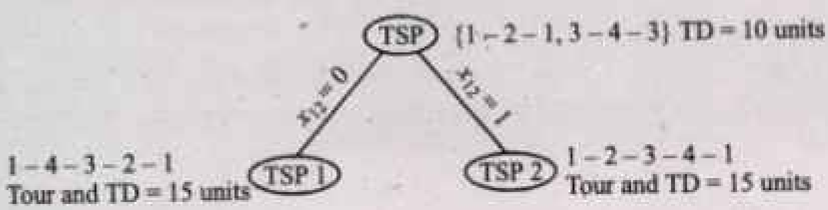
Let $x_{12} = 1 \Rightarrow$ Delete row 1 and column 2 from $[D_0]$ and put $d_{21} = \infty$ to prevent subtour. The resultant transformed matrix is obtained as follows :

$$\begin{array}{c|ccc|} & 1 & 3 & 4 \\ \hline 2 & \infty & 2 & 5 \\ 3 & 4 & \infty & 0 \\ 4 & 3 & 0 & \infty \\ \hline \end{array}$$

$$\Downarrow$$

$$\begin{array}{c|ccc|} & 1 & 3 & 4 \\ \hline 2 & \infty & 0 & 3 \\ 3 & 1 & \infty & 0 \\ 4 & 0 & 0 & \infty \\ \hline \end{array} \quad \text{(Applying Hungarian Algorithm)}$$

The solution is 1 – 2, 2 – 3, 3 – 4, 4 – 1 *i.e.*, 1 – 2 – 3 – 4 – 1 which is a tour and TD = 3 + 5 + 2 + 5 = 15 units from [D]. The above calculations is presented in the following tree diagram :



Since, there are two tours with same TD, the given problem has two solutions.

GAME THEORY

NOTES

3.6 INTRODUCTION

The mathematical theory of games was invented by John Von Neumann and Oskar Morgenstern (1944). Game theory is the study of the ways in which strategic interactions among rational players produce **outcomes** with respect to the **preferences** (or utilities) of those players, none of which might have been intended by any of them.

Game theory has found its applications in various fields such as Economics, Social Science, Political Science, Biology, Computer Science etc.

The famous example of a game is the **Prisoner's Dilemma** game. Suppose that the police have arrested two people whom they know have committed an armed robbery together. Unfortunately they lack enough admissible evidence to get a jury to convict. They do, however, have enough evidence to send each prisoner away for two years for theft. The chief inspector now makes the following offer to each prisoner. If you will confess the robbery implicating your partner and he does not also confess, then you shall go free and he will get ten years. If you both confess, you shall each get 5 years. If neither of you confess, then you shall each get two years for the theft.

3.7 BASIC DEFINITIONS

We assume that players are economically rational *i.e.*, a player can (*i*) assess outcomes, (*ii*) choose actions that yield their most preferred outcomes, given the actions of the other players.

(*i*) **Game :** All situations in which at least one player can only act to maximize his utility through anticipating the responses to his actions by one or more other players is called a game.

(*ii*) **Strategy :** A strategy is a possible course of action open to the player.

(*iii*) **Pure strategy :** A pure strategy is defined by a situation in which a course of action is played with probability one.

(*iv*) **Mixed strategy :** A mixed strategy is defined by a situation in which no course of action is taken with probability one.

(*v*) **Payoff matrix (or Reward matrix) :** A payoff matrix is an array in which any (*i, j*)th entry shows the outcome. Positive entry is the gain and negative entry is the loss for the row-player.

Matrix games are referred to as 'normal form' or 'strategic form' games, and games as trees are referred to as 'extensive form' games. The two sorts of games are not equivalent.

(vi) **Maximin criterion :** This is a criterion in which a player will choose the strategies with the largest possible payoff given an opponent's set of minimising countermoves.

(vii) **Minimax criterion :** This is a criterion in which a player will choose the strategies with the smallest possible payoff given an opponent's set of maximising countermoves.

(viii) **Saddle point :** If a payoff matrix has an entry that is simultaneously a maximum of row minima and a minimum of column maxima, then this entry is called a **saddle point** of the game and the game is said to be **strictly determined.**

(ix) **Value of the game :** If the game has a saddle point then the value at that entry is called the value of the game. If this value is zero then the game is said to be **fair.**

(x) **Zero-sum game :** A zero-sum game is a game in which the interests of the players are diametrically opposed *i.e.*, what one player wins the other loses. When two person play such game then it is called **two person zero-sum game.**

In this chapter, we shall consider only matrix games.

Note. If in a game the total payoff to be divided among players is invariant *i.e.*, it does not depend upon the mix of strategies selected, then the game is called **constant-sum** game.

3.8 TWO-PERSON ZERO-SUM GAME WITH PURE STRATEGIES

To identify the saddle point and value of game the following procedure to be adopted on the payoff matrix :

(i) Identify the minimum from each row and place a symbol * in that cell/entry.

Take the maximum of these minima.

(ii) Identify the maximum from each column and place a symbol × in that cell/entry.

Take the minimum of these maxima.

(iii) If both the symbols * and × occurs in a/an cell/entry, then that cell/entry is called saddle point and the value in that cell/entry is called value of the game (v).

Also v = Maximum (row minima) = Minimum (column maxima). There may be more than one saddle point but the value of the game is unique.

Example 7. *Solve the following game :*

		Player B			
		B1	*B2*	*B3*	*B4*
	A1	1	5	4	2
Player A	*A2*	2	3	5	3
	A3	3	4	5	3

NOTES

Solution. The calculations are displayed in the following table :

		Player B				
		B1	B2	B3	B4	Min.
	A1	1•	5×	4	2	1
Player A	A2	2•	3	5×	3×	2
	A3	3•×	4	5×	3•×	3
Max.		3	5	5	3	

Max. (Row Min.) = 3,

Min. (Column Max.) = 3

In the above game, there are two saddle points at (A3, B1) and (A3, B4).

The value of the game is 3. Here the optimal strategy for player A is A3 and the optimal strategy for player B is B1 and B4.

Example 8. *Determine the solution of the following game :*

		Player B				
		B1	*B2*	*B3*	*B4*	*B5*
	A1	*0*	*1*	*7*	*8*	*2*
Player A	*A2*	*6*	*4*	*5*	*5*	*4*
	A3	*7*	*3*	*2*	*1*	*2*
	A4	*1*	*4*	*1*	*4*	*5*

Solution. In the given game, player A has 4 strategies and player B has 5 strategies. The calculations are displayed in the following table :

	B1	*B2*	*B3*	*B4*	*B5*	Row Min.
A1	0*	1	7×	8×	2	0
A2	6	4*×	5	5	4*	4
A3	7×	3	2	1*	2	1
A3	1*	4×	1*	4	5×	1
Col. Max.	7	4	7	8	5	

Max. (Row Min.) = 4, Min. (Col Max.) = 4

In this game there is one saddle point at (A2, B2)

The value of the game is 4.

The optimal strategy for player A is A2

and the optimal strategy for player B is B2.

3.9 TWO-PERSON ZERO-SUM GAME WITH MIXED STRATEGIES

NOTES

Consider the following game :

		Player B	
		I	II
Player A	I	a_{11}	a_{12}
	II	a_{21}	a_{22}

If this game does not have saddle point, then we assume that both players use mixed strategies.

Let player A select strategy I with probability p and strategy II with probability $1 - p$. Suppose player B select strategy I, then the expected gain to player A is given by $a_{11}\, p + a_{21}\, (1 - p)$.

If player B select strategy II, then the expected gain to player A is given by $a_{12}\, p + a_{22}\, (1 - p)$.

The optimal plan for player A requires that its expected gain to be equal for each strategies of player B. Thus we obtain

$$a_{11}\, p + a_{21}\, (1 - p) = a_{12}\, p + a_{22}\, (1 - p)$$

$$\Rightarrow \qquad p = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}.$$

Similarly, let player B selects strategy I with probability q and strategy II with probability $1-q$. The expected loss to player B with respect to the strategies of player A are

$$a_{11}\, q + a_{12}\, (1 - q) \text{ and } a_{21}\, q + a_{22}\, (1 - q).$$

By equating the expected losses of player B we obtain

$$q = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}.$$

The value of game v is found by substituting the value of p in one of the equations for the expected gain of A and on simplification, we obtain

$$v = \frac{a_{11}\, a_{22} - a_{12}\, a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}.$$

Example 9. *Determine the solution of the following game :*

$$\begin{array}{cc} & \textit{Player B} \\ & \begin{array}{cc} B1 & B2 \end{array} \\ \textit{Player A} \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \end{array}$$

Solution. Clearly the given game has no saddle point. So the players have to use mixed strategies.

Let the mixed strategies for A as $S_A = \begin{pmatrix} A1 & A2 \\ p_1 & p_2 \end{pmatrix}$

where $$p_2 = 1 - p_1$$

and the mixed strategies for B as $S_B = \begin{pmatrix} B1 & B2 \\ q_1 & q_2 \end{pmatrix}$

NOTES

where $$q_2 = 1 - q_1$$

$$\therefore \quad p_1 = \frac{4-2}{(3+4)-(1+2)} = \frac{2}{4} = \frac{1}{2}, p_2 = 1 - p_1 = \frac{1}{2}$$

$$q_1 = \frac{4-1}{(3+4)-(1+2)} = \frac{3}{4}, q_2 = 1 - q_1 = \frac{1}{4}$$

$$v = \frac{12-2}{(3+4)-(1+2)} = \frac{10}{4} = \frac{5}{2}.$$

Thus the optimal strategy for A is $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}$

and for B is $S_B = \begin{pmatrix} B1 & B2 \\ 3/4 & 1/4 \end{pmatrix}$

and the value of the game is 5/2.

3.10 ODDS METHOD

This method can be used only for 2 × 2-matrix games. In this method, we ensure that sum of column odds and row odds is equal.

Finding out Odds

Step I. Take first row and find out the difference between values of cell (1, 1) and that of cell (1, 2) place this value in front of second row on right side.

Step II. Take second row, find out the difference between the value of cell (2, 1) and that of cell (2, 2). Place it in front of the first row on the right side.

Step III. Take first column, find out the difference between the value of cell (1, 1) and that of value of cell (2, 1). Place it below the second column.

Step IV. Take second column, find out the difference between the value of cell (1, 2) and that of the value of cell (2, 2). Place this value below the first column.

Example 10. *Consider a modified form of "matching biased coins" game problem. The matching player is paid ₹ 8 if two coins turn both heads and ₹ 1 if the coins turn both tail. The non-matching player is paid ₹ 3 when the two coins do not match. Given the choice of being the matching or non-matching players, which one would you choose and what would be your strategy?*

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Solution. Let us prepare the pay-off matrix.

		Player B	
		$\boxed{\text{H}}$	1
Player A	H	8	−3
	T	−3	$\boxed{1}$

Let us see if the saddle point exists. Minimum of row one is −3 and similarly minimum of row two is also −3, a circle has been put around these figures. Maximum of column is 8 and that of column 2 is 1. A square has been put around these two figures. There is no value, which is the lowest in its row and maximum of its column. Hence no saddle point exists.

So, both the player will use mixed strategy.

Use of Odds Method

		Player B		
		B_1	B_2	
		H	T	Odds
Player A	A_1	8	−3	4
	A_2	−3	1	11
	Odds	4	11	

(a) Take first row – difference between the cell A_1 B_1 and A_1 B_2
$8 - (-3) = 8 + 3 = 11$ place it in front of second row.

(b) Take second row – difference between the cell A_2 B_1 and A_2 B_2
$-3 - 1 = -4$ (ignore sign)

(c) Take first column $-3 - 1 = -4$ (ignore sign)

(d) Take second column $8 - (-3) = 11$

Value of the Game

For finding out the value of the game, following formula is used:

		Player B		
		B_1	B_2	Odds
Player A	A_1	a_1	a_2	$(b_1 - b_2)$
	A_2	b_1	b_2	$(a_1 - a_2)$
	Odds	$(a_2 - b_2)$	$(a_1 - b_1)$	

$$\text{Value V} = \frac{a_1(b_1 - b_2) + b_1(a_1 - a_2)}{(b_1 - b_2) + (a_1 - a_2)}$$

$$\text{Probability of } A_1 = \frac{b_1 - b_2}{(b_1 - b_2) + (a_1 - a_2)},\ A_2 = \frac{a_1 - a_2}{(b_1 - b_2) + (a_1 - a_2)}$$

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$$\text{Probability of } B_1 = \frac{a_2 - b_2}{(a_2 - b_2) + (a_1 - b_1)},\ B_2 = \frac{a_1 - b_1}{(a_2 - b_2) + (a_1 - b_1)}$$

$$\text{Game value} = \frac{8 \times 4 - 3 \times 11}{4 + 11} = \frac{-1}{15}$$

$$\text{Probabilities of } A_1 = \frac{4}{15},\ A_2 = \frac{11}{15}$$

$$\text{Probabilities of } B_1 = \frac{4}{15},\ B_2 = \frac{11}{15}$$

3.11 DOMINANCE RULES

(a) For rows : (*i*) In the payoff matrix if all the entries in a row i_1 are **greater than or equal** to the corresponding entries of another row i_2, then row i_2 is said to be dominated by row i_1. In this situation row i_2 of the payoff matrix can be deleted.

e.g., $i_2 = (1, 2, -1)$ is dominated by $i_1 = (2, 2, 1)$, hence $(1, 2, -1)$ can be deleted.

(*ii*) If sum of the entries of any two rows is greater than or equal to the corresponding entry of a third row, then that third row is said to be dominated by the above two rows and hence third row can be deleted.

(b) For columns : (*i*) In the payoff matrix if all the entries in a column j_1 are less than or equal to the corresponding entries of another column j_2, then column j_2 is said to be dominated by column j_1. In this situation column j_2 of the payoff matrix can be deleted.

e.g., $j_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is dominated by $j_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ can be deleted.

(*ii*) If sum of the entries of any two columns is less than or equal to the corresponding entry of a third column, then that third column is said to be dominated by the above two columns and hence third column can be deleted.

Example 11. *Using the rules for dominance solve the following game :*

$$\begin{array}{cc} & \textit{Player B} \\ & \begin{array}{ccc} I & II & III \end{array} \\ \textit{Player A} \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 5 & 2 & -2 \\ 2 & 3 & -1 \\ 3 & -2 & 3 \end{bmatrix} \end{array}$$

Solution. The given game has no saddle point. Let us apply the rules for dominance. It is observed that column 1 is dominated by column 3. Hence delete column 1 and the payoff matrix is reduced as follows :

$$\begin{array}{cc} & \begin{array}{cc} II & III \end{array} \\ \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 2 & -2 \\ 3 & -1 \\ -2 & 3 \end{bmatrix} \end{array}$$

NOTES

Again, row 1 is dominated by row 2. Hence delete row 1 and the payoff matrix is reduced to a 2 × 2 matrix.

$$\begin{array}{c} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{cc} \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \end{array}$$

Let the mixed strategy for player A be $S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ 0 & p_1 & p_2 \end{pmatrix}$ with $p_2 = 1 - p_1$ and the mixed strategy for player B be

$$S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ 0 & q_1 & q_2 \end{pmatrix} \text{ with } q_2 = 1 - q_1$$

$$p_1 = \frac{3-(-2)}{(3+3)-(-1-2)} = \frac{5}{9}, p_2 = \frac{4}{9}$$

$$q_1 = \frac{3-(-1)}{(3+3)-(-1-2)} = \frac{4}{9}, q_2 = \frac{5}{9}$$

$$v = \frac{9-2}{(3+3)-(-1-2)} = \frac{7}{9}.$$

Hence the optimal mixed strategies are

$$S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ 0 & 5/9 & 4/9 \end{pmatrix},$$

$$S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ 0 & 4/9 & 5/9 \end{pmatrix},$$

$$v = 7/9.$$

Example 12. *Solve the following game:*

$$\begin{array}{cc} & \textit{Player B} \\ & \begin{array}{cccc} I & II & III & IV \end{array} \\ \textit{Player A} \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 4 & 3 & 0 & 3 \\ 3 & 4 & 3 & 0 \\ 4 & 3 & 4 & 3 \\ 0 & 5 & 4 & 4 \end{bmatrix} \end{array}$$

Solution. The given game has no saddle point. Let us apply the rules for dominance to reduce the size of the payoff matrix. It is observed that row II is dominated by row III, hence row I can be deleted and the payoff matrix reduces as follows :

$$\begin{array}{c} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} 3 & 4 & 3 & 0 \\ 4 & 3 & 4 & 3 \\ 0 & 5 & 4 & 4 \end{bmatrix} \end{array}$$

It is observed that column III is dominated by column I. Hence column III can be deleted. Also it is observed that column II is dominated by column IV. Hence column IV can also be deleted. Hence the payoff matrix reduces as follows :

$$\begin{array}{c} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}\begin{array}{c} \begin{array}{cc} \text{I} & \text{IV} \end{array} \\ \begin{bmatrix} 3 & 0 \\ 4 & 3 \\ 0 & 4 \end{bmatrix} \end{array}$$

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Here row II is dominated by row III. Hence row II can be deleted and the payoff matrix reduces to 2 × 2 matrix.

$$\begin{array}{c} \\ \text{III} \\ \text{IV} \end{array}\begin{array}{c} \begin{array}{cc} \text{I} & \text{IV} \end{array} \\ \begin{bmatrix} 4 & 3 \\ 0 & 4 \end{bmatrix} \end{array}$$

Let the mixed strategies for A be $S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & p_1 & p_2 \end{pmatrix}$

with $\qquad p_2 = 1 - p_1$

and the mixed strategies for B be $S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ q_1 & 0 & 0 & q_2 \end{pmatrix}$

with $\qquad q_2 = 1 - q_1.$

$$\therefore \quad p_1 = \frac{4-0}{(4+4)-(0+3)} = \frac{4}{5}, \; p_2 = 1 - p_1 = \frac{1}{5}$$

$$q_1 = \frac{4-3}{(4+4)-(0+3)} = \frac{1}{5}, \; q_2 = 1 - q_1 = \frac{4}{5}$$

$$v = \frac{16-0}{(4+4)-(0+3)} = \frac{16}{5}$$

Hence the optimal mixed strategies are

$$S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & 0 & 4/5 & 1/5 \end{pmatrix},$$

$$S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 1/5 & 0 & 0 & 4/5 \end{pmatrix}$$

and $\qquad v = \frac{16}{5}.$

Note. If we add a fixed number x to each element of the payoff matrix, then the strategies remain unchanged while the value of the game is increased by x.

3.12 GRAPHICAL METHOD FOR GAMES

(a) Let us consider a $2 \times n$ game *i.e.*, the payoff matrix will consist of 2 rows and n columns. So player A (or, row-player) will have two strategies. Also assume that there is no saddle point. Then the problem can be solved by using the following procedure :

(*i*) Reduce the size of the payoff matrix using the rules of dominance, if it is applicable.

(*ii*) Let p be the probability of selection of strategy I and $1 - p$ be the probability of selection of strategy II by player A.

Write down the expected gain function of player A with respect to each of the strategies of player B.

(*iii*) Plot the gain functions on a graph. Keep the gain function on y-axis and p on x-axis. Here p will take the value 0 and 1.

(*iv*) Find the **highest intersection point** in the **lower boundary** (*i.e.*, lower envelope) of the graph. Since player A is a maximin player, then this point will be a maximin point.

(*v*) If the number of lines passing through the maximin point is only two, then obtain a 2×2 payoff matrix by retaining the columns corresponding to these two lines. Go to step (*vii*) else go to step (*vi*).

(*vi*) If more than two lines passing through the maximin point then identify two lines with opposite slopes and form the 2×2 payoff matrix as described in step (*v*).

(*vii*) Solve the 2×2 game.

Example 13. *Consider the following game and solve it using graphical method.*

		Player B				
		I	*II*	*III*	*IV*	*V*
Player A	*I*	3	1	6	–1	5
	II	–2	4	–1	2	1

Solution. It is observed that there is no saddle point. Column V is dominated by column I and column II is dominated by column IV. Therefore delete column V and column II and the payoff matrix is reduced as follows :

		Player B		
		I	III	IV
Player A	I	3	6	–1
	II	–2	–1	2

Let p be the probability of selection of strategy I and $(1-p)$ be the probability of selection of strategy II by player A. Therefore, the expected gain (or payoff) function to player A with respect to different strategies of player B is given as follows:

B's strategy	A's expected gain function	A's expected gain $p = 0$	$p = 1$
I	$3p - 2(1-p) = 5p - 2$	–2	3
III	$6p - (1-p) = 7p - 1$	–1	6
IV	$-p + 2(1-p) = -3p + 2$	2	–1

Now the A's expected gain function is plotted in Fig. 3.1. It is observed that line I and IV passes through the highest point of the lower boundary. Hence, we can form 2 × 2 payoff matrix by taking the columns due to I and IV for player A and it is displayed below :

NOTES

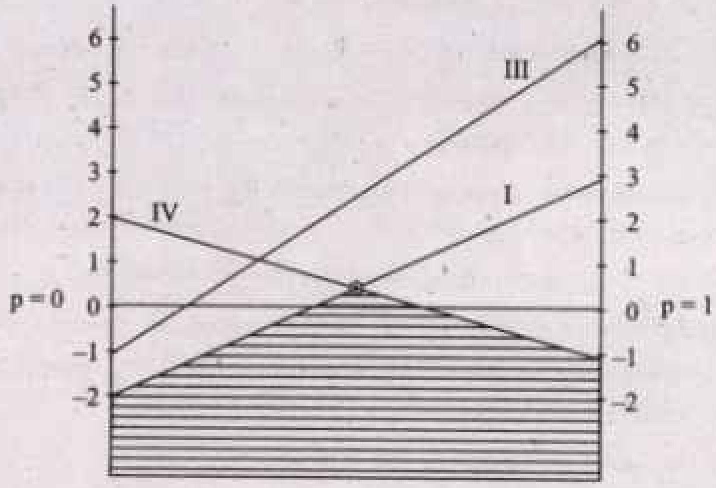

Fig. 3.1

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} \text{I} & \text{IV} \end{array} \\ \text{Player A} & \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \end{array}$$

Let the mixed strategies for A be $S_A = \begin{pmatrix} \text{I} & \text{II} \\ p_1 & p_2 \end{pmatrix}$

with $\qquad p_2 = 1 - p_1$

and the mixed strategies for B be $S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ q_1 & 0 & 0 & q_2 & 0 \end{pmatrix}$

with $\qquad q_2 = 1 - q_1$

Therefore,

$$p_1 = \frac{2-(-2)}{(3+2)-(-1-2)} = \frac{1}{2}, p_2 = 1 - p_1 = \frac{1}{2}$$

$$q_1 = \frac{2-(-1)}{(3+2)-(-1-2)} = \frac{3}{8}, q_2 = 1 - q_1 = \frac{5}{8}$$

$$v = \frac{6-2}{(3+2)-(-1-2)} = \frac{1}{2}$$

∴ The optimal mixed strategies for A is

$$S_A = \begin{pmatrix} \text{I} & \text{II} \\ 1/2 & 1/2 \end{pmatrix}$$

NOTES

the optimal mixed strategies for B is

$$S_B = \begin{pmatrix} I & II & III & IV & V \\ 3/8 & 0 & 0 & 5/8 & 0 \end{pmatrix}$$

and value of game $= \frac{1}{2}$.

(b) Let us consider a $m \times 2$ game *i.e.*, the payoff matrix will consist of m rows and 2 columns. Also assume that there is no saddle point. Then the problem can be solved by using the following procedure :

(*i*) Reduce the size of the payoff matrix using the rules of dominance, if it is applicable.

(*ii*) Let q be the probability of selection of strategy I and $1-q$ be the probability of selection of strategy II by the player B.

Write down the expected gain function of player B with respect to each of the strategies of player A.

(*iii*) Plot the gain functions on a graph. Keep the gain function on y-axis and q on x-axis. Here q will take the value 0 and 1.

(*iv*) Find the **lowest intersection point** in the **upper boundary** (*i.e.*, upper envelope) of the graph. Since player B is a minimax player, then this point will be a minimax point.

(*v*) If the number of lines passing through the minimax point is only two, then obtain a 2×2 payoff matrix by retaining the rows corresponding to these two lines. Go to step (*vii*) else goto step (*vi*).

(*vi*) If more than two lines passing through the minimax point then identify two lines with opposite slopes and form a 2×2 payoff matrix as described in step (*v*).

(*vii*) Solve the 2×2 game.

Example 14. *Consider the following game and solve it using graphical method.*

$$\begin{array}{cc} & \textit{Player B} \\ & \begin{array}{cc} I & II \end{array} \\ \textit{Player A} \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & -1 \\ 5 & -2 \end{bmatrix} \end{array}$$

Solution. The given game does not have saddle point. Also it is observed that none of the rows can be deleted using the rules of dominance.

Let q be the probability of selection of strategy I and $1-q$ be the probability of selection of strategy II by player B. Therefore, the expected gain (or payoff) function to player B with respect to different strategies of player A is given below:

NOTES

A's strategy	B's expected gain function	B's expected gain $q = 0$	$q = 1$
I	$2q + (1 - q) = q + 1$	1	2
II	$q + 3(1 - q) = -2q + 3$	3	1
III	$4q - (1 - q) = 5q - 1$	–1	4
IV	$5q - 2(1 - q) = 7q - 2$	–2	5

Now the B's expected gain function is plotted in Fig. 3.2.

It is observed that the line II and IV passes through the lowest point of the upper boundary. Hence, we can form 2 × 2 payoff matrix by taking the rows due to II and IV for player B and it is displayed below :

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \text{Player A} & \begin{array}{c} \text{II} \\ \text{IV} \end{array} & \begin{array}{|cc|} \hline 1 & 3 \\ 5 & -2 \\ \hline \end{array} \end{array}$$

Let the mixed strategies for A be $S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} & \text{IV} \\ 0 & p_1 & 0 & p_2 \end{pmatrix}$

with $$p_2 = 1 - p_1$$

and the mixed strategies for B be $S_B = \begin{pmatrix} \text{I} & \text{II} \\ q_1 & q_2 \end{pmatrix}$

with $$q_2 = 1 - q_1$$

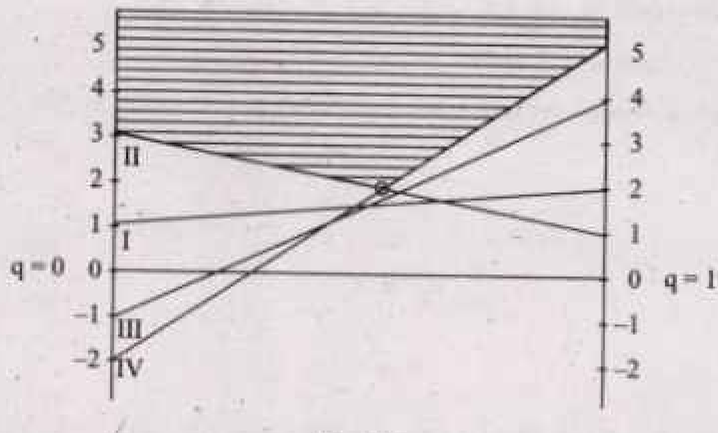

Fig. 3.2

Therefore,
$$p_1 = \frac{-2-5}{(1-2)-(5+3)} = \frac{7}{9}, p_2 = 1 - p_1 = \frac{2}{9}$$

$$q_1 = \frac{-2-3}{(1-2)-(5+3)} = \frac{5}{9}, q_2 = 1 - q_1 = \frac{4}{9}$$

$$v = \frac{-2-15}{(1-2)-(5+3)} = \frac{17}{9}$$

∴ The optimal mixed strategies for A is

$$S_A = \begin{pmatrix} I & II & III & IV \\ 0 & 7/9 & 0 & 2/9 \end{pmatrix}$$

NOTES

the optimal mixed strategies for B is

$$S_B = \begin{pmatrix} I & II \\ 5/9 & 4/9 \end{pmatrix} \text{ and value of game } = \frac{1}{2}.$$

3.13 LINEAR PROGRAMMING METHOD FOR GAMES

The linear programming method is used in solving mixed strategies games of dimensions greater than (2 × 2) size. Consider an $m \times n$ payoff matrix in which player A (*i.e.*, the row player) has m strategies and player B (*i.e.*, the column player) has n strategies. The elements of payoff matrix be $\{(a_{ij}); i = 1, 2, ..., m \text{ and } j = 1, 2, ..., n\}$. Let p_i be the probability of selection of strategy i by player A and q_j be the probability of selection of strategy j by player B.

LPP FOR PLAYER A

B's strategy	Expected gain function for A
1	$\sum_{i=1}^{m} a_{i1} p_i$
2	$\sum_{i=1}^{m} a_{i2} p_i$
– – –	– – –
n	$\sum_{i=1}^{m} a_{in} p_i$

Let $$v = \text{Min.} \left\{ \sum_i a_{i1} p_i, \sum_i a_{i2} p_i, ..., \sum_i a_{in} p_i \right\}$$

Since the player A is maximin type, the LPP can be written as follows :

Maximize v

Subject to,

$$\sum_i a_{i1} p_i \geq v$$

$$\sum_i a_{i2} p_i \geq v$$

– – – – – –

$$\sum_i a_{in} p_i \geq v$$

$$p_1 + p_2 + \ldots + p_m = 1$$

$$\text{all } p_i \geq 0$$

⇒ Maximize v

NOTES

Subject to, $\sum_i a_{i1}(p_i / v) \geq 1$

$\sum_i a_{i2}(p_i / v) \geq 1$

$- - - - - -$

$\sum_i a_{in}(p_i / v) \geq 1$

$$\frac{p_1}{v} + \frac{p_2}{v} + ... + \frac{p_m}{v} = 1$$

all $\qquad p_i \geq 0$

Set $\qquad p_i/v = x_i, i = 1, 2, ..., m$. Therefore

$$\text{Maximize } v = \text{Minimize } \left(\frac{1}{v}\right)$$

$$= \text{Minimize } \left(\frac{p_1}{v} + \frac{p_2}{v} + ... + \frac{p_m}{v}\right)$$

$$= \text{Minimize } (x_1 + x_2 + ... + x_m)$$

Subject to, $\sum_i a_{i1} x_i \geq 1$

$\sum_i a_{i2} x_i \geq 1$

$- - - - - -$

$\sum_i a_{in} x_i \geq 1$

and $\qquad x_i \geq 0, i = 1, 2, ..., m.$

LPP FOR PLAYER B

A's strategy	Expected loss/gain function to B
1	$\sum_j a_{1j} q_j$
2	$\sum_j a_{2j} q_j$
- - - - -	- - - - -
m	$\sum_j a_{mj} q_j$

Let $\qquad u = \text{Max.} \left\{\sum_j a_{1j} q_j, \sum_j a_{2j} q_j, ..., \sum_j a_{mj} q_j\right\}$

Since the player B is minimax type, the LPP can be written as follows :

Minimize u

Subject to, $\sum_j a_{1j} q_j \leq u$

$\sum_j a_{2j} q_j \leq u$

$- - - - - -$

$\sum_j a_{mj} q_j \leq u$

$$q_1 + q_2 + \ldots + q_n = 1$$

all $\quad q_j \geq 0$

$\Rightarrow$ Minimize u

Subject to, $\quad \sum_j a_{1j}(q_j/u) \leq 1$

$$\sum_j a_{2j}(q_j/u) \leq 1$$

$$\text{------}$$

$$\sum_j a_{mj}(q_j/u) \leq 1$$

$$\frac{q_1}{u} + \frac{q_2}{u} + \ldots + \frac{q_n}{u} = 1$$

all $\quad q_j \geq 0$

Set $\quad q_j/u = y_j, j = 1, 2, \ldots, n$. Therefore

$$\text{Minimize } u = \text{Maximize}\left(\frac{1}{u}\right)$$

$$= \text{Maximize}\left(\frac{q_1}{u} + \frac{q_2}{u} + \ldots + \frac{q_n}{u}\right)$$

$$= \text{Maximize } (y_1 + y_2 + \ldots + y_n)$$

Subject to, $\quad \sum_j a_{1j} y_j \leq 1$

$$\sum_j a_{2j} y_j \leq 1$$

$$\text{-----}$$

$$\sum_j a_{mj} y_j \leq 1$$

and $\quad y_j \geq 0, \ j = 1, 2, \ldots, n.$

Note. 1. In the above approach we may face two problems. Value of the game may be zero or less than zero. First case constraints will become infinite and in the second case, the type of each constraint will get changed. Therefore to obtain a non-negative value of the game, a constant c = Max. {abs. (negative values)} + 1 is to be added to each elements in the payoff matrix. The optimal strategy will not change. However the value of the original game will be the value of the new game minus constant.

2. The above LPP formulations for player A and B are primal-dual pair. So solving one problem, we can read the solution of the other problem from the optimal table.

Example 15. *Solve the following game by linear programming technique :*

$$\begin{array}{cc} & \textit{Player B} \\ \textit{Player A} & \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$$

NOTES

Solution. The game has no saddle point. Since the payoff matrix has negative values, let us add a constant $c = 2$ to each element. The revised payoff matrix is given below:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 4 \\ 3 & 3 & 1 \end{bmatrix} \end{array}$$

Let the strategies of the two players be

$$S_A = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ p_1 & p_2 & p_3 \end{pmatrix}, \quad S_B = \begin{pmatrix} \text{I} & \text{II} & \text{III} \\ q_1 & q_2 & q_3 \end{pmatrix}$$

where $\quad p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$.

The LPP for player A:

$$\text{Maximize } v = \text{Minimize } \frac{1}{v} = x_1 + x_2 + x_3$$

Subject to,
$$\begin{aligned} x_1 + x_2 + 3x_3 &\geq 1 \\ x_1 + 3x_2 + 3x_3 &\geq 1 \\ 3x_1 + 4x_2 + x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

where $\quad x_j = p_j/v, j = 1, 2, 3.$

The LPP for player B:

$$\text{Minimize } u = \text{Maximize } \frac{1}{u} = y_1 + y_2 + y_3$$

Subject to,
$$\begin{aligned} y_1 + y_2 + 3y_3 &\leq 1 \\ y_1 + 3y_2 + 4y_3 &\leq 1 \\ 3y_1 + 3y_2 + y_3 &\leq 1 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

where $\quad y_j = q_j/u, j = 1, 2, 3.$

Let us now solve the problem for player B.

The standard form can be written as follows :

$$\text{Maximize } \frac{1}{u} = y_1 + y_2 + y_3 + 0.s_1 + 0.s_2 + 0.s_3$$

Subject to,
$$\begin{aligned} y_1 + y_2 + 3y_3 + s_1 &= 1 \\ y_1 + 3y_2 + 4y_3 + s_2 &= 1 \\ 3y_1 + 3y_2 + y_3 + s_3 &= 1 \end{aligned}$$

$$y_1, y_2, y_3 \geq 0, s_1, s_2, s_3 \text{ slacks} \geq 0.$$

Iteration 1

		c_j	1	1	1	0	0	0	Min.	
c_B	x_B	Soln.	y_1	y_2	y_3	s_1	s_2	s_3	Ratio	
0	s_1	1	1	1	3	1	0	0	1	
0	s_2	1	1	3	4	0	1	0	1	
0	s_3	1	3	3	1	0	0	1	1/3	→
$z_j - c_j$			–1	–1	–1	0	0	0		
			↑							

Iteration 2

		c_j	1	1	1	0	0	0	Min.	
c_B	x_B	Soln.	y_1	y_2	y_3	s_1	s_2	s_3	Ratio	
0	s_1	2/3	0	0	8/3	1	0	–1/3	$\frac{2}{8}=\frac{1}{4}$	
0	s_2	2/3	0	2	11/3	0	1	–1/3	2/11	→
1	y_1	1/3	1	1	1/3	0	0	1/3	1	
$z_j - c_j$			0	0	–2/3	0	0	1/3		
					↑					

Iteration 3

		c_j	1	1	1	0	0	0	Min.
c_B	x_B	Soln.	y_1	y_2	y_3	s_1	s_2	s_3	Ratio
0	s_1	2/11	0	–16/11	0	1	–8/11	–1/11	
1	y_3	2/11	0	6/11	1	0	3/11	–1/11	
1	y_1	3/11	1	9/11	0	0	–1/11	4/11	
$z_j - c_j$			0	4/11	0	0	2/11	3/11	

$\therefore\ y_1^* = \frac{3}{11}, y_2^* = 0, y_3^* = \frac{2}{11}$, Max. $\frac{1}{u} = \frac{5}{11} \Rightarrow u^* = \frac{11}{5} = v^*$,

original $u^* = \frac{11}{5} - 2 = \frac{1}{5} =$ original v^*

Using duality, $x_1^* = 0,\ x_2^* = \frac{2}{11},\ x_3^* = \frac{3}{11}$

Now, $q_1 = y_1.u = \frac{3}{11}.\frac{11}{5} = \frac{3}{5}$, $p_1 = x_1 . v = 0$

$q_2 = y_2.u = 0$, $p_2 = x_2.v = \frac{2}{11}.\frac{11}{5} = \frac{2}{5}$

$q_3 = y_3.u = \frac{2}{11}.\frac{11}{5} = \frac{2}{5}$, $p_3 = x_3.v = \frac{3}{11}.\frac{11}{5} = \frac{3}{5}$

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$\therefore$

$$S_A = \begin{pmatrix} I & II & III \\ 0 & 2/5 & 3/5 \end{pmatrix},$$

$$S_B = \begin{pmatrix} I & II & III \\ 3/5 & 0 & 2/5 \end{pmatrix} \text{ and } v^* = \frac{1}{5}.$$

3.14 SUMMARY

- This is an efficient algorithm for solving the assignment problem developed by the Hungarian mathematician König. Here the optimal assignment is not affected if a constant is added or subtracted from any row or column of the balanced assignment cost matrix.
- When the objective of the assignment is to maximize, the problem is called 'Max-type assignment problem'. This is solved by converting the profit matrix to an opportunity loss matrix by subtracting each element from the highest element of the profit matrix.
- There are various types of routing problems which occurs in a network. The most widely discussed problem is the 'Travelling Salesman Problem (TSP)'.
- All situations in which at least one player can only act to maximize his utility through anticipating the responses to his actions by one or more other players is called a game.
- A mixed strategy is defined by a situation in which no course of action is taken with probability one.
- If a payoff matrix has an entry that is simultaneously a maximum of row minima and a minimum of column maxima, then this entry is called a **saddle point** of the game and the game is said to be **strictly determined.**
- If the game has a saddle point then the value at that entry is called the value of the game.
- A zero-sum game is a game in which the interests of the players are diametrically opposed *i.e.*, what one player wins the other loses. When two person play such game then it is called **two person zero-sum game.**
- The linear programming method is used in solving mixed strategies games of dimensions greater than (2×2) size.

3.15 REVIEW QUESTIONS

Solve the following travelling salesman problems using branch and bound algorithm :

1.

		To			
		1	2	3	4
From	1	∞	10	6	4
	2	8	∞	5	8
	3	7	5	∞	2
	4	4	10	2	∞

NOTES

2.

From \ To	1	2	3	4	5
1	∞	4	9	5	10
2	4	∞	7	6	8
3	10	6	∞	5	4
4	5	6	5	∞	3
5	8	7	4	3	∞

3.

From \ To	1	2	3	4	5
1	∞	5	2	7	4
2	5	∞	3	5	6
3	2	3	∞	4	1
4	7	5	4	∞	3
5	4	6	1	3	∞

4. Solve the following games :

(*a*)

Player A \ Player B	B1	B2	B3	B4
A1	0	1	–4	6
A2	3	4	4	5
A3	2	0	3	–2

(*b*)

	B1	B2	B3	B4	B5
A1	2	0	–2	0	5
A2	1	1	4	1	2
A3	5	0	–2	–3	–2
A4	–4	–2	4	4	2

(*c*)

	B1	B2	B3	B4	B5
A1	2	2	1	3	5
A2	3	4	7	5	2
A3	5	5	8	5	6
A4	2	8	0	2	9

(*d*)

	B1	B2	B3	B4	B5	B6
A1	3	2	–1	1	5	–2
A2	2	4	0	–2	3	1
A3	5	1	2	1	2	3

NOTES

5. Determine the values of a and b such that the following game is determinable :

	B1	B2	B3
A1	2	a	3
A2	–3	2	b
A3	1	5	4

6. Determine the value of a such that the following game is determinable:

	B1	B2	B3
A1	a	1	2
A2	0	–2	a
A3	3	a	1

7. Solve the folloiwng two person zero-sum games :

(*a*)

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} B1 & B2 \end{array} \\ \text{Player A} & \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} B1 & B2 \end{array} \\ \text{Player A} & \begin{array}{c} A1 \\ A2 \end{array} & \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} \end{array}$$

8. Using the rules for dominance solve the following games :

(*a*)

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cccc} I & II & III & IV \end{array} \\ \text{Player A} & \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 3 & 0 & -3 & 1 \\ 11 & 2 & 0 & 5 \\ -1 & 5 & 1 & -4 \\ 6 & 6 & -4 & 1 \end{bmatrix} \end{array}$$

(*b*)

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{ccc} I & II & III \end{array} \\ \text{Player A} & \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} 3 & 8 & 2 \\ 8 & 3 & 7 \\ 7 & 2 & 6 \end{bmatrix} \end{array}$$

(*c*)

$$\begin{array}{cc} & \begin{array}{cccc} B1 & B2 & B3 & B4 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \end{array} & \begin{bmatrix} -1 & 6 & 2 & 1 \\ 3 & 2 & 4 & 5 \\ 2 & 0 & 3 & 6 \end{bmatrix} \end{array}$$

(*d*)

$$\begin{array}{cc} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} -3 & 3 & 2 & -5 & 4 \\ -3 & 4 & 7 & 0 & 7 \\ 0 & 1 & 4 & 8 & 7 \end{bmatrix} \end{array}$$

(*e*)

$$\begin{array}{cc} & \begin{array}{ccccc} B1 & B2 & B3 & B4 & B5 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 \\ 4 & 3 & 1 & 3 & 2 \\ 4 & 3 & 4 & -1 & 2 \end{bmatrix} \end{array}$$

(*f*)

$$\begin{array}{cc} & \begin{array}{ccccc} B1 & B2 & B3 & B4 & B5 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \end{array} & \begin{bmatrix} 8 & 7 & 6 & -1 & 2 \\ 12 & 10 & 12 & 0 & 4 \\ 14 & 6 & 8 & 14 & 16 \end{bmatrix} \end{array}$$

NOTES

9. Use dominance property to reduce the game in 2 × 4 game and then solve graphically

Player B

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{Player A I} & 4 & 0 & 5 & -1 \\ \text{II} & 0 & 2 & -1 & 3 \\ \text{III} & -2 & 0 & -3 & 1 \end{array}$$

10. Solve the following games graphically :

(*a*) Player B

$$\begin{array}{c|ccc} & \text{I} & \text{II} & \text{III} \\ \hline \text{Player A I} & 3 & 8 & 5 \\ \text{II} & 6 & 2 & 7 \\ \text{III} & 4 & 5 & 6 \end{array}$$

(*b*)

$$\begin{array}{c|cccc} & B1 & B2 & B3 & B4 \\ \hline A1 & 2 & 2 & 3 & -1 \\ A2 & 4 & 3 & 2 & 6 \end{array}$$

(*c*)

$$\begin{array}{c|cc} & B1 & B2 \\ \hline A1 & 2 & 7 \\ A2 & 3 & 5 \\ A3 & 11 & 2 \end{array}$$

(*d*)

$$\begin{array}{c|cc} & B1 & B2 \\ \hline A1 & 1 & -3 \\ A2 & 3 & 5 \\ A3 & -1 & 6 \\ A4 & 4 & 1 \\ A5 & 2 & 2 \\ A6 & -5 & 0 \end{array}$$

11. Solve the following games by linear programming :

(*a*) Player B

$$\text{Player A} \begin{bmatrix} 1 & 6 \\ 4 & -5 \\ -5 & 3 \end{bmatrix}$$

(*b*) Player B

$$\text{Player A} \begin{array}{c|ccc} & B1 & B2 & B3 \\ \hline A1 & 1 & -1 & 2 \\ A2 & -2 & 3 & -1 \end{array}$$

(*c*) Player B

$$\text{Player A} \begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 0 & 2 \end{bmatrix}$$

(*d*) Player B

$$\text{Player A} \begin{bmatrix} -2 & 3 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

(*e*) Player B

$$\text{Player A} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -3 \end{bmatrix}$$

(*f*) Player B

$$\text{Player A} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

NOTES

(g)

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 4 & 2 & 6 \\ 6 & 8 & 0 \\ 9 & 5 & 1 \end{bmatrix} \end{array}$$

12. Two competitive brands rely on advertising for securing a greater shares of the market. They select three media: TV, Newspaper and Mobile Phone. The expected change in their market shares depends on the type of media chosen. Consider the following payoff matrix of brand A:

$$\begin{array}{ccc} & & \text{Brand B} \\ & & \begin{array}{ccc} \text{TV} & \text{Newspaper} & \text{Mobile Phone} \end{array} \\ \text{Brand A} & \begin{array}{r} \text{TV} \\ \text{Newspaper} \\ \text{Mobile Phone} \end{array} & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

Find the optimal solution for both brands.

13. An MNC has decided to establish a plant in either Singapore, Denmark or India. The degree of competition in the next five years is not certain. The company's expected return will depend on whether this competition is weak, mild or strong as shown in the following matrix:

$$\begin{array}{rc} & \begin{array}{ccc} \text{Weak} & \text{Mild} & \text{Strong} \end{array} \\ \begin{array}{r} \text{Singapore} \\ \text{Denmark} \\ \text{India} \end{array} & \begin{bmatrix} 16 & 13 & 4 \\ 13 & 11 & 6 \\ 11 & 9 & 8 \end{bmatrix} \end{array}$$

If the company's managing board is conservative, where should they decide to establish the plant?

ANSWERS

1. 1 – 4 – 3 – 2 – 1, TD = 19 units.
2. 1 – 4 – 5 – 3 – 2 – 1, TD = 22 units.
3. 1 – 3 – 5 – 4 – 2 – 1 and 1 – 2 – 4 – 5 – 3 – 1, TD = 16 units.
4. (*a*) Saddle Point (A2, B1), $v = 3$
   (*b*) Saddle Point (A2, B2), $v = 1$
   (*c*) Saddle Point (A3, B1) and (A3, B4), $v = 5$
   (*d*) Saddle Point (A3, B4), $v = 11$
5. $2 < a < 5$, $-3 < b < 4$ **[Hint.** Ignore a and b to find saddle point]
6. $a = 1$ [Saddle point at (A1, B2)]
7. (*a*) $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}$, $S_B = \begin{pmatrix} B1 & B2 \\ 1/4 & 3/4 \end{pmatrix}$, $v = 3.5$

   (*b*) $S_A = \begin{pmatrix} A1 & A2 \\ 4/7 & 3/7 \end{pmatrix}$, $S_B = \begin{pmatrix} B1 & B2 \\ 3/7 & 4/7 \end{pmatrix}$, $v = \frac{5}{7}$

8. (*a*) $S_A = \begin{pmatrix} I & II & III & IV \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV \\ 0 & 0 & \frac{9}{10} & \frac{1}{10} \end{pmatrix}, v = \frac{1}{2}$

(*b*) $S_A = \begin{pmatrix} I & II & III \\ 2/5 & 3/5 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III \\ 0 & 1/2 & 1/2 \end{pmatrix}, v = 5$

(*c*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 1/8 & 7/8 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}, v = 2.5$

(*d*) $S_A = \begin{pmatrix} I & II & III \\ 0 & 0 & 1 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV & V \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, v = 0$

(*e*) $S_A = \begin{pmatrix} A1 & A2 & A3 & A4 \\ 0 & 0 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 & B5 \\ 0 & 0 & \frac{4}{7} & \frac{3}{7} & 0 \end{pmatrix}, v = \frac{13}{7}$

(*f*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 0 & \frac{4}{9} & \frac{5}{9} \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 & B5 \\ 0 & \frac{7}{9} & 0 & \frac{2}{9} & 0 \end{pmatrix}, v = \frac{70}{9}$

9. $S_A = \begin{pmatrix} I & II & III \\ 3/8 & 5/8 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III & IV \\ 0 & 3/4 & 1/4 & 0 \end{pmatrix}, v = \frac{5}{4}$

10. (*a*) $S_A = \begin{pmatrix} I & II & III \\ 4/9 & 5/9 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II & III \\ 2/3 & 1/3 & 0 \end{pmatrix}, v = \frac{14}{3}$

(*b*) $S_A = \begin{pmatrix} A1 & A2 \\ 1/2 & 1/2 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 & B4 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}, v = \frac{5}{2}$

(*c*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 9/14 & 0 & 5/14 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 5/14 & 9/14 \end{pmatrix}, v = \frac{73}{14}$

(*d*) $S_A = \begin{pmatrix} A1 & A2 & A3 & A4 & A5 & A6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 4/5 & 1/5 \end{pmatrix}, v = \frac{17}{5}$

11. (*a*) $S_A = \begin{pmatrix} I & II & III \\ 9/14 & 5/14 & 0 \end{pmatrix}, S_B = \begin{pmatrix} I & II \\ 11/14 & 3/14 \end{pmatrix}, v = \frac{29}{14}$ (3It)

(*b*) $S_A = \begin{pmatrix} A1 & A2 \\ 5/7 & 2/7 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 4/7 & 3/7 & 0 \end{pmatrix}, v = \frac{1}{7}$ (4It)

(*c*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 1/2 & 1/2 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 \\ 3/4 & 1/4 \end{pmatrix}, v = \frac{5}{2}$ (3It)

(*d*) $S_A = \begin{pmatrix} A1 & A2 \\ 3/8 & 5/8 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 5/8 & 3/8 & 0 \end{pmatrix}, v = -\frac{1}{8}$ (3It)

(*e*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 5/12 & 5/12 & 1/6 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 4/9 & 2/9 & 1/3 \end{pmatrix}, v = \frac{1}{3}$ (4It)

(*f*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 2/9 & 5/9 & 2/9 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 2/9 & 5/9 & 2/9 \end{pmatrix}, v = \frac{1}{9}$ (4It)

(*g*) $S_A = \begin{pmatrix} A1 & A2 & A3 \\ 2/3 & 1/3 & 0 \end{pmatrix}, S_B = \begin{pmatrix} B1 & B2 & B3 \\ 0 & 1/2 & 1/2 \end{pmatrix}, v = 1$ (3It)

12. $S_A = \begin{pmatrix} T & N & M \\ 0 & 0 & 1 \end{pmatrix}, S_B = \begin{pmatrix} T & N & M \\ 2/5 & 3/5 & 0 \end{pmatrix}, v = 2$

13. In India with an expected return of 8.

CHAPTER 4 SEQUENCING PROBLEM AND QUEUING THEORY

NOTES

★ STRUCTURE ★

SEQUENCING PROBLEM

4.1 INTRODUCTION

A sequence is the order in which different jobs are to be performed. When there is a choice that a number of tasks can be performed in different orders, then the problem of sequencing arises. Such situations are very often encountered by manufacturing units, overhauling of equipments or aircraft engines, maintenance schedule of a large variety of equipment used in a factory, customers in a bank or car servicing garage and so on.

The basic concept behind sequencing is to use the available facilities in such a manner that the cost (and time) is minimized. The sequencing theory has been developed to solve difficult problems of using limited number of facilities in an optimal manner to get the best production and minimum costs.

Terms Commonly used

1. **Job** : These have to be sequenced, hence there should be a particular number of jobs (groups of tasks to be performed) say n to be processed.
2. **Machine** : Jobs have to be performed or processed on machines. It is a facility which has some processing capability.
3. **Loading** : Assigning of jobs to facilities and committing of facilities to jobs without specifying the time and sequence.
4. **Scheduling** : When the time and sequence of performing the job is specified, it is called *scheduling*.

5. **Sequencing :** Sequencing of operations refers to a systematic procedure of determining the order in which a series of jobs will be processed in a definite number, say k, facilities or machines.
6. **Processing Time :** Every operation that is required to be performed requires definite amount of time at each facility or machine when processing time is definite and certain, scheduling is easier as compared to the situation in which it is not known.
7. **Total Elapsed Time :** It is the time that lapses between the starting of first job and the completion of the last one.
8. **Idle Time :** The time for which the facilities or machine are not utilized during the total elapsed time.
9. **Technological Order :** It is the order which must be followed for completing a job. The requirement of the job dictates in which order various operations have to be performed, for example, painting cannot be done before welding.
10. **Passing not Allowed :** If '*n*' jobs have to be processed through '*m*' machines in a particular order of M_1 , M_2 , M_3 then each job will go to machine M_1 first and then to M_2 and finally to M_3. This order cannot be passed.
11. **Static Arrival Pattern :** If all the jobs to be done are received at the facilities simultaneously.
12. **Dynamic Arrival Pattern :** Here the jobs keep arriving continuously.

Assumptions

In sequencing problems, the following assumptions are made :

(*i*) All machines can process only one job at a time.

(*ii*) No time is wasted in shifting a job from one machine to other.

(*iii*) Processing time of job on a machine has no relation with the order in which the job is processed.

(*iv*) All machines have different capability and capacity.

(*v*) All jobs are ready for processing.

(*vi*) Each job when put on the machine is completed.

(*vii*) All jobs are processed in specified order as soon as possible.

4.2 TYPES OF SEQUENCING PROBLEMS

The following types of sequencing problems will be discussed in this chapter :

(*a*) *n* jobs one machine case

(*b*) *n* jobs two machines case

(*c*) *n* jobs '*m*' machine case

(*d*) Two jobs '*m*' machines case.

The solution of these problems depends on many factors such as :

(*a*) The number of jobs to be scheduled

(*b*) The number of machines in the machine shop

(*c*) Type of manufacturing facility (slow shop or fast shop) (*d*) Manner in which jobs arrive at the facility (static or dynamic)

(*e*) Criterion by which scheduling alternatives are to be evaluated.

NOTES

As the number of jobs (n) and the number of machines (m) increases, the sequencing problems become more complex. In fact, no exact or optimal solutions exist for sequencing problems with large n and m. Simulation seems to be a better solution technique for real life scheduling problems.

n-Jobs One Machine Case

This case of a number of jobs to be processed on one facility is very common in real life situations. The number of cars to be serviced in a garage, number of engines to be overhauled in one workshop, number of patients to be treated by one doctor, number of different jobs to be machined on a lathe, etc., are the cases which can be solved by using the method under study. In all such cases we are all used to *'first come first served'* principle to give sense of satisfaction and justice to the waiting jobs. But if this is not the consideration, it is possible to get more favourable results in the interest of effectiveness and efficiency. The following assumptions are applicable :

(*a*) The job shop is static.

(*b*) Processing time of the job is known.

The implication of the above assumption that job shop is static will mean that new job arrivals do not disturb the processing of n jobs already being processed and the new job **arrivals** wait to be attended to in next batch.

Shortest Processing Time (SPT) Rule

This rule says that jobs are sequenced in such a way that the job with least processing time is picked up first, followed by the job with the next Smallest Processing Time (SPT) and so on. This is referred to as *shortest processing time sequencing*. However, when the importance of the jobs to be performed varies, a different rule called Weight-Scheduling (Weight Scheduling Process Time) rule is used. Weights are allotted to jobs, greater weight meaning more important job. Let W_i be the weight allotted. By dividing the processing time by the weight factor, the tendency to move important job to an earlier position in the order is achieved.

$$\text{Weighted Mean low Time, (WMFT)} = \frac{\sum_{i=1}^{n} W_i f_i}{\sum_{i=1}^{n} W_i}$$

where f_i = flow time of job $i = W_i + t_i$

t_i = processing time of job i

WSPT rule for minimizing Weighted Mean-Flow Time (WMFT) puts n jobs in a sequence such that

$$\frac{t[1]}{W[1]} \leq \frac{t[2]}{W[2]} \leq \dots \leq \frac{t[n]}{W[n]}$$

The numbers in brackets above define the position of the jobs in the optimal sequence.

Example 1. *Consider the 8 jobs with processing times, due dates and importance weights as shown below.*

8 jobs one machine case data

Task (i)	Processing Time (t_i)	Due Date (d_i)	Importance Weight (W_i)	$\frac{t_i}{W_i}$
1	5	15	1	5.0
2	8	10	2	4.0
3	6	15	3	2.0
4	3	25	1	3.0
5	10	20	2	5.0
6	14	40	3	4.7
7	7	45	2	3.5
8	3	50	1	3.0

Solution. From processing time t_i in the table the SPT sequence is 4–8–1–3–7–2–5–6 resulting in completion of these jobs at times 3, 6, 14, 20, 27, 36, 46, 60 respectively.

$$\text{WMFT} = \frac{3+6+14+20+27+36+46+60}{8} = 26.5 \text{ hours}$$

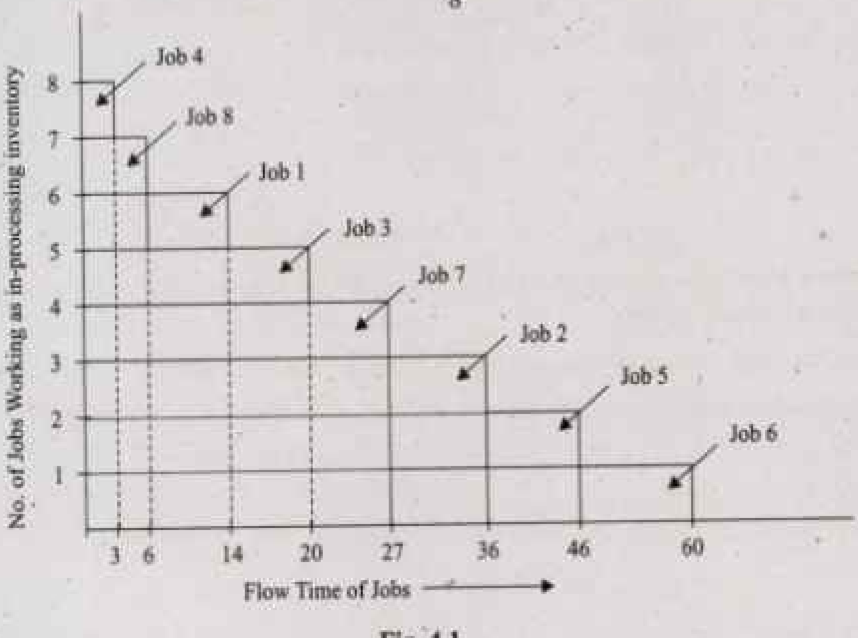

Fig. 4.1

The sequence is shown graphically above from which the number of tasks waiting as in-process inventory is seen to be 8 during 0–3, 7 during 3–6, 6 during 6–14, 5 during 14–20, 4 during 20–27, 3 during 27–36, 2 during 36–46 and one during 46–60. Thus, the average inventory is given by

$$\text{Average inventory} = \frac{8\times3+7\times3+6\times8+5\times6+4\times7+3\times9+2\times10+1\times14}{60}$$

$$= \frac{24+21+48+30+28+27+20+14}{60} = \frac{212}{60} = 3.53 \text{ jobs}$$

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Weight Scheduling Process Time

If the important weights W_i were to be considered the WSPT could be used to minimize the Weighted Mean Flow Time (WMFT) to yield the sequence 3–4–8–2–7–6–5–1. This results by first choosing job with minimum $\frac{t_i}{W_i}$ in the table. The respective flow time of jobs in this sequence are 6, 9, 12, 21, 28, 42, 52, 58. Mean flow time is hours

$$\text{WMFT} = \frac{6\times3+9\times1+12\times1+21\times3+28\times2+42\times3+52\times2+58\times1}{3+1+1+3+2+3+2+1}$$

$$= \frac{18+9+12+63+56+126+104+58}{16} = \frac{446}{16} = 27.85 \text{ hours}$$

Example 2. *Eight jobs A, B, C, D, E, F, G and H arrive at one time to be processed on a single machine. Find out the optimal job sequence, when their operation time is give in the table below.*

Job (n)	Operation Time in Minutes
A	16
B	12
C	10
D	8
E	7
F	4
G	2
H	1

Solution. For determining the optimal sequence, the jobs are selected in a non-descending operation time as follows :

Non-decreasing operation time sequence is H → G → F → E → D → C → B → A.

Total processing time

$H = 1$

$G = 1 + 2 = 3$

$F = 1 + 2 + 4 = 7$

$E = 1 + 2 + 4 + 7 = 14$

$D = 1 + 2 + 4 + 7 + 8 = 22$

$C = 1 + 2 + 4 + 7 + 8 + 10 = 32$

$B = 1 + 2 + 4 + 7 + 8 + 10 + 12 = 44$

$A = 1 + 2 + 4 + 7 + 8 + 10 + 12 + 16 = 60$

Average processing time = Total time/number of jobs = 183/8 = 23 minutes

In case the jobs are processed in the order of their arrival, *i.e.*, A → B → C → D → E → F → G → H the total processing time would have been as follows :

$A = 16$

$B = 16 + 12 = 28$

$C = 16 + 12 + 10 = 38$

$D = 16 + 12 + 10 + 8 = 46$

$E = 16 + 12 + 10 + 8 + 7 = 53$

$F = 16 + 12 + 10 + 8 + 7 + 4 = 57$

$G = 16 + 12 + 10 + 8 + 7 + 4 + 2 = 59$

$H = 16 + 12 + 10 + 8 + 7 + 4 + 2 + 1 = 60$

Average processing time = 357/8 = 44·6, which is much more than the previous time.

Priority Sequencing Rules

The following priority sequencing rules are generally followed in production/service system :

1. **First Come First Served (FCFS) :** As explained earlier, it is followed to avoid any heart burns and avoidable controversies.
2. **Earliest Due Date (EDD) :** In this rule, top priority is allotted to the waiting job, which has the earliest due/delivery date. In this case the order of arrival of the job and processing time it takes is ignored.
3. **Least Slack Rule (LS) :** It gives top priority to the waiting job whose slack time is the least. Slack time is the difference between the length of time remaining until the job is due and the length of its operation time.
4. **Average Number of Jobs in the System :** It is defined as the average number of jobs remaining in the system (waiting or being processed) from the beginning of sequence through the time when the last job is finished.
5. **Average Job Lateness :** Jobs lateness is defined as the difference between the actual completion time of the job and its due date. Average job lateness is sum of lateness of all jobs divided by the number of jobs in the system. This is also called *Average Job Tardiness*.
6. **Average Earliness of Jobs :** If a job is completed before its due date, the lateness value is negative and the magnitude is referred as earliness of job. Mean earliness of the job is the sum of earliness of all jobs divided by the number of jobs in the system.
7. **Number of Tardy Jobs :** It is the number of jobs which are completed after the due date.

Sequencing *n* Jobs through Two Machines

The sequencing algorithm for this case was developed by Johnson and is called *Johnson's Algorithm*. In this situation n jobs must be processed through machines M_1 and M_2. The processing time of all the n jobs on M_1 and M_2 is known and it is required to find the sequence, which minimizes the time to complete all the jobs.

Johnson's algorithm is based on the following assumptions :

(*i*) There are only two machines and the processing of all the jobs is done on both the machines in the same order, *i.e.*, first on M_1 and then on M_2.

(*ii*) All jobs arrive at the same time (static arrival pattern) have no priority for job completion.

Johnson's algorithm involves following steps :

1. List operation time for each job on machine M_1 and M_2.
2. Select the shortest operation or processing time in the above list.

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3. If minimum-processing time is on M_1, place the corresponding job first in the sequence. If it is on M_2, place the corresponding job last in the sequence. In case of tie in shortest processing time, it can be broken arbitrarily.
4. Eliminate the jobs which have already been sequenced as result of step 3.
5. Repeat steps 2 and 3 until all the jobs are sequenced.

Example 3. *Six jobs are to be sequenced, which require processing on two machines M_1 and M_2. The processing time in minutes for each of the six jobs on machines M_1 and M_2 is given below. All the jobs have to be processed in sequence M_1, M_2. Determine the optimum sequence for processing the jobs so that the total time of all the jobs is minimum. Use Johnson's algorithm.*

Jobs		1	2	3	4	5	6
Processing Time	Machine M_1	30	30	60	20	35	45
	Machine M_2	45	15	40	25	30	70

Solution.

Step I. Operation time or processing time for each jobs on M_1 and M_2 is provided in the question.

Step II. The shortest processing time is 15 for job 2 on M_2.

Step III. As the minimum processing time is on M_2, job 2 has to be kept last as follows :

					2

Step IV. We ignore job 2 and find out the shortest processing time of rest of jobs. Now the least processing time is 20 minutes on machine M_1 for job 4. Since it is on M_1, it is to be placed first as follows :

4					2

The next minimum processing time is 30 minutes for job 5 on M_2 and Job 1 on M_1. So, job 5 will be placed at the end. Job 1 will be sequenced earlier as shown below.

4	1			5	2

The next minimum processing time is 40 minutes for job 3 on M_2, hence it is sequenced as follows :

4	1		3	5	2

Job 6 has to be sequenced in the gap or vacant space. The complete sequencing of the jobs is as follows.

4	1	6	3	5	2

The minimum time for six jobs on machine M_1 and M_2 can be shown with the help of a Gantt chart as shown below.

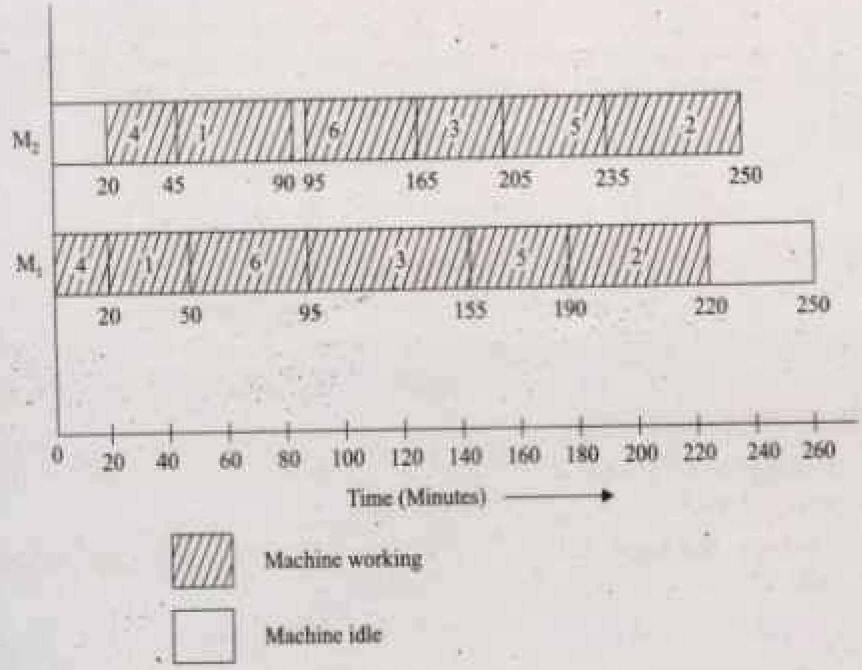

Fig. 4.2

The above figure shows idle time for M_1 (30 minutes) after the last job (2) has been processed. Idle time for M_2 is 20 minutes before job 4 is started and 5 minutes before processing 6 and finishing job 1. The percentage utilization of M_1 = 250 – 30/250 = 88% and M_2 = 250 – 25/250 = 90%.

Example 4. *A manufacturing company has 5 different jobs on two machines M_1 and M_2. The processing time for each of the jobs on M_1 and M_2 is given below. Decide the optimal sequence of processing of the jobs in order to minimize total time.*

Job No.	Processing Time	
	M_1	M_2
1	8	6
2	12	7
3	5	11
4	3	9
5	6	14

Solution. The shortest processing time is 3 on M_1 for job 4 so it will be sequenced as follows :

4				

Next is job 3 with time 5 and M_1 , hence job 3 will be sequenced as

4	3			

Next minimum time is for jobs 1 on M_2 this will be sequenced last.

4	3			1

NOTES

After eliminating jobs 4, 3 and 1, the next with minimum time is job 5 on M_1 so it will be placed as

4	3	5		1

Now, job 2 will be sequenced in the vacant space.

4	3	5	2	1

n Jobs 3 Machines Case

Johnson's algorithm which we have just applied can be extended and made use of in *n* jobs 3 machine case, if the following conditions hold good :

(*a*) Maximum processing time for a job on machine M_1 is greater than or equal to maximum processing time for the same job.

or

(*b*) Minimum processing time for a job on machine M_3 is greater than or equal to maximum processing time for a job on machine M_2.

The following assumptions are made :

(*a*) Every job is processed on all the three machines M_1, M_2 and M_3 in the same order, *i.e.*, the job is first processed on M_1 then on M_2 and then on M_3.

(*b*) The passing of jobs is not permitted.

(*c*) Processing time for each job on the machine M_1, M_2 and M_3 are known.

In this procedure two dummy machines M_1' and M_2' are assumed in such a manner that the processing time of jobs on these machines can be calculated as

Processing time of jobs on M_1' = Processing time $(M_1 + M_2)$

Processing time of a jobs on M_2' = Processing time $(M_2 + M_3)$

After this Johnson's algorithm is applied on M_1' and M_2' to find out the optimal sequencing of jobs.

Example 5. *In a manufacturing process three operations have to be performed on machines M_1, M_2 and M_3 in order M_1, M_2 and M_3. Find out the optimum sequencing when the processing time for four jobs on three machines is as follows :*

Job	M_1	M_2	M_3
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12

Solution.

Step I. As the minimum processing time for job 2 on M_1 > maximum processing time for job 2 on M_2, Johnson's algorithm can be applied to this problem.

Step II. Let us combine the processing time of M_1 and M_2 and M_3 to form two dummy machines M_1' and M_2'. This is shown matrix below.

Job	M'_1	M'_2
1	11 (3 + 8)	21 (8 + 13)
2	18 (12 + 6)	20 (6 + 14)
3	9 (5 + 4)	13 (4 + 9)
4	8 (2 + 6)	18 (6 + 12)

Step III. Apply Johnson's algorithm. Minimum time of 8 occurs for job 4 on M_1' hence it is sequenced first.

4	3	1	

The next minimum time is for job 3 on M_1' so it is sequenced next to job 4. Next is job 1 and so on. So the optimal sequencing is

4	3	1	2

Example 6. *Four Jobs 1, 2, 3 and 4 are to be processed on each of the five machines M_1, M_2, M_3, M_4 and M_5 in the order M_1, M_2, M_3, M_4, and M_5. Determine total minimum elapsed time if no passing off is allowed. Also find out the idle time of each of the machines. Processing time are given in the matrix below.*

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
1	8	4	6	3	9
2	7	6	4	5	10
3	6	5	3	2	8
4	9	2	1	4	6

Solution.

Step I. Find out if the condition minimum $e_i \geq$ maximum b_i, c_i and d_i is satisfied.

Job	Machines				
	M_1	M_2	M_3	M_4	M_5
1	8	4	6	3	9
2	7	6	4	5	10
3	6	5	3	2	8
4	9	2	1	4	6
	Minimum 6	Maximum 6	Maximum 6	Maximum 5	Minimum 6

This condition is satisfied hence we can convert the problem into four jobs and two fictitious machines M_1' and M_2'.

$$M_1' = a_i + b_i + c_i + d_i, \quad M_2' = b_i + c_i + d_i + e_i$$

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Step II.

Job	M_1'	M_2'
1	21 (8 + 4 + 6 + 3)	22 (4 + 6 + 3 + 9)
2	22 (7 + 6 + 4 + 5)	25 (6 + 4 + 5 + 10)
3	16 (6 + 5 + 3 + 2)	18 (5 + 3 + 2 + 8)
4	16 (9 + 2 + 1 + 4)	13 (2 + 1 + 4 + 6)

Step III. The optimal sequence can be determined as minimum of processing time of 13 occurs on M_2' for job 4 it will be processed last. Next minimum time is for job 3 on machine M_1' so it will be processed earliest. Next shortest time is for machine 1 on M_1', so it will be sequenced next to job 3 and so on.

3	1	2	4

Step IV. Total time can be calculated with the help of the matrix shown below.

Job	M_1		M_2		M_3		M_4		M_5	
	In	Out	In	Out	In	Out	In	Out	In	Out
1	0	8	8	12	12	18	18	21	21	30
2	8	15	15	21	21	25	25	30	30	40
3	15	21	21	26	26	29	29	32	40	48
4	21	30	30	32	30	31	32	36	48	54

Hence total minimum elapsed time is 51.

Idle time for machines $M_1 = 24$ hours

$M_2 = 3 + 4 + 22 = 29$

$M_3 = 3 + 1 + 1 + 23 = 28$

$M_4 = 4 + 18 = 22$

Two Jobs '*m*' Machines Case

1. Two axis to represent job 1 and 2 are drawn at right angles to each other. Same scale is used for X and Y-axes. X-axis represents the processing time and sequence of job 1 and Y-axis represents the processing time and sequence of job 2. The processing time on machines are laid out in the technological order of the problem.
2. The area which represents processing times of jobs 1 and 2 and is common to both the jobs is shaded. As processing of both the jobs on it machine is not feasible, the shaded area represents the unfeasible region in the graph.
3. The processing of both the jobs 1 and 2 are represented by a continued path which consists of horizontal, vertical and 45 degree diagonal region. The path starts at the lower left corner and stops at upper right corner and the shaded area is avoided. The path is not allowed to pass through shaded area which as brought out in step II represents both the jobs being processed simultaneously on the same machine.

   Any vertical movement represents that job 2 is in progress and job 1 is waiting to be processed. Horizontal movement along the path indicates that job 1 is in progress and job 2 is idle waiting to be processed. The diagonal movement of the path indicates that both the jobs are being processed on different machines simultaneously.

4. A feasible path maximizes the diagonal movement minimizes the total processing time.

5. Minimum elapsed time for any job = processing time of the job + idle time of the same job.

Example 7. *The operation time of two jobs 1 and 2 on 5 machines M_1, M_2, M_3, M_4 and M_5 is given in the following table. Find out the optimum sequence in which the jobs should be processed so that the total time used is minimum. The technological order of use of machine for job 1 is M_1, M_2, M_3, M_4 and M_5 for job 2 is M_3, M_1, M_4, M_5 and M_2.*

Time Hours

Job	M_1	M_2	M_3	M_4	M_5
1	1	2	3	5	1
Job	M_3	M_1	M_4	M_5	M_2
2	3	4	2	1	5

Solution. Job 1 preceeds job 2 on machine M_1, job 1 preceeds job 2 on machine M_2, job 2 preceeds job 1 on machine M_3, job 1 preceeds job 2 on M_4 and job 2 preceeds job 1 on M_5.

The minimum processing time for jobs 1 and 2, total processing time for job 1 + idle time for Job 1 = 12 + 3 = 15 hours.

Total processing time for job 2 + idle time for job 2 = 15 + 0 = 15 hours.

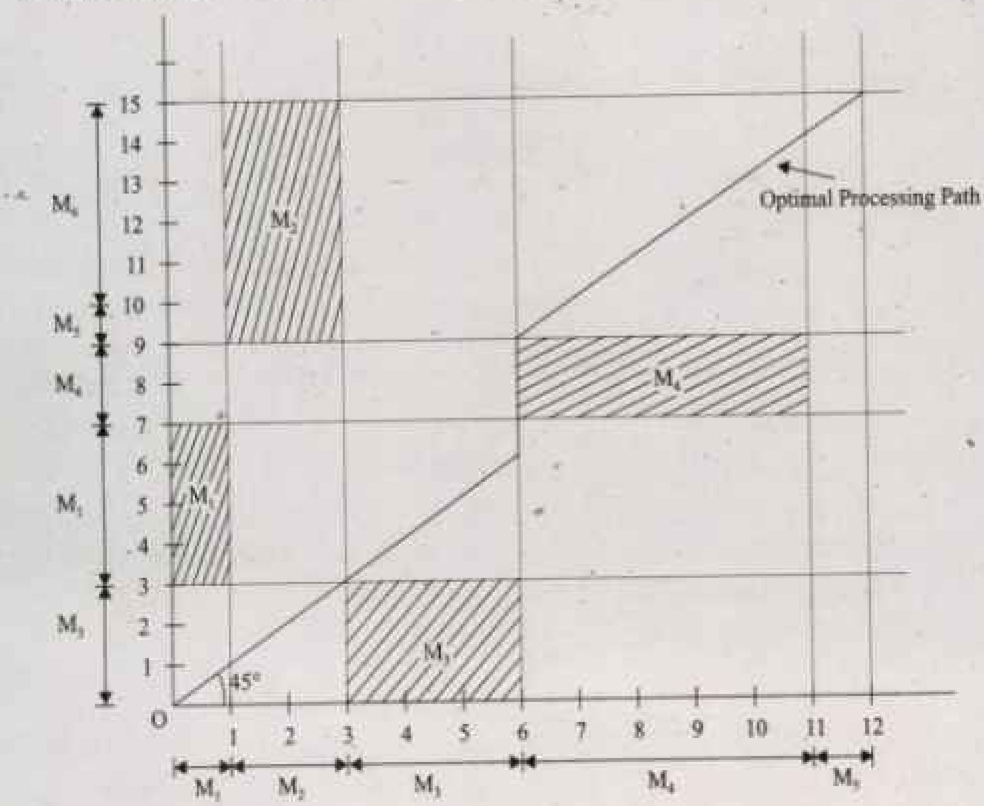

Fig. 4.3

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Example 8. *Use geographical method to minimize time added to process the following jobs on the machines shown, i.e., for each machine find the job which should be done first. Also calculate the total time elapsed to complete both the jobs.*

Job 1	Sequence Time	A	B	C	D	E
		3	4	2	6	2
Job 2	Sequence Time	B	C	A	D	E
		5	4	3	2	6

Solution. The information provided in the problem can be used to draw the following diagram. The shaded area is of the overlap and need to be avoided.

The path that minimizes the idle time for Job 1 is an optimal path. Also the ideal (optimal) path should minimize the idle time for Job 2. For working out the elapsed time, we have to add the idle time for either of the two jobs to that time which is taken for processing of that job. It can be seen that idle time for the chosen path for Job 1 is 5 hours and for Job 2 it is 2 hours, the elapsed time can be calculated as

Processing time for Job 1 + idle time for Job 1 = 17 + (2 + 3) = 22 hours

Processing time for Job 2 + idle time for Job 2 = 20 + 2 = 22 hours.

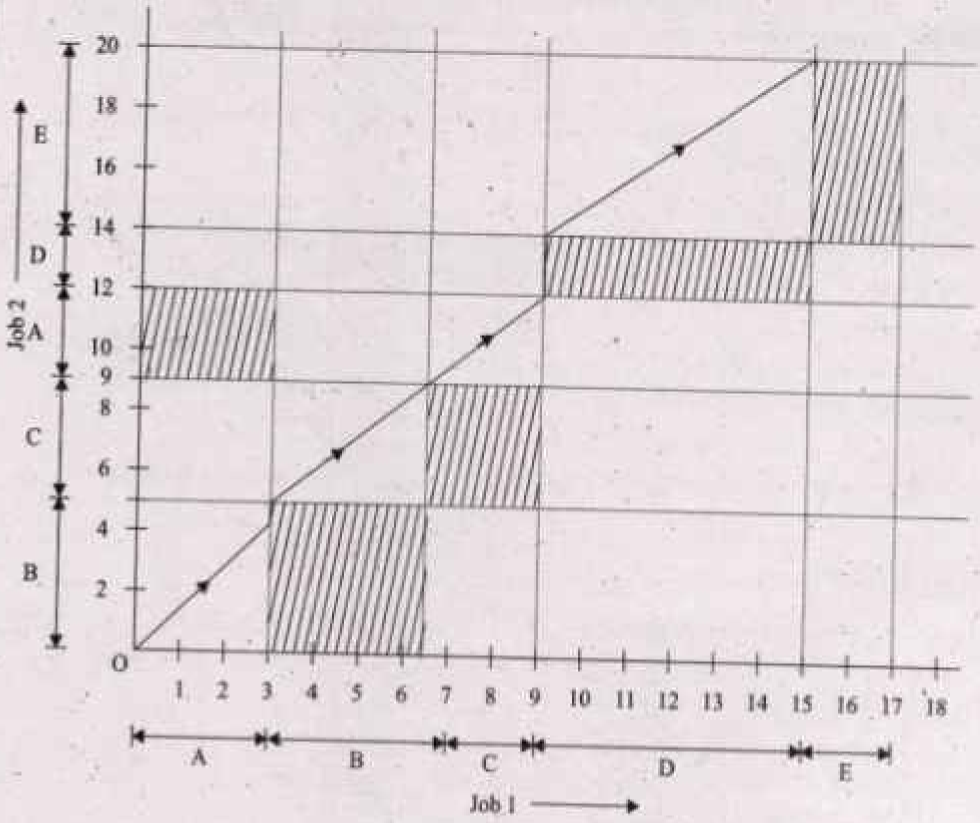

Fig. 4.4

QUEUING THEORY

4.3 INTRODUCTION

Queuing theory has been used for many real life applications to a great advantage. It is so because many problems of business and industry can be assumed/simulated to be arrival-departure or queuing problems. In any practical life situations, it is not possible to accurately determine the arrival and departure of customers when the number and types of facilities as also the requirements of the customers are not known. Queuing theory techniques, in particular, can help us to determine suitable number and type of service facilities to be provided to different types of customers. Queuing theory techniques can be applied to problems such as :

(*a*) Planning, scheduling and sequencing of parts and components to assembly lines in a mass production system.

(*b*) Scheduling of workstations and machines performing different operations in mass production.

(*c*) Scheduling and dispatch of war material of special nature based on operational needs.

(*d*) Scheduling of service facilities in a repair and maintenance workshop.

(*e*) Scheduling of overhaul of used engines and other assemblies of aircrafts, missile systems, transport fleet, etc.

(*f*) Scheduling of limited transport fleet to a large number of users.

(*g*) Scheduling of landing and take-off from airports with heavy duty of air traffic and limited facilities.

(*h*) Decision of replacement of plant, machinery, special maintenance tools and other equipment based on different criteria.

Special **benefit** which this technique enjoys in solving problems such as above are :

(*i*) Queuing theory attempts to solve problems based on a scientific understanding of the problems and solving them in optimal manner so that facilities are fully utilised and waiting time is reduced to minimum possible.

(*ii*) Waiting time (or queuing) theory models can recommend arrival of customers to be serviced, setting up of workstations, requirement of manpower, etc., based on probability theory.

Limitation of Queuing Theory

Though queuing theory provides us a scientific method of understanding the queues and solving such problems, the theory has certain limitations which must be understood while using the technique, some of these are :

(*a*) Mathematical distributions, which we assume while solving queuing theory problems, are only a close approximation of the behaviour of customers, time between their arrival and service time required by each customer.

(*b*) Most of the real life queuing problems are complex situation and are very difficult to use the queuing theory technique, even then uncertainty will remain.

(*c*) Many situations in industry and service are multi-channel queuing problems. When a customer has been attended to and the service provided, it may still have to get some other service from another service point and may have to fall in queue once again. Here the departure of one channel queue becomes the arrival of the other channel queue. In such situations, the problem becomes still more difficult to analyse.

(d) Queuing model may not be the ideal method to solve certain very difficult and complex problems and one may have to resort to other techniques like Monte-Carlo simulation method.

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4.4 IMPORTANT TERMS USED IN QUEUING THEORY

Following are some important terms used in queuing theory :

1. **Arrival Pattern :** It is the pattern of the arrival of a customer to be serviced. The pattern may be regular or at random. Regular interval arrival patterns are rare, in most of the cases, arrival of the customers cannot be predicted. Regular pattern of arrival of customers follows Poisson's distribution.

2. **Poisson's Distribution :** It is discrete probability distribution which is used to determine the number of customers in a particular time. It involves allotting probability of occurrence of the arrival of a customer. Greek letter λ (*lamda*) is used to denote mean arrival rate. A special feature of the Poisson's distribution is that its mean is equal to the variance. It can be represented with the notation as explained below.

$$P(n) = \text{Probability of } n \text{ arrivals (customers)}$$

$$\lambda = \text{Mean arrival rate}$$

$$e = \text{Constant} = 2.71828$$

$$P(n) = \frac{e^{-\lambda}(\lambda)^n}{\lfloor n}, \text{ where } n = 0, 1\ 2, \ldots\ldots$$

Notation $\lfloor\ $ or : is called the factorial and it means that

$\lfloor n$ or $n! = n(n-1)(n-2)(n-3)$ 2, 1

Poisson's distribution tables for different values of n is available and can be used for solving problems where Poisson's distribution is used. However, it has certain limitations because of which its use is restricted. It assumes that arrivals are random and independent of all other variables or parameters. Such can never be the case.

3. **Exponential Distribution :** This is based on the probability of completion of a service and is the most commonly used distribution in queuing theory. In queuing theory, our effort is to minimize the total cost of queue and it includes cost of waiting and cost of providing service. A queue model is prepared by taking different variables into consideration. In this distribution system, no maximization or minimization is attempted. Queue models with different alternatives are considered and the most suitable for a particular is attempted. Queue models with different alternatives are considered and the most suitable for a particular situation is selected.

4. **Service Pattern :** We have seen that arrival pattern is random and poissons distribution can be used for use in queue model. Service pattern are assumed to be exponential for purpose of avoiding complex mathematical problem.

5. **Channels :** A service system has a number of facilities positioned in a suitable manner. These could be

   (a) *Single channel single phase system :* This is very simple system where all the customers wait in a single line in front of a single service facility and depart after service is provided. In a shop if there is only one person to attend to a customer, is an example of the system.

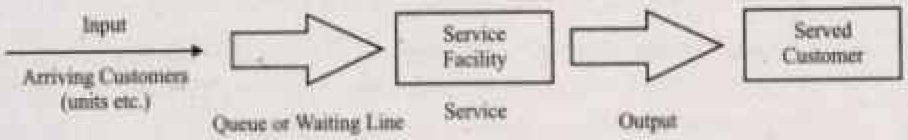

Fig. 4.5

(*b*) *Service in series* : Here the input gets serviced at one service station and then moves to second and or third and so on before going out. This is the case when a raw material input has to undergo a number of operations like cutting, turning drilling etc.

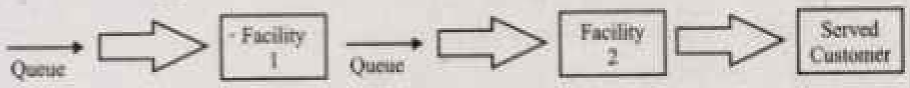

Fig. 4.6

(*c*) *Multi-parallel facility with a single queue* : Here the service can be provided at a number of points to one queue. This happens when in a grocery store, there are 3 persons servicing the same queue or a service station having more than one facility of washing cars. This is shown in Fig. 4.7.

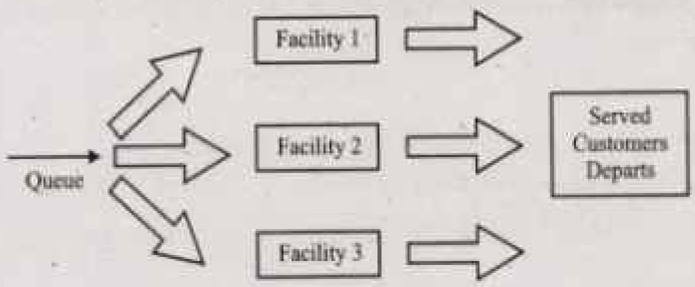

Fig. 4.7

(*d*) *Multiple parallel facilities with multiple queue* : Here there are a number of queues and separate facility to service each queue. Booking of tickets at railway stations, bus stands, etc., is a good example of this. This is shown in Fig. 4.8.

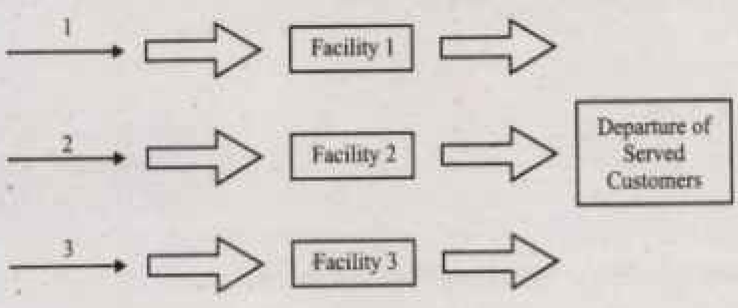

Fig. 4.8

6. **Service Time :** Service time, *i.e.*, the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider. The arrival pattern is random so also is the service time required by different customers. For the sake

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of simplicity the time required by all the customers is considered constant under the distribution. If the assumption of exponential distribution is not valid, Erlang Distribution is applied to the queuing model.

7. **Erlang Distribution :** It has been assumed in the queuing process we have seen that service is either constant or it follows negative exponential distribution in which case the standard deviation σ (sigma) is equal to its mean. This assumption makes the use of the exponential distribution simple. However, in cases where σ and mean are not equal, Erlang distribution developed by AK Erlang is used. In this method, the service time is divided into number of phases assuming that total service can be provided by different phases of service. It is assumed that service time of each phase follows the exponential distribution, *i.e.*, σ = mean.

8. **Traffic Intensity or Utilisation Rate :** This is the rate of at which the service facility is utilised by the components.

   If λ = mean arrival rate and

   (Mue) μ = Mean service rate, then utilisation rate $(p) = \lambda/\mu$ it can be easily seen from the equation that $p > 1$ when arrival rate is more than the service rate and new arrivals will keep increasing the queue. $p < 1$ means that service rate is more than the arrival rate and the waiting time will keep reducing as μ keeps increasing. This is true from the commonsense.

9. **Idle Rate :** This is the rate at which the service facility remains unutilised and is lying idle.

   Idle rate = 1 – utilisation rate = $1 - p = \left(1 - \frac{\lambda}{\mu}\right) \times$ total service facility = $\left(1 - \frac{\lambda}{\mu}\right) \times \frac{\lambda}{\mu}$.

10. **Expected Number of Customers in the System :** This is the number of customers in queue plus the number of customers being serviced and is denoted by $E_n = \frac{\lambda}{(\mu - \lambda)}$.

11. **Expected Number of Customers in Queue (Average queue length) :** This is the number of expected customers minus the number being serviced and is denoted by E_q.

$$E_q = E_n - p = \frac{\lambda}{(\mu - \lambda)} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

12. **Expected Time spent by customer in system :** It is the time that a customer spends waiting in queue plus the time it takes for servicing the customer and is denoted by E_t

$$E_t = \frac{E_n}{\lambda} = \frac{\frac{\lambda}{(\mu - \lambda)}}{\lambda} = \frac{1}{(\mu - \lambda)}.$$

13. **Expected waiting Time in queue :** It is known that E_t = expected waiting time in queue + expected service time, therefore expected waiting time in queue $(E_w) = E_t - \frac{1}{\mu}$.

14. **Average Length of Non-empty Queue :** $E_l = \frac{\mu}{(\mu - \lambda)} = \frac{1}{(\mu - \lambda)} - \frac{1}{\mu} = \frac{\lambda}{\lambda(\mu - \lambda)}$.

15. **Probability that customer wait is zero :** It means that the customer is attended to for servicing at the point of arrival and the customer does not wait at all. This depends upon the utilization rate of the service or idle rate of the system, $p_0 = 0$ persons waiting in the queue $= 1 - \frac{\lambda}{\mu}$ and the probability of 1, 2, 3, .. , n persons waiting in the queue will be given by

$$p_1 = p_0\left(\frac{\lambda}{\mu}\right)^1, p_2 = p_1\left(\frac{\lambda}{\mu}\right)^2, p_n = p_0\left(\frac{\lambda}{\mu}\right)^n.$$

16. **Queuing Discipline :** All the customers get into a queue on arrival and are then serviced. The order in which the customer is selected for servicing is known as queuing discipline. A number of systems are used to select the customer to be served. Some of these are :

    (a) *First in First Served (FIFS) :* This is the most commonly used method and the customers are served in the order of their arrival.

    (b) *Last in First Served (LIFS) :* This is rarely used as it will create controversies and ego problems amongst the customers. Any one who comes first expects to be served first. It is used in store management, where it is convenient to issue the store last received and is called Last In First Out (LIFO).

    (c) *Service in Priority (SIP) :* The priority in servicing is allotted based on the special requirement of a customer like a doctor may attend to a serious patient out of turn, so may be the case with a vital machine which has broken down. In such cases the customer being serviced may be put on hold and the priority customer attended to or the priority may be on hold and the priority customer waits till the servicing of the customer already being serviced is over.

17. **Customer Behaviour :** Different types of customers behave in different manner while they are waiting in queue, some of the patterns of behaviour are :

    (a) *Collusion :* Some customers who do not want to wait they make one customer as their representative and he represents a group of customers. Now only the representative waits in queue and not all members of the group.

    (b) *Balking :* When a customer does not wait to join the queue at the correct place which he warrants because of his arrival. They want to jump the queue and move ahead of others to reduce their waiting time in the queue. This behaviour is called balking.

    (c) *Jockeying :* This is the process of a customer leaving the queue which he had joined and goes and joins another queue to get advantage of being served earlier because the new queue has lesser customers ahead of him.

    (d) *Reneging :* Some customers either do not have time to wait in queue for a long time or they do not have the patience to wait, they leave the queue without being served.

18. **Queuing Cost Behaviour :** The total cost a service provider system incurs is the sum of cost of providing the services and the cost of waiting of the customers. Suppose the garage owner wants to install another car washing facility so that the waiting time of the customer is reduced. He has to manage a suitable compromise in his best interest. If the cost of adding another facility is more than offset by reducing

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the customer waiting time and hence getting more customers, it is definitely worth it. The relationship between these two costs is shown below.

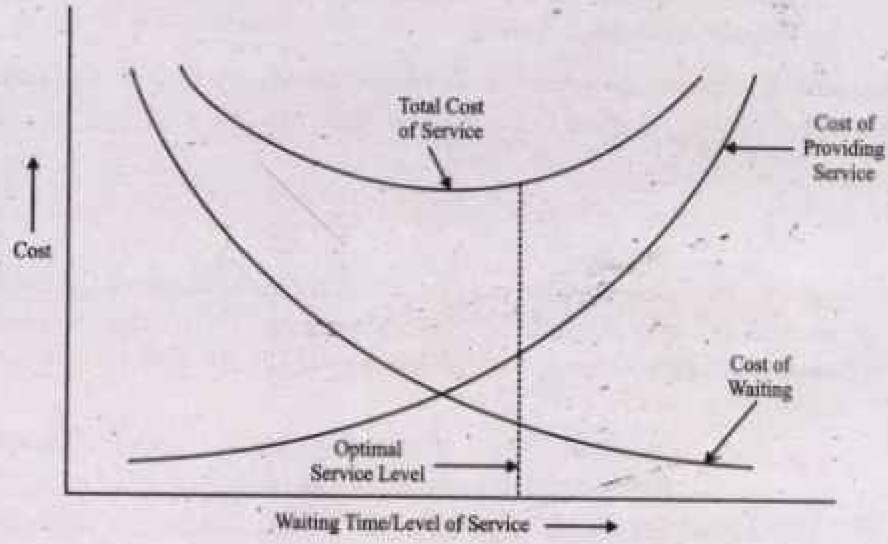

Fig. 4.9

4.5 TYPES OF QUEUING MODELS

Different types of models are in use. The three possible types of categories are :

(a) ***Deterministic model*** **:** Where the arrival and service rates are known. This is rarely used as it is not a practical model.

(b) ***Probabilistic model*** **:** Here both the parameters, *i.e.*, the arrival rate as also the service rate are unknown and are assumed random in nature. Probability distribution, *i.e.*, Poissons, Exponential or Erlang distributions are used.

(c) ***Mixed model*** **:** Where one of the parameters out of the two is known and the other is unknown.

4.6 SINGLE CHANNEL QUEUING MODEL (ARRIVAL — POISSON AND SERVICE TIME EXPONENTIAL)

This is the simplest queuing model and is commonly used. It makes the following assumptions :

(a) Arriving customers are served on First Come First Serve (FCFS) basis.

(b) There is no Balking or Reneging. All the customers wait the queue to be served, no one jumps the queue and no one leaves it.

(c) Arrival rate is constant and does not change with time.

(d) New customers arrival is independent of the earlier arrivals.

(e) Arrivals are not of infinite population and follow Poisson's distribution.

(f) Rate of serving is known.

(g) All customers have different service time requirements and are independent of each other.

(*h*) Service time can be described by negative exponential probability distribution.

(*i*) Average service rate is higher than the average arrival rate and over a period of time the queue keeps reducing.

Example 9. *Assume a single channel service system of a library in a school. From past experiences it is known that on an average every hour 8 students come for issue of the books at an average rate of 10 per hour. Determine the following :*

(*a*) *Probability of the assistant librarian being idle.*

(*b*) *Probability that there are at least 3 students in system.*

(*c*) *Expected time that a student is in queue.*

Solution.

(*a*) Probability that server is idle = $\left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)$ in this example $\lambda = 8$, $\mu = 10$

$$p_0 = \frac{8}{10}\left(1-\frac{8}{10}\right) = 16\% = 016.$$

(*b*) Probability that at least 3 students are in the system

$$E_n = \left(\frac{\lambda}{\mu}\right)^{3+1} = \left(\frac{8}{10}\right)^4 = 0.4$$

(*c*) Expected time that a students is in queue

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{64}{(10\times 2)} = 3.2 \text{ hours.}$$

Example 10. *Self-help canteen employs one cashier at its counter, 8 customers arrive every 10 minutes on an average. The cashier can serve at the rate of one customer per minute. Assume Poisson's distribution for arrival and exponential distribution for service patterns. Determine*

(*a*) *Average number of customers in the system;*

(*b*) *Average queue length;*

(*c*) *Average time a customer spends in the system.*

Solution. Arrival rate $\lambda = \frac{8}{10}$ customers/minute

Service rate $\mu = 1$ customer/minute

(*a*) Average number of customers in the system

$$E_n = \frac{\lambda}{\mu-\lambda} = \frac{0.8}{1-0.8} = 4$$

(*b*) Average queue length

$$E_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(0.8)^2}{1\times 0.2} = 3\times 2.$$

(*c*) Average time a customer spends in the queue

$$E_w = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{0.8}{1\times 0.2} = 4 \text{ minutes.}$$

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Example 11. *Arrival rate of telephone calls at telephone booth are according to Poisson distribution, with an average time of 12 minutes between two consecutive calls arrival. The length of telephone calls is assumed to be exponentially distributed with mean 4 minutes.*

(a) Determine the probability that person arriving at the booth will have to wait.

(b) Find the average queue length that is formed from time to time.

(c) The telephone company will install second booth when convinced that an arrival would expect to have to wait at least 5 minutes for the phone. Find the increase in flows of arrivals which will justify a second booth.

(d) What is the probability that an arrival will have to wait for more than 15 minutes before the phone is free?

(e) Find the fraction of a day that the phone will be in use.

Solution. Arrival rate $\lambda = 1/12$ minute

Service rate $\mu = 1/4$ minute.

(*a*) Probability that a person will have to wait $= \dfrac{\lambda}{\mu} = \dfrac{\frac{1}{12}}{\frac{1}{4}} = \dfrac{1}{12} \times 4 = \dfrac{1}{3} = 0.33$

(*b*) Average queue length $= E_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{\frac{1}{144}}{\frac{1}{4}\left(\frac{1}{4}-\frac{1}{12}\right)} = \dfrac{1}{144} \times 4 \times \dfrac{12}{2} = 1$ person.

(*c*) Average waiting time in the queue $E_w = \dfrac{\lambda_1}{\mu(\mu-\lambda_1)} = \dfrac{\lambda_1}{\frac{1}{4}(\mu-\lambda_1)}$

$$5 = \frac{\lambda_1}{\frac{1}{4}\left(\frac{1}{4}-\lambda_1\right)} \cdot \quad \frac{5}{16} = \left(\frac{5}{4}+1\right)\lambda_1$$

$$\lambda_1 = \frac{5}{16} \times \frac{4}{9} = \frac{5}{36} \text{ arrivals/minute}$$

Increase in flow of arrivals $= \dfrac{5}{36} - \dfrac{1}{12} = \dfrac{1}{18}$ minutes

(*d*) Probability of waiting time > 15 minutes.

$$= \frac{\lambda}{\mu} e^{(\lambda-\mu)15} = \frac{\frac{1}{12}}{\frac{1}{4}} e^{\left(\frac{1}{12}-\frac{1}{4}\right)15} = \frac{1}{3} e^{-\frac{15}{6}} = \frac{1}{3} e^{-2.5}$$

(*e*) Fraction of a day that phone will be in use $= \dfrac{\lambda}{\mu} = 0.33$.

Example 12. *An electricity bill receiving window in a small town has only one cashier who handles and issues receipts to the customers. He takes on an average 5 minutes per customer. It has been estimated that the persons coming for bill payment have no set pattern but on an average 8 persons come per hour. The management receives a lot of complaints regarding customers waiting for long in queue and so decided to find out.*

(*a*) *What is the average length of queue?*

(*b*) *What time on an average, the cashier is idle?*

(*c*) *What is the average time for which a person has to wait to pay his bill?*

(*d*) *What is the probability that a person would have to wait for at least 10 minutes?*

Solution. Making use of the usual notations

$$\lambda = 8 \text{ persons/hour}$$

$$\mu = 10 \text{ persons/hour}$$

(*a*) Average queue length $= \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{64}{10(10-8)} = 3.2$ persons

(*b*) Probability that cashier is idle $= p_0 = 1 - \dfrac{\lambda}{\mu} = 1 - \dfrac{8}{10} = 0.2$, *i.e.*, the cashier would be idle for, 20% of his time.

(*c*) Average length of time that a person is expected to wait in queue.

$$E_w = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{10(10-8)} = 24 \text{ minutes}$$

(*d*) Probability that a customer will have to wait for at least 10 minutes.

$$p(8) = \frac{\lambda}{\mu} \times e^{-(8-10)\times\frac{1}{6}} = \frac{8}{10}e^{-33}, t = \frac{1}{6} \text{ hours.}$$

Example 13. *ABC Diesel engineering works gets on an average 40 engines for overhaul per week, the need of getting a diesel engine overhauled is almost constant and the arrival of the repairable engines follows Poissons's distribution.*

However, the repair or overhaul time is exponentially distributed. An engine not available for use costs ₹ 500 per day. There are six working days and the company works for 52 weeks per year. At the moment the company has established the following overhaul facilities.

	Facilities	
	1	2
Installation Charges	1200000	1600000
Operating Expenses / year	200000	350000
Economic life (years)	8	10
Service Rate/Week	50	80

The facilities scrap value may be assumed to be nil. Determine which facility should be preferred by the company, assuming time value of money is zero?

Solution. Let us work out the total cost of using both the facilitates.

Facility 1 : $\lambda = 40$/week, $\mu = 50$/week

Total annual cost = Annual capital cost + Annual operating cost + Annual cost of lost time of overhaul able engines.

Expected annual lost time = (Expected time spent by repairable engines in system) × (Expected number of arrivals in a year).

$$E_t = \frac{1}{\mu-\lambda}(\lambda \times \text{ number of weeks}) = \frac{1}{(50-40)} \times 40 \times 52 = 208 \text{ weeks.}$$

Cost of the lost time = ₹ 208 × 6 × 500 = 624000

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Total annual cost

$$= \frac{1200000}{8} + 200000 + 624000 = 150000 + 200000 + 624000$$

$$= ₹\, 974000$$

Facility 2 : Annual capital cost

$$= \frac{1600000}{10} + 350000 + \text{cost of lost engine availability time}$$

Cost o lost availability time = $E_t \times (\lambda \times \text{number of weeks}) = \frac{1}{(\mu-\lambda)} \times (\lambda \times \text{number of weeks})$

Here $\lambda = 40$

$\mu = 80$

Hence, cost of lost availability time = $\frac{1}{80-40} \times (40 \times 52) = \frac{2080}{40} = 52$ weeks/years.

Cost of lost time = 52 × 6 × 52 = ₹ 162245

Total cost = $\frac{1600000}{10} + 350000 + 162245 =$ ₹ 672245

Hence, facility No. 2 should be preferred to facility number one.

4.7 MULTI-CHANNEL QUEUING MODEL (ARRIVAL POISSON AND SERVICE TIME EXPONENTIAL)

This is a common facilities system used in hospitals or banks where there are more than one service facilities and the customers arriving for service are attended to by these facilities on first come first serve basis. It amounts to parallel service points in front of which there is a queue. This shortens the length of the queue if there was only one service station. The customer has the advantage of shifting from a longer queue where he has to spend more time to shorter queue and can be serviced in lesser time. Following assumptions are made in this model :

(*a*) The input population is infinite, *i.e.*, the customers arrive out of a large number and follow Poisson's distribution.

(*b*) Arriving customers form one queue.

(*c*) Customer are served on First Come First Served (FCFS) basis.

(*d*) Service time follows an exponential distribution.

(*e*) There are a number of service station (K) and each one provides exactly the same service.

(*f*) The service rate of all the service stations put together is more than arrival rate.

In this analysis we will use the following notations.

λ = Average rate of arrival

μ = Average rate of service of each of the service stations

K = Number of service stations

Kμ = Mean combined service rate of all the service stations.

Hence ρ(*rho*) the utilisation factor for th system = $\frac{\lambda}{K\mu}$.

(*a*) Probability that system will be idle $p_0 = \left[\sum_{n=0}^{K-1}\frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} + \frac{\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k(1-\rho)}\right]^{-1}$

(*b*) Probability of *n* customers in the system.

$$\rho_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{\lfloor n} \times \rho_0 n \le k$$

$$\rho_n = \frac{\left(\frac{\lambda}{n}\right)^n}{\lfloor k} K^{n-k} \times \rho_0 \quad n > k$$

(*c*) Expected number of customers in queues or queue length

$$E_q = \frac{\left(\frac{\lambda}{\mu}\right)^k \rho}{\lfloor k(1-l)^2} \times \rho_0.$$

(*d*) Expected number of customers in the system = $E_n = E_q + \frac{\lambda}{\mu}$

(*e*) Average time a customer spends in queue

$$E_w = \frac{E_q}{\lambda}$$

(*f*) Average time a customer spends in waiting line

$$= E_w + .\frac{1}{\mu}$$

Example 14. *A workshop engaged in the repair of cars has two separate repair lines assembled and there are two tools stores one for each repair line. Both the stores keep in identical type of tools. Arrival of vehicle mechanics has a mean of 16 per hour and follows a Poisson distribution. Service time has a mean of 3 minutes per machine and follows an exponential distribution. Is it desirable to combine both the tool stores in the interest of reducing waiting time of the machie and improving the efficiency?*

Solution. $\lambda = 16/\text{hour}$

$$\mu = 1\frac{1}{3} \times 60 = 20 \text{ hours}$$

Expected waiting time in queue, $E_w = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{16}{20(20-16)} = 0.2$ hour = 12 minutes.

If we combine the two tools stores.

λ = Mean arrival rate = 16 + 16 = 32 / hour K = 2, n = 1.

μ = Mean service rate 20/hour

Expected waiting time in queue, $E_w = \frac{E_q}{\lambda} = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k}{\lfloor k-1(K\mu-\lambda)^2} \times \rho_0$.

NOTES

$$\text{where } \rho_0 = \left[\sum_{n=0}^{k-1} \frac{\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k}\left\{1-\frac{\lambda}{k\mu}\right\}}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1} \frac{\left(\frac{32}{20}\right)^n}{\underline{|n}} + \frac{\left(\frac{32}{20}\right)^2}{\underline{|2}-\left\{1-\frac{32}{2\times 24}\right\}}\right]^{-1}$$

$$= 0.182$$

$$E_w = \frac{E_q}{\lambda} \times p_0$$

$$\frac{E_q}{\lambda} = \frac{32\left(\frac{32}{20}\right)}{\underline{|2-1}(40-32)^2} = \frac{32}{25}$$

$$\text{Hence. } E_w = \frac{32}{25} \times 0.182 = 14 \text{ minutes.}$$

Since the waiting time in queue has increased, it is not desirable to combine both the tools stores. Present system is more efficient.

Example 15. *A bank has three different single window service counters. Any customer can get any service from any of the three counters. Average time of arrival of customer is 12 per hour and it follows Poisson's distribution. Also, on average the bank officer at the counter takes 4 minutes for servicing the customer. The bank is considering the option of installing ATM, which is expected to be more efficient and service the customer twice as the bank officers do at present. If the only consideration of the bank is to reduçe the waiting time of the customer, which system is better?*

Solution. The existing system is multi-channel system, using the normal notations here

$$\lambda = 12/\text{hour} = \frac{60}{4} = 15/\text{hour}$$

Average time a customer spends in the queue waiting to be served.

E_q = Average number of customer in he queue waiting to be served.

$$E_q = \frac{\lambda\mu\left(\frac{\mu}{\mu}\right)^k}{\underline{|k-1}(k\mu-\lambda)^2} \times \rho_0$$

$$\text{or} \quad E_w = \frac{E_q}{\lambda} = \frac{\mu\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k-1}(k\mu-\lambda)^2} \times \rho_0$$

$$\text{where } \rho_0 = \left[\sum_{n=0}^{k=1} \frac{\frac{\lambda}{\mu}}{\underline{|k}} + \frac{\left(\frac{\lambda}{\mu}\right)^k}{\underline{|k}\left\{1-\frac{\lambda}{k\mu}\right\}}\right]^{-1}$$

Here $k = 3$

$$\rho_0 = \left[1 + \frac{12}{15\times 6} + \frac{\left(\frac{16}{25}\right)^3}{6\left\{1-\frac{12}{45}\right\}}\right]^{-1}$$

$$\rho_0 = [1 + 0.133 + 0.06]^{-1} = [1.193]^{-1} = 0.83$$

$$E_w = \frac{15\left(\frac{12}{15}\right)^3}{\lfloor 2(18)^2} \times \rho_0 = 15 \times \frac{64}{(125\times 2\times 324)} \times \rho_0$$

$$= 15\times 64\times \frac{0.83}{(250\times 324)} = 0.009 \text{ hour}$$

$$= 0.33 \text{ second.}$$

Proposed System

$$E_w = \frac{\lambda}{\mu(\mu-\lambda)} \text{ here } \lambda = 12\text{/hour}, \mu = 15\text{/hour}, E_w = \frac{12}{15(15-12)} = \frac{12}{45}\times 60$$

$$= 16 \text{ minutes}$$

Hence, it is better to continue with the present system rather than installing ATM purely on the consideration of customer waiting time.

Example 16. *At a polyclinic three facilities of clinical laboratories have been provided for blood testing. Three lab technicians attend to the patients. The technicians are equally qualified and experienced and they take 30 minutes to serve a patient. This average time follows exponential distribution. The patients arrive at an average rate of 4 per hour and this follows Poisson's distribution. The management is interested in finding out the following :*

(a) *Expected number of patients waiting in the queue.*

(b) *Average time that a patient spends at the polyclinic.*

(c) *Probability that a patient must wait before being served.*

(d) *Average percentage idle time for each of the lab technicians.*

Solution. In this example

$$\lambda = 4\text{/hour}$$

$$\mu = \frac{1}{30}\times 60 = 2\text{ / hour}$$

$$K = 3$$

ρ_0 = Probability that there is no patient in the system.

$$= \left[\sum_{n=1}^{k-1} \frac{1}{\lfloor n}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{\lfloor n}\frac{\left(\frac{\lambda}{\mu}\right)^k}{\left(1-\frac{\lambda}{k\mu}\right)}\right]^{-1}$$

$$= \left[\frac{1}{\lfloor 0} + \frac{2}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{1}{16}(2)^3 \times \frac{1}{1} - \frac{4}{6}\right]^{-1} = \left[1 + \frac{2^1}{\lfloor 2} + \frac{2^2}{\lfloor 2} + \frac{(2)^3}{2\times\frac{2}{6}}\right]^{-1}$$

NOTES

$$= \left[1+1+2\frac{8\times 6}{4}\right]^{-1} = (26)^{-1} = 0.038$$

(a) Expected number of patients waiting in the queue

$$E_q = \frac{1}{\lfloor k-1}\left(\frac{\lambda}{\mu}\right)^k \frac{\lambda\mu}{(k\mu-k)^2}\times p_0$$

$$= \left[\frac{1}{2}\times 8\times\frac{8}{4}\right]\times 0.038 = 8\times 0.038 = 0.304 \text{ or one patient}$$

(b) Average time a patient spends in the system

$$= \frac{E_q}{\lambda}+\frac{1}{\mu}=\frac{0.304}{4}+\frac{1}{2}=0.076+0.5 = 0576 \text{ hours} = 35 \text{ minutes}$$

(c) Probability that a patient must wait

$$p(n\geq k) = \frac{1}{\lfloor k}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{\left(\frac{1-\lambda}{k\mu}\right)}\times p_0$$

$$= \frac{1}{6}\times 8\times 8\times 0.038$$

$$= 0.40$$

(d) p (idle technician) = $\frac{3}{3}p_0+\frac{2}{3}p_1+\frac{1}{3}p_2$ when $p_n = \frac{1}{\lfloor n}\left(\frac{\lambda}{\mu}\right)^n p_0$

p_0 = when all the 3 technician are idle (no one is busy)

p_1 = when only one technician is idle (two are busy)

p_2 = when two technicians are idle (only one busy)

$$p \text{ (idle technician)} = \frac{3}{3}\times 0.038+\frac{2}{3}\times\left(\frac{4}{2}\right)\times 0.038+\frac{1}{3}\times\frac{1}{\lfloor 2}(2)^2\times 0.038$$

$$= 0.038 + 0.05 + 0.025$$

$$= 0.113$$

4.8 POISSON ARRIVAL AND ERLANG DISTRIBUTION FOR SERVICE

We have assumed in our earlier problems that the two service pattern distributions follow exponential distribution in a manner that its standard deviation is equal to its mean. But there are many situations where these two will vary, we must use a model which is more relevant and applicable to real life situations. In this method the service is considered in a number of phases each with a service time $1/\mu$ and time taken in each phase is exponentially distributed. With same mean time of $1/\mu$, with different channels we get different distribution. The method makes the follows assumptions :

(a) The arrival pattern follows Poisson distribution.

(b) One unit completes service in all the phases and only then the other unit is served.

(c) In each phase the service follows exponential distribution.

Fig. 4.10

The following formulae are used in this method :

1. Expected number of customer in the system

$$E_n = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - k)} + \frac{\lambda}{\mu} = E_q + \frac{\lambda}{\mu}$$

2. Expected number of customers in the queue (or Average queue length

$$E_q = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

3. Average waiting time of a customer in queue

$$E_t = k + \frac{1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)}$$

4. Expected waiting time of a customer in the system

$$E_t = \frac{k+1}{2k} \times \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

Example 17. *Maintenance of machine can be carried out in 5 operations which have to be performed in a sequence. Time taken for each of these operations has a mean time of 5 minutes and follows exponential distribution. The breakdown of machine follows Poisson distribution and the average rate of breakdown is 3 per hour. Assume that there is only one mechanic available, find out the average idle time for each machine breakdown.*

Solution. K = 3

Arrival $\lambda = \frac{3}{60} = 1/20$ machines/hour

Total service time for one machine = 5 × 3 = 15 minutes

Service rate μ = 1/15 machines/hour

$$\rho = \text{ Utilisation rate/traffic intensity} = \frac{\lambda}{k\mu} = \left(\frac{1}{20} \times 3\right) \times 15 = \frac{1}{4} = 0.25$$

$$\text{Expected idle time for machine} = k + \frac{1}{2}k = \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

$$= \frac{4}{6} \times \frac{1}{20} \times \frac{1}{20} \times 15\left(\frac{1}{15} - \frac{1}{20}\right) + \frac{1}{\frac{1}{15}}$$

$$= \frac{1}{600} + 15 \times 60 + 15 = 1.5 + 15 = 16.5 \text{ minutes.}$$

Example 18. *In a restaurant, the customers are required to collect the coupons after making the payment at one counter, after which he moves to the second counter where he collects the snacks and then to the third counter, where he collects the cold drinks. At each* *counter he spends $1 \times 1/2$ minutes on an average and this time of service at each counter is exponentially distributed. The arrival of customer is at the rate of 10 customers per hour and it follows Poisson's distribution. Determine*

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(a) Average time a customer spends waiting in the restaurant;

(b) Average time the customer is in queue.

Solution. λ= 10 customer/our

μ = Total service time for one customer

$$= \frac{3}{2}\times 3 = \frac{9}{4} \text{ customers}$$

$$= \frac{4}{9}\times 60 = \frac{80}{3} \text{ hours.}$$

(*a*) Average time a customer spends waiting in the restaurant $E_t = k + \frac{1}{2k}\times\frac{\lambda}{\mu(\mu-\lambda)}$

$$\frac{4}{9} = 10\times\frac{3}{80}\times\frac{80}{3}-10 = \frac{1}{4}\times\frac{3}{50} = \frac{3}{200} \text{ minutes or } \frac{3}{200}\times 600 = 0.9 \text{ minute.}$$

(*b*) Average time the customers in queue

$$\frac{1}{\mu} = \frac{1}{\frac{80}{3}} = \frac{3}{80}\times 60 = \frac{9}{4} = \text{minutes.}$$

4.9 SUMMARY

- A sequence is the order in which different jobs are to be performed. When there is a choice that a number of tasks can be performed in different orders, then the problem of sequencing arises.
- The basic concept behind sequencing is to use the available facilities in such a manner that the cost (and time) is minimized. The sequencing theory has been developed to solve difficult problems of using limited number of facilities in an optimal manner to get the best production and minimum costs.
- **Job :** These have to be sequenced, hence there should be a particular number of jobs (groups of tasks to be performed) say *n* to be processed.
- **Machine :** Jobs have to be performed or processed on machines. It is a facility which has some processing capability.
- **Loading :** Assigning of jobs to facilities and committing of facilities to jobs without specifying the time and sequence.
- **Scheduling :** When the time and sequence of performing the job is specified, it is called *scheduling*.
- **Total Elapsed Time :** It is the time that lapses between the starting of first job and the completion of the last one.
- **Idle Time :** The time for which the facilities or machine are not utilized during the total elapsed time.
- **Static Arrival Pattern :** If all the jobs to be done are received at the facilities simultaneously.

- **Dynamic Arrival Pattern :** Here the jobs keep arriving continuously.
- Queuing theory has been used for many real life applications to a great advantage. It is so because many problems of business and industry can be assumed/simulated to be arrival-departure or queuing problems.
- Queuing theory techniques, in particular, can help us to determine suitable number and type of service facilities to be provided to different types of customers.
- **Arrival Pattern :** It is the pattern of the arrival of a customer to be serviced.
- **Poisson's Distribution :** It is discrete probability distribution which is used to determine the number of customers in a particular time.
- **Exponential Distribution :** This is based on the probability of completion of a service and is the most commonly used distribution in queuing theory.
- **Service Pattern :** We have seen that arrival pattern is random and poissons distribution can be used for use in queue model.
- **Channels :** A service system has a number of facilities positioned in a suitable manner.
- **Service Time :** Service time, *i.e.*, the time taken by the customer when the facility is dedicated to it for serving depends upon the requirement of the customer and what needs to be done as assessed by the facility provider.
- **Idle Rate.** This is the rate at which the service facility remains unutilised and is lying idle.

4.10 REVIEW QUESTIONS

1. What is no passing rule in a sequencing algorithm?
2. Explain the four elements that characterize a sequencing problem.
3. Explain the principal assumptions made while dealing with sequencing problems.
4. Describe the method of processing '*n*' jobs through two machines.
5. Give Johnson's procedure for determining an optimal sequence for processing *n* items on two machines. Give justification of the rules used in the procedure.
6. Explain the method of processing '*m*' jobs on three machines A, B, C in the order ABC.
7. Explain the graphical method to solve the two jobs *m*-machines sequencing problem with given technological ordering for each job. What are the limitations of the method?
8. A Company has 8 large machines, which receive preventive maintenance. The maintenance team is divided into two crews A and B. Crew A takes the machine 'Power' and replaces parts according to a given maintenance schedule. The second crew resets the machine and puts it back into operation. At all times 'no passing' rule is considered to be in effect. The servicing times for each machine are given below.

Machine	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
Crew A	5	4	22	16	15	11	9	4
Crew B	6	10	12	8	20	7	2	21

Determine the optimal sequence of scheduling the factory maintenance crew to minimize their idle time and represent it on a chart.

NOTES

9. Use graphical method to find the minimum elapsed total time sequence of 2 jobs and 5 machines, when we are given the following information :

Job 1	Sequence	A	B	C	D	E
	Time (hours)	2	3	4	6	2
Job 2	Sequence	C	A	D	E	B
	Time (hours)	4	5	3	2	6

10. Two jobs are to be processed on four machines a, b, c and d. The technological order for these jobs on machines is as follows :

Job 1	a	b	c	d
Job 2	d	b	a	c

Processing times are given in the following table :

	Machines			
Job	*a*	*b*	*c*	*d*
1	4	6	7	3
2	4	7	5	8

Find the optimal sequence of jobs on each of machines.

11. A machine shop has four machines A, B, C and D. Two jobs must be processed through each of these machines. The time (in hours) taken on each of these machines and the necessary sequence of jobs trough the shop are given below.

Job 1	Sequence	A	B	C	D
	Time (hour(s)	2	4	5	1
Job 2	Sequence	D	B	C	A
	Time (hours)	6	4	2	3

Use graphic method to obtained total minimum elapsed time.

12. What is a queue? Give an example and explain the basic concept of queue.

13. Define a queue. Give a brief description of the type of queue discipline commonly faced.

14. (*a*) Explain the single channel and multi-channel queuing models.

(*b*) Draw a diagram showing the physical layout of a queuing system with a multi server, multi-channel service facility.

15. (*a*) Give some applications of queuing theory.

(*b*) State three applications of waiting line theory in business enterprises.

16. With respect to the queue system, explain the following :

(*i*) Input process, (*ii*) Queue discipline, (*iii*) capacity of the system, (*iv*) Holding time, (*v*) Balking and (*vi*) Jockeying.

17. Briefly explain the important characteristic of queuing system.
18. What do you understand by :
    (*a*) (*i*) queue length, (*ii*) traffic intensity, (*iii*) the service channels?
    (*b*) (*i*) steady and transient state and (*ii*) utilization factor?
19. Show that if the inter-arrival times are exponentially distributed, the number of arrivals in a period of time is a Poisson process and conversely.
20. Consider the pure birth process, where the system starts with K customers at $t = 0$. Derive the equation describing the system and then show that

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{(n-k)!};\ n = k, k+1, \ldots$$

21. Consider the pure birth process, where the number of departures in some time interval follows a Poisson distribution. Show that the line between successive departures is exponential.
22. If $\lambda \Delta t$ is the probability of a single arrival during a small interval of time Δt, and if the probability of more than one arrival is negligible, prove that the arrivals follows the Poisson's law.
23. (*a*) Derive Poisson's process assuming that the number of arrival, in non-overlapping intervals, are statistically independent and then apply the binomial distribution.
    (*b*) What are the various queuing models available?
24. Explain (*i*) Single queue, single server queuing system, and (*ii*) Single queue, multiple servers in series queues.
    [Hint. GD indicates that discipline is general, *i.e.*, it may be FCFS or LCFS or SIRO]
25. For a (M/M/1) : (∞/F/FO) queuing model, in the steady-state case, obtain expressions for the mean and variance of queue length in terms of relevant parameters : λ and μ.
26. For a (M/M/1) : (∞/F/FO) queuing model in the steady-state case, show that
    (*a*) The expected number of units in the system and in the queue is given by

$$E(n) = \frac{\lambda}{(\mu - \lambda)} \text{ and } E(m) = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

    (*b*) (*i*) Expected waiting time of an arrival in the queue is $\frac{\rho}{\mu(1-\rho)}$.
    (*ii*) Expected waiting time the customer spends in the system (including services) is

$$\frac{1}{(\mu - \lambda)}$$

27. Define busy period of a queuing system. Obtain the busy period distribution for the simple (M/M/1) : (∞/FCFS) queue.
    What is the condition that the busy period will terminate eventually?
28. Derive the differential-differential equations for the queuing model (M/M/1) : (N/FCFS) and solve the same.
29. For a (M/M/1) : (N/FIFO) queuing model :
    (*i*) find the expression for E(*n*),
    (*ii*) derive the formula for P_n and E(*n*) when $\rho = 1$.

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30. (*a*) For a (M/M/C) : (∞ / FCFS) queuing model, derive the expression for

(*i*) the steady state equation,

(*ii*) probability that a customer will not have to wait,

(*iii*) expected number of customer in the queue,

(*iv*) expected number of customers in the system,

(*v*) expected waiting time of a customers in the system,

(*vi*) probability of server to be idle.

(*b*) Giving clearly the assumptions, derive the steady state distribution of queue length in (M/M/K) queuing model.

31. For (M/M/C) ; (N/FCFS), derive the steady-state equations describing the situation for N = C; then show that the expression for P*n* is given by

$$Pn = \begin{cases} \dfrac{P_0\left(\frac{\lambda}{\mu}\right)^n}{n!}, & 0 \le n \le C \\ 0, & \text{otherwise} \end{cases}$$

where $$P_0^{-1} = \sum_{n=0}^{c} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}.$$

32. For Erlang is distribution with parameters μ and K prove that the mode is at $\dfrac{(K-1)}{K\mu}$, the mean is $\dfrac{1}{\mu}$, and the variance is $\dfrac{1}{K\mu^2}$.

33. For (M/G/1) : (∞/FCFS) queuing model, derive the Pollaczek-Khintchine (P-K) formula for expected number of customers in the system.

34. Show that for the special case of exponential service time with mean $\dfrac{1}{\mu}$, the results of (M/G/1) model reduce to those of the (M/M/1) model.

35. Under the standard queuing model nomenclature indicate what do you mean by the following :

M/M/S, D/M/1, M/G_I/S, G/G/S and E_k/GI/S

36. Write short notes on :

(*i*) Cost-profit models in queuing theory.

(*ii*) Non-Poisson queues.

(*iii*) Information requirement, assumption and objectives of queuing models.

37. A foreign bank is considering opening a drive in window for customer service. Management estimates that customers will arrive for service at the rate of 12 per hour. The teller whom it is considering to staff the window and serve customers at the rate of one every three minutes. Assuming Poisson arrival and Exponent ice service find :

(*a*) Utilization of teller;

(*b*) Average number in the system;

(*c*) Average waiting time in the line;

(*d*) Average waiting time in the system.

CHAPTER 5 REPLACEMENT PROBLEM AND PROJECT MANAGEMENT

★ STRUCTURE ★

REPLACEMENT PROBLEM

5.1 INTRODUCTION

Replacement of old plant and equipment and items of use like bulbs/tube-lights, refrigerators/heating tools/gadgets, etc., is a necessity. All these items are designed for performance up to the desired level for a particular time (years/hours) or particular number of operations. For example, when a refrigerator is given the warranty for 7 years the manufacturer knows that the design of the refrigerator is such that it will perform up to desired level of efficiency without breakdown for that period. Similarly, bulbs/tube-lights may have been designed for say 10,000 on-off operations. But all these need to be replaced after a particular period/number of operations. The equipment is generally replaced because of the following reasons :

(*i*) When the item/equipment fails and does not perform its function it is meant for.

(*ii*) Item/equipment has been in use for sometimes and is expected to fail soon.

(*iii*) The item/equipment in use has deteriorated in performance and needs expensive repairs *i.e.*, it has gone beyond the economic repair situations. The cost of maintenance and repair of equipment keeps increasing with the age of the equipment. When it becomes uneconomical to continue with an old equipment, it must be replaced by a new equipment.

(*iv*) Improved technology has given access to much better (convenient to use) and technically superior (using less power) products. This is the case of obsolescence. The equipment needs to be replaced not because it does not perform up to the standards it is designed for but because new equipment is capable of performance of much higher standards.

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It should be understood that all replacement decisions involve high financial costs. The financial decisions of such nature will depend upon a large number of factors, like the cost of new equipment, value of scrap, availability of funds, cost of funds that have to be arranged, tax benefits, government policy, etc.

When making replacement decisions, the management has to make certain assumptions, these are :

(*i*) The quality of the output remains unchanged.

(*ii*) There is no change in the maintenance costs.

(*iii*) Equipments perform to the same standards.

Let us discuss some of the common replacement problems.

Replacement of items, which deteriorate with time without considering the change in money value

Most of the machinery and equipment having moving parts deteriorate in their performance with passage of time. The cost of maintenance and repair keeps increasing with passage of time and a stage may reach when it is more economical (in overall analysis) to replace the item with a new one. For example, a passenger car is bound to wear out with time and its repair and maintenance cost may go to such level that the owner has to replace it with a new one.

Let C = Capital cost of the item,

$S(t)$ = Scrap value of the item after t years of use,

$O(t)$ = Operating and maintenance cost of the equipment at time t,

n = Number of years the item can be used,

$TC(n)$ = Total cost of using the equipment for n years,

$$TC(n) = C - S(t) + \sum_{t=1}^{n} O(t)$$

$$\text{Average } TC(n) = \frac{1}{n\left[C - S(t)\sum_{t=1}^{n} O(t)\right]}$$

Time 't' in this case is a discrete variable.

In this case as long as the average TC (n) is minimum, the equipment can remain in use for that number of years. If average total cost keeps decreasing up to ith year and starts increasing from (i + 1)th year then ith year may be considered as most economic year for replacement of the equipment.

The concept of depreciation cost also must be understood here. As the years pass by, the cost of the equipment or items keeps decreasing. How much the cost keeps decreasing can be calculated by two methods commonly used, *i.e.*, straight line depreciation method and the diminishing value method.

Example 1. *A JCB excavator operator purchases the machine for ₹ 1500000. The operating cost and the resale value of the machine is given below.*

Year	1	2	3	4	5	6	7	8
Operating Cost (in ₹)	30000	32000	36000	40000	45000	52000	60000	70000
Resale value (in lakhs of ₹)	12	10	8	5	4.5	4	3	2

When should the machine be replaced?

Solution.

C = 1500000

O (t) = Operating cost

S (t) = Resale value

t = Time

n = Number of years after when the machine is to be replaced.

Let us draw a table showing the various variables required to make decision. This is shown in the table below.

Year	O (t) (in thousand of Rupees)	Cumulative O (t)	Resale value S (t) (in thousands of Rupees)	Depreciation C – S (t) (in thousands of Rupees)	Total cost TC (n) (in thousands of Rupees)	Average TC (n) (in thousands of Rupees)
1	30	30	1200	300	330	330
2	32	62	1000	500	562	281
3	36	98	800	700	798	266
4	40	138	500	1000	1138	284.5
5	45	183	450	1050	1233	246.6
6	52	235	400	1100	1335	222.5
7	60	295	300	1200	1495	213.6
8	70	365	200	1300	1665	208

In third year the minimum average cost is 266000 as shown in the table above. So, replacement should take place at the end of three year.

Example 2. *A new tempo costs ₹ 100000 and may be sold at the end of year at the following prices :*

Year	1	2	3	4	5	6
Selling Price (₹)	60000	45000	32000	22000	10000	2000

The corresponding annual operating costs are :

Year	1	2	3	4	5	6
Cost/Year (₹)	10000	12000	15000	20000	30000	45000

NOTES

It is not only possible to sell the tempo after use but also to buy a second hand tempo. It may be cheaper to do so than to buy a new tempo.

Age of tempo	0	1	2	3	4	5
Purchase Price (₹)	100000	60000	45000	33000	20000	10000

What is the age to buy and to sell so as to minimize average annual cost?

Solution. Cost of new tempo = ₹ 100000

Let us find out the average cost per year of the new tempo.

Year of Service (1)	Operating Cost O (t) (2)	Cumulative Operating Cost O (t) (3)	Resale Value S (t) (4)	Depreciation C – S (t) (5)	Total Cost TC (n) (6) = (3) + (5)	Average TC (n) (7) = (6) ÷ (1)
1	10000	10000	60000	40000	50000	50000
2	12000	22000	45000	55000	77000	38500
3	15000	37000	32000	68000	105000	35000
4	20000	57000	22000	78000	135000	33750
5	30000	87000	10000	90000	177000	35400
6	45000	132000	2000	98000	230000	35000

Average cost is minimum at the end of fourth year; hence the new tempo should be replaced after 4 years.

Let us now find out the average total cost of second hand tempo.

Year of Service (1)	Operating Cost O (t) (2)	Cumulative Operating Cost O (t) (3)	Resale Value S (t) (4)	Depreciation C – S (t) (5)	Total Cost TC (n) (6) = (3) + (5)	Average TC (n) (7) = (6) ÷ (1)
0			100000	100000	100000	
1	10000	10000	60000	40000	50000	50000
2	12000	22000	45000	55000	77000	38500
3	15000	37000	32000	68000	105000	35000
4	20000	57000	20000	78000	137000	33750
5	30000	87000	10000	90000	177000	35400

The tempo may be replaced by second hand tempo at the end of third year and the owner can save ₹ (35000–34666), *i.e.*, ₹ 334 instead of buying a new one.

Example 3. (*a*) *Machine A costs ₹ 9000. Annual operating costs are ₹ 200 for the first year and then increases ₹ 2000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?*

(*b*) *Machine B costs ₹ 10000. Annual operating cost are ₹ 400 for the first year and then increases by ₹ 800 every year. You now have a machine of type A which is one year old. Should you replace it with B, if so, when?*

Solution. (*a*) Let us assume that there is no scrap value of the machine Average total cost can be computed as

Year (*n*)	Operating Cost O (₹)	Cumulative Operating Cost Σ O (₹)	Depreciation C − S (₹)	Total Cost	Average Cost
1	200	200	9000	9200	9200
2	2200	2400	for all years	11400	5700
3	4200	6600		15600	5200
4	6200	12800		21800	5450
5	8200	21000		30000	6000

It can be seen that the best age for replacement is third year.

(b) For machine B, the average cost can be calculated as follows :

Year (*n*)	Operating Cost O (₹)	Cumulative Operating Cost Σ O (₹)	Depreciation C − S (₹)	Total Cost	Average Cost
1	400	400	10000	10400	10400
2	1200	1600	for all year	11600	5800
3	2000	3600		13600	4533
4	2800	6400		16400	4100
5	3600	10000		20000	**4000**
6	4400	14400		24400	4066

Since the minimum average cost for machine B is lower than for machine A, machine B should be replaced by machine A. Minimum average cost is (₹ 4000), it should be replaced when it exceeds ₹ 4000. In case of one year old machine ₹ 2200/- will be spent next year and ₹ 4200 the following year. We should keep machine A for one year.

Replacement policy of an equipment/item whose operating cost increases with time and money value also changes with time

In previous examples, we assumed that the money value does not change and remains constant but it is well-known that as the equipment deteriorates and operating costs keep increasing, the money value keeps decreasing with time. Hence, we must calculate the *Net Present Value* (NPV) of the money to be spent a few years hence. Otherwise the resale value, the operating costs, which are to take place in future, will not be realistic and management will not be able to take optimal decisions.

Let C = Initial cost of item/equipment

O C = Operating cost

R = Rate of interest

A rupee invested at present will be equivalent to $(1 + r)$ a year after $(1 + r)^2$ two years hence and $(1 + r)^n$ in n years time. It means that making a payment of one rupee after n years is equivalent to paying $(1 + r)^n$ now. The quantity $(1 + r)^{-n}$ is called the *present worth* or *present value* of one rupee spent n years from now.

Present value of a rupee $V = (1 + r)^{-1} = 1/1 + r$ is called *discount rate* and is always less than 1.

Then, yearwise present value of expenditure in future years can be calculated as

$$\text{Present value } (n) = (c + oc_1) + oc_2\, v + oc_3\, v^2 + \ldots\ldots\ldots + oc_n\, v^{n-1} + (c + oc_1)\, v^n + oc_2\, v^{n+1}$$
$$+ oc_3\, v^{n+2} + \ldots\ldots\ldots + oc_n\, v^{2n-1} + (c + oc_1)\, v^{2n} + oc_2\, v^{2n+1} + \ldots\ldots\ldots + oc_n\, v^{3n-1}$$

NOTES

Steps Involved in Calculation of Replacement Policy When Time Value Changes

Step I. Find out the present value factor at the given rate and multiply it with the operating/ maintenance cost of the equipment/items for different years.

Step II. Work out the total cost by adding the cumulative present value to the original cost for all the years.

Step III. Cumulate the discount factors.

Step IV. Divide the total cost by corresponding value of the cumulated discount factor for every year.

Step V. Find out the value of last column that exceeds the total cost. Equipment/item will be replaced in the latest year.

These steps will be explained with the help of an example.

Example 4. *The yearly cost of two machines X and Y, when money value is neglected is shown below. Find which machine is more economical if money value is 10% per year.*

Year	1	2	3
Machine X (₹)	2400	1600	1800
Machine Y (₹)	3200	800	1800

Solution. It may be seen that the total cost for each machine X and Y is ₹ 5800 (2400 + 1600 + 1800) or (3200 + 800 + 1800). When the money value is not discounted the machines are equally good, total costwise, when money value is not changed with time, with money value 10% per year, the discount rate, it changes as follows :

$$V = \frac{1}{1+r} = \frac{1}{1+0.10} = \frac{1}{1.1} = 0.9091$$

Discounted costs are obtained by multiplying the original costs with 0·9091 after one year. Total costs of machines X and Y are calculated as shown below.

Year	1	2	3	Total Cost (₹)
Machine X	2400	1600 × 0.9091 = 1440	1800 × 0.9081 = 1620	5450
Machine Y	3200	800 × 0.9091 = 720	1800 × 0.9081 = 1620	5540

The total cost of machine X is less than that of machine Y, machine X is more economical.

Example 5. *A manufacturer is offered two machines X and Y. Machine X is priced at ₹ 10000 with running cost of ₹ 1000 for first four years and increasing by ₹ 400 in fifth year and subsequent years. Machine Y which has the same capacity and performance as X costs ₹ 8000 but has maintenance cost of ₹ 1200 per year for first five years increasing by ₹ 400 in the sixth and subsequent years. If cost of money is 10% per year, which is a more economical machine? Assume running cost is incurred at the beginning of the year.*

Solution. $PV = \frac{1}{1+0{\cdot}10} = 0.909$

Machine X, C = 1000.

Year	OC	PV factors	PV of OC	C + Cumulative PV of OC	Cumulative PV Factor	Weighted Average Cost
1	1000	1·00	1000	11000	1·00	11000
2	1000	0·909	909	11909	1·909	5984·5
3	1000	0·826	826	12735	2·735	4656·30

4	1000	0·751	751	13486	3·486	3868·6
5	1400	0·683	956	14442	4·169	3464
6	1800	0·621	1116	15558	4·790	3248
7	2200	0·564	1240	16798	5·355	3137
8	2600	0·513	1326	18124	5·868	30886
9	3000	0·466	1398	19532	6·334	**3084**
10	3400	0·424	1429	21951	6·759	3247

Machine Y, C = 800.

Year	OC	PV factors	PV of OC	C + Cumulative PV of OC	Cumulative PV Factor	Weighted Average Cost
1	1200	1.00	1200	9200	1.00	9200
2	1200	0.909	1090.8	10290.8	1.909	5390.67
3	1200	0.826	991.2	11282	2.735	4125
4	1200	0.751	901.2	12183.2	3.486	3494.8
5	1200	0.683	819.6	13002.8	4.169	3119
6	1600	0.621	993.6	13996.4	4.790	2922
7	2000	0.564	1128	15124.4	5.355	2824.35
8	2400	0.513	1231.8	16355.6	5.868	**2787.25**
9	2800	0.466	1304.8	17660.4	6.334	2788.2
10	3200	0.424	1356.8	19017.2	6.759	2813.6

It can be seen that weighted average cost of machine X is minimum, *i.e.*, ₹ 3084 in ninth year. Where as the weighted average cost of machine Y is minimum in 8th year, *i.e.*, ₹ 2787.25 so it is advisable to purchase machine Y.

5.2 REPLACEMENT POLICY FOR EQUIPMENT/POLICY THAT BREAKS DOWN/FAILS SUDDENLY

As an equipment or item, which is made of a number of components ages with time, it deteriorates in its functional efficiency and the performance standard are reduced. However, in real life situation there are many such items whose performance does not deteriorate with time but fail suddenly without any warning. This can cause immense damage to the system or equipment and inconvenience to the user. When the item deteriorates with time, one is expecting reduced performance but other items, which may fail without being expected to stop performing, can create a lot of problems. A minor component in an electronic device or equipment like TV, fridge or washing machine, costs very little and may be replaced in no time but the entire equipment fails suddenly if the component fails. Hence the cost of failure in terms of the damage to the equipment and the inconvenience to the user is much more than the cost of the item.

If it is possible to know exactly the life of the component, it is possible to predict that the component and hence equipment is likely to fail after performance of so many hours or miles, etc. This is the concept of preventive maintenance and preventive replacement. If the equipment is inspected at laid down intervals to know its conditions, it may not be possible to expect the failure of the item. The cost of failure must be brought down to minimum, preventive maintenance is cheap but avoids lots of problems. In many cases, it may not be possible to know the time of failure by direct inspection. In such cases the probability of failure can be determined from the past experience. Finding the Mean Time Between Failure (MTBF) of the equipment in past is one good way of finding this probability. It is possible by using the probabilities to find the number of items surviving up to certain time period or the number of items failing in a particular time period.

NOTES

In situations, when equipment/item fails without any notice, two types of situations arise.

(a) *Individual Replacement Policy*. In this case an item is replaced immediately when it fails.

(b) *Group Replacement Policy*. In this policy all the items are replaced irrespective of the fact whether the items have failed or not, of course, any item failing before the time fixed for group replacement is also replaced.

Individual Replacement Policy

In this policy, a particular time 't' is fixed to replace the item whether it has failed or not. It can be done when one knows that an item has been in service for a particular period of time and has been used for that time period. In case of moving parts like bearings, this policy is very useful to know when the bearing should be replaced whether it fails or not. Failure of a bearing can cause a lot of damage to the equipment in which it is fitted and the cost of repairing the equipment is much more than the cost of bearing if it had been replaced well in time. If it is possible to find out the optimum service life 't' the sudden failure and hence loss to the equipment and production loss, etc, can be avoided. However, when we replace items on a fixed interval of preventive maintenance period certain items may be left with residual useful life which goes waste. Such items could still perform for another period of time (not known) and so the utility of items has been reduced. Consider the case of a city corporation wanting to replace its street lights. If individual replacement policy is adopted then replacement can be done simultaneously at every point of failure. If group replacement policy is adopted then many lights with residual life will be replaced incurring unnecessary costs.

Analysis of the cost of replacement in case of items/equipments that fail without warning is similar to finding out the probability of human deaths or finding out the liability of claims of Life Insurance Company on the death of a policy holder.

The probability of failure or survival at different times can be found out by using mortality tables or life tables.

The problem of human births and deaths as also individual problems where death is equivalent to failure and birth is equivalent to replacement can also be studied as part of the replacement policy. For solving such problems, we make the following assumptions :

(a) All deaths or part failures are immediately replaced by births or part replacements and

(b) There are no other, except the ones under consideration, entries or exits.

Let us find out the rate of deaths that occur during a particular time period assuming that each item in a system fails just before a particular time period. The aim is to find out the optimum period of time during which an item can be replaced so that the costs incurred are minimum. Mortality or life tables are used to find out the probability destination of lifespan of items in the system.

Let $f(t)$ – number of items surviving at time $(t-1)$ n = Total number of items with system under consideration. The probability of failure of items between 't' and $(t-1)$ can be foud out by

$$P = \left(\frac{(t-1) - f(t)}{n}\right).$$

Replacement Policy

Let the service life time of an item be T and n = number of items in a system which need to be replaced whenever any of these fails or reaches T.

$$F(t) = \text{number of items surviving at T}$$

$$F'(t) = 1 - f(t) \text{ number of items that have failed}$$

$$O(t) = \text{Total operating time}$$

$$Cf = \text{Cost of replacement after failure of item}$$

$$CPM = \text{Cost of preventive maintenance}$$

Cost of replacement after failure of service time $T = n \times f'(t) \times C_f$

Also cost of replacement for item replaced before failure $= n\,[1 - f'(T)]\,C_{pm}$

$$= n + f'(T)\,cf + n\,[1 - f'(T)\,C_{pm}$$

Hence we can replace an item when the total replacement cost given above is minimum where

$$O(t) = f(t)\,dt$$

Group Replacement Policy

Under this policy, all items are replaced at a fixed interval 't' irrespective of the fact they have failed or not and at the same time keep replacing the items as and when they fail. This policy is applicable to a case where a large number of identical low cost items which are more and more likely to fail at a time. In such cases, *i.e.*, like the case of replacement of street lights, bulbs, it may be economical to replace all items at fixed intervals.

Let n = total number of items in the system

$N(t)$ = number of items that fail during time t

$C(t)$ = Cost of group replacement after time t

$C(t)/t$ = average cost per unit time

Cg = Cost of group replacement

Cf = Cost of replacing one item on failure

$$C(t) = n\,Cg + Cf(n_1 + n_2 + \dots\dots\dots + n_{t-1})$$

$$F(t) = \text{Average cost per unit time} = C(t)/t = n\,Cg + Cf(n_1 + n_2 + \dots\dots + n_{t-1})/t$$

We have to minimize average cost per unit time, so optimum group replacement time would be that period which minimize this time.

It can be concluded that the best group replacement policy is that which makes replacement at the end of 't'th period if the cost of individual replacement for the same period is more than the average cost per unit time.

Example 6. *The following mortality rates have been observed for certain type of light bulbs :*

End of week	1	2	3	4	5
Percentage Failing	10	20	50	70	100

There are 1000 bulbs in use and it costs ₹ 10 to replace an individual bulb which has burnt out. If all the bulbs are replaced simultaneously, it would cost ₹ 5 per bulb. It is proposed to replace all the bulbs at fixed intervals whether they have fixed or not and to continue replacing fused bulbs as and when they fail. At what intervals should all the bulbs be replaced so that the proposal is economical?

NOTES

Solution. Average life of a bulb in weeks = Probability of failure at the end of week × number of bulbs

$$= (1 \times 10/100 + 2 \times 10/100 + 3 \times 30/100 + 4 \times 20/100 + 5 \times 30/100)$$
$$= 0.10 + 0.2 + 0.9 + 0.8 + 1.5 = 3.5$$

Average number of replacement per week = $\dfrac{\text{Number of bulbs}}{\text{Average life}} = \dfrac{1000}{3.5} = 285$

Cost per week @ ₹ 10 per bulb = 285 × 10 = ₹ 2850

Let n_1, n_2, n_3, n_4 and n_5 be the number of bulbs being replaced at the end of first, second, third, fourth and fifth week respectively then

n_1 = number of bulbs in the beginning of the first week × probability of the bulbs failing during first week = 1000 × 10/100 = 100.

n_2 =(number of bulbs in the beginning × probability of the bulbs failing during second week) + number of bulbs replaced in first week × probability of these replaced bulbs failing in second week.

$= 1000 \times (20 - 10)/100 + 100 \times 10/100 = 100 + 10 = 110$

n_3 = (number of bulbs in the beginning × probability of the bulbs failing during third week) + number of bulbs being replaced in first week × probability of these replaced bulbs failing in second week) + number of bulbs being replaced in second week × probability of those failing in third week)

$= 100 \times (50 - 20)/100 + 100 \times (20 - 10)/100 + 110 + 10/100 = 300 + 10 + 11 = 321$

$n_4 = 1000 \times (70 - 50)/100 + 100 \times (50 - 20)/100 + 110 + 20 - 10/100 + 321 \times 10/100$

$= 200 + 30 + 11 + 32 = 273$

$n_5 = 100 \times 30/100 + 100 \times 20/100 + 110 \times 30/100 + 321 \times 10/100 + 273 \times 10/100$

$= 300 + 20 + 33 + 32 + 28 = 413$

The economics of individual or group replacement can be worked out as shown in the table below.

End of Week	No. of Bulbs Failing	Cumulative No. of Failed Bulbs	Cost of Individual Replacement	Cost of Group Replacement	Total Cost	Average Total Cost
1	100	100	1000	5000	6000	6000
2	110	220	2200	5000	7200	3600
3	321	541	5410	5000	10410	3470
4	273	814	8140	5000	13140	3285
5	413	1227	12270	5000	17270	3454

Individual replacement cost was worked out to be ₹ 2850. Minimum average cost per week corresponding to 4th week is ₹ 3285, it is more than individual replacement cost. So it will be economical to follow individual replacement policy.

Example 7. *The computer system has a large number of transistors. These are subject to a mortality rate given below :*

Period	Age of Failure in Hours	Probability of Failure
1	0 – 400	0.05
2	400 – 800	0.15
3	800 – 1200	0.35
4	1200 – 1600	0.45

If the transistors are replaced individually the cost per transistor is ₹ 20. But if it can be done as a group at a specific interval determined by the preventive maintenance policy of the user, then the cost per transistor comes down to ₹ 10. Should the transistor be replaced individually or as a group?

Solution. Let us assume that a block of 400 hours is the one period and total number of transistors in the system are 1600.

Find out the average failure of transistors

Average failure = Number of transistors /Average mean life

$= 1600 / (0.05 \times 1 + 0.15 \times 2 + 0.35 \times 3 + 4 \times 0.45)$

$= 1600 / (0.05 + 0.30 + 1.05 + 1.80)$

$= 1600/3.2 = 500$

If cost of individual replacement policy is adopted, Cost = ₹ 500 × 20 = ₹ 10000, now we must find out the failure of transistors per period of block of 400 hours.

$n_1 = n_0 p_1 = 1600 \times 0.05 = 80$

$n_2 = n_0 p_2 + n_1 p_1 = 1600 \times 0.15 + 80 \times 0.05 = 240 + 4 = 244$

$n_3 = n_3 = n_0 p_3 + n_1 p_2 + n_2 p_1 = 1600 \times 0.35 + 80 \times 0.15 + 244 \times 0.05$

$= 560 + 12 + 12.2 = 585$

$n_4 = n_0 p_4 + n_1 p_3 + n_2 p_2 + n_3 p_1 = 1600 \times 0.45 + 80 \times 0.35 + 244 \times 0.15 + 585 \times 0.05$

$= 720 + 28 + 36.6 + 29.25 = 814$

Now, average cost of group replacement must be found.

Period 400 hours Block	Failure of ICs during Month	Cumulative Failure	Individual Replacement Cost @ 20	Group Replacement Cost @ 10	Total Cost	Average Total Cost
1	80	80	1600	1600 × 10 = 16000	17600	17600
2	244	324	6480		222480	11240
3	585	909	18180		34180	11393
4	814	1723	34460		50460	12615

The minimum cost of group replacement is ₹ 11240 for an interval of 400 hours. Individual replacement is optimal policy since the cost is ₹ 10000, which is less than the group replacement cost.

NOTES

Manpower replacement policy (Staffing policy)

All organizations face the problem of initial recruitment and filling up of vacancies caused by promotions, transfer, employee quitting their jobs or retirement and deaths. The principle of replacement used in industry for replacement of parts, etc, can also be used for recruitment and promotion policies, which are laid down as personnel policy of an organization. The assumption made in such case is that the destination of manpower is already decided. Few examples will illustrate this point.

Example 8. *An arı unit requires 200 men, 20 Junior Commissioned Officers (JCOs) and 10 officers. Men are recruited at the age of 18 and JCOs and officers are selected out of these. If they continue in service, they retire at the age of 40. At present there are 800 jawans and every year 20 of them retire. How many jawans should be recruited every year and at what age promotions should take place ?*

Solution. If 800 jawans had been recruited for the past 22 years (age of recruitment 40 years – age of entry 18 years), the total number of them serving up to age of 39 years = 20 × 22 = 440

Total number of jawans required = 200 + 20 + 10 = 230

Total number of jawans to be recruited every year in order to maintain a strength of 230

$$= 800/440 \times 230 = 418$$

Let a jawan be promoted at age of X, then up to X – 1 year, number of jawans recruited is 200 out of 230. Hence out of 800, jawans required = 200/230 × 800 = 696.

696 will be available up to 5 years as 20 retire every year and (800 – 20 × 5) = 700. Hence promotion of jawans is due in sixth year.

Out of 230 jawans required, 20 are JCOs, therefore if recruited 800, number of JCOs = 20/230 × 800 = 70 approximately.

In a recruitment of 800, total number of men and JCOs = 697 + 70 = 766

Number of officers required = 800 – 766 = 34

This number will be available in 20 years of service, so promotion of JCOs to officers is due in 21 year of service.

Example 9. *College X plans to raise the strength of its faculty to 150 and then keep it at that level. The wastage of faculty due to retirement, quitting, deaths, etc. based on the length of service of the faculty member s as given below.*

Block years	1	2	3	4	5	6	7	8	9	10
% of teachers	0	5	10	15	20	30	35			
Those who live up to end of year	0	5	35	60	65	70	85	100		

(*i*) Find the number of faculty members to be recruited every year.

(*ii*) If there are 10 posts of Head of Departments (HODs) for which length of service is the only criterion of promotion, what is the average length of service after which a new faculty member should expect promotion?

Solution. Let us assume that the recruitment per year is 100. These 100 teachers join initially in the block of 0 – 5 years, will serve for 35 years and will become 0 in 7th block of 5 years, *i.e.*, at the service of 35 years. Those 100 who join between the block of year 5 – 10 will serve for 30 years and become 15, the third set of teachers will become 30 after 25 years of service and soon.

Year	No. of Faculty Members
0	100
5	95
10	65
15	40
20	35
25	30
30	15
35	0

Hence, if 100 faculty members are required every year, the total number of staff members after 35 years (7 block of 5 years) of service = 380

To maintain staff strength of 150, the number to be recruited every year = $100/380 \times 150 = 40$

If the college recruits 40 every year, then they want 10 as HODs. Hence if the college recruits 100 every year then they will need HODs = $10/40 \times 100 = 25$.

It can be seen from the above table that $0 + 15 + 30 \geq 25$, *i.e.*, the promotion of the newly recruited faculty member as HODs can be done after 20 years of service.

PROJECT MANAGEMENT

5.3 INTRODUCTION

Programming Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques used in project management. Project management is necessary to ensure that a project is completed within the stipulated budget, within the allocated time and perform to satisfaction.

PERT was developed by US Navy in 1958 for managing its Polaris Missile Project. It is very useful device for planning time and resources of a project. Polaris Missile project involved 3000 separate contracting organizations and was regarded as the most complex project experience till that time.

Parallel efforts, at almost the same time, were undertaken by Du Pont Company, which developed Critical Path Method (CPM) to plan and control the maintenance of chemical plants. These methods were subsequently widely used by Du Pont for many engineering functions.

5.4 PROJECT MANAGEMENT

NOTES

Definition of Terms Commonly used in PERT and CPM

Activity

Activity is the smallest unit of productive efforts to be planned, scheduled and controlled. It is an identifiable part of the project, which consumes time and resources. In fact, a project is a combination of inter-related activities, which must be performed in a certain order for its completion. The project is divided into different activities by the work breakdown into smaller work contents. In network (arrow diagram) an activity is represented by an arrow, the tail that represents the start and the head, the finish of the activity. The length, shape and direction of the arrow have no relation to the size of the activity.

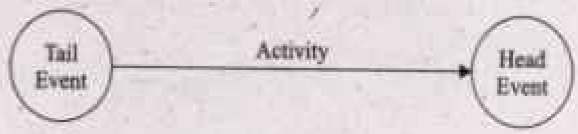

Fig. 5.1

Event

An event is an instant of time at which an activity starts and finishes. An event is represented by a node, *i.e.*, O. The beginning of an activity is *Tail Event* and finishing of an event is *Head Event.*

Path

An unbroken chain of activity arrows connecting the initial event to some other event is called a path.

Predecessor Activity

This is an activity that must be completed immediately before the start of another activity.

Successor Activity

Activity, which cannot be started until one or more activities are completed but immediately succeeds them is *called successor activity of a project.*

Dummy Activity

As seen in the definition of activities, all activities take some time and resources. *A dummy activity is the one which is introduced in the network for communication when two or more activities have the same head and tail events. It means that two or more activities share the same start and finish nodes simultaneously. A dummy activity takes no time and requires no resources.* It is shown as a dotted line in Figure 5.2. Figure 5.3 shows wrong representation.

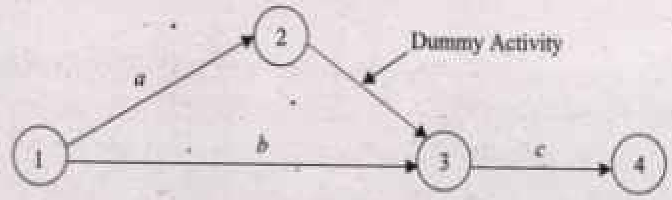

Fig. 5.2. Correct Representation

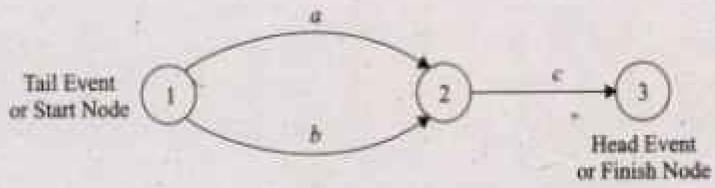

Fig. 5.3. Wrong Representation

Let us assume that the start of activity C depends upon the completion of activity A and B and the start of activity D depends only on the finish of activity. For this situation, we draw wrong and right representation in Figure 5.4 to understand the introduction of dummy activity in network.

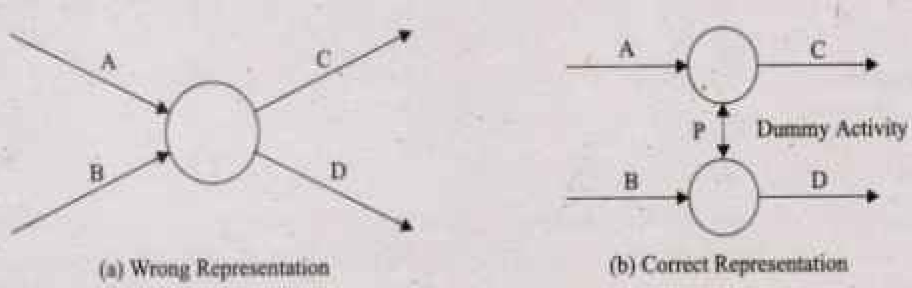

Fig. 5.4

5.5 NETWORK (ARROW DIAGRAM)

A network is the graphical representation of logically and sequentially connected arrows representing activities and nodes representing events of a project.

Looping

Sometimes, due to errors in network logic, a situation of looping or cycling error occurs in which no activity can be completed as all the activities of the network are interlinked. In such situations, there is need to re-examine the network logic and redraw the network. To understand looping, see Figure 5.5.

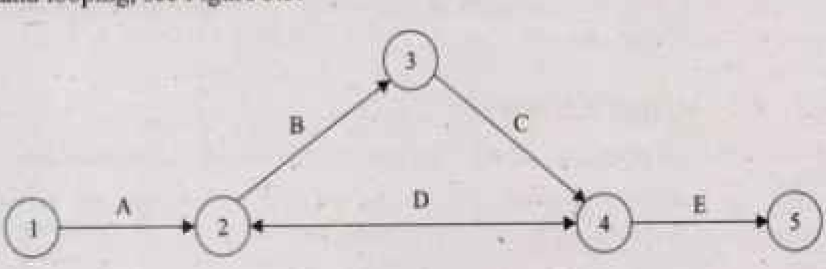

Fig. 5.5

Activity B cannot start until activity D is completed and activity D depends on the completion of activity C but C is dependent on the completion of activity B. Thus activities B, C and D form a loop and the network cannot proceed. Such condition can be avoided by checking the precedence relationship of the activities and numbering them in a logical sequence.

Dangling

NOTES

In a network all activities except the final activity has a successor activity. A situation may occur when an activity other than the final activity, does not have a successor activity. The situation is shown in Figure 5.6.

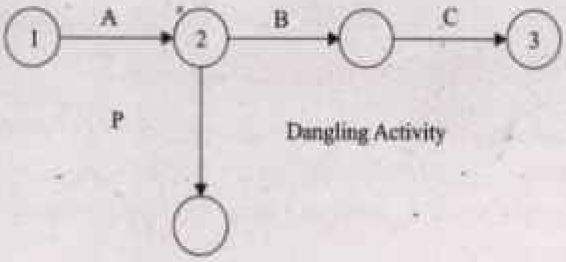

Fig. 5.6

In must be remembered that except the first node and last node, all nodes must have at least one activity entering it and one activity leaving it.

Construction of Networks

Construction of a network is a simple procedure of putting all the events and activities in a logical and sequential manner to meet the requirement of a particular project/ problem. Difficulty occurs only when the basic rules are ignored. The following steps are helpful in constructing the network :

(*a*) Divide the project into activities by following the procedure of Work Breakdown Structure (WBS).

(*b*) Decide the start event, and the end event of project for all the activities. This is called *establishing the precedence order* and is the most important part of drawing the network.

(c) The activities decided by the precedence order are put in a logical sequence by using the graphical representation notations. Logical sequence can be decided by asking the following questions :

- (*i*) What are the activities that must be completed before the start of a particular activity? (Predecessor Activities)
- (*ii*) What activities must follow the activity already drawn ? (Successor Activities)
- (*iii*) Are there any activities which must be performed simultaneously with a particular activity ?

Rules to Construct a Network

1. Activities are represented by arrows ⟶ and events are represented by circles O.
2. Each activity is represented by one and only one arrow. The tail of the arrow represents the start and head the end of the activity.
3. Each activity must start and end in a node.
4. Arrow representing activities must be kept straight and should not be shown curved or bent.
5. Angles between arrows should be as larges as possible to make the activities clearly distinguishable from each other.
6. Arrows should not cross each other.

7. Event Number 1 represents the start of the project. There will be no activities (arrows) entering this node.
8. All events (nodes) should be numbered in an ascending order.
9. No events numbers can be repeated.
10. Dangling is not permitted.
11. Dummy activities also must follow the above rules, even though they do not consume any resource or time.

Numbering of events

For numbering of the events, Fulkerson's Rule is very helpful.

(*a*) Initial or start event, having no preceeding event is numbered 1.

(*b*) Numbering of other events is done from left to right or from top to bottom as 2, 3, 4, etc.

(*c*) The events, which has been numbered are ignored or deleted. This will result in new initial events; these must be numbered in ascending order.

(*d*) Continue numbering all the events till we reach the last event out of which no activity (arrow) will emerge. It will be allotted the highest number, as it is the end event.

The numbering of activities is illustrated with the help of Figure 5.7.

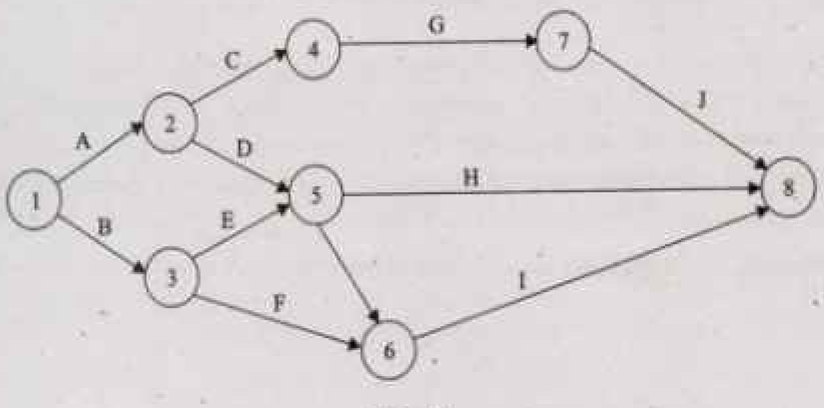

Fig. 5.7

Skipping of Event Numbers

In large projects in which the activities run into hundreds, it is not always possible to list all the activities at the initial stage and some additional activities may have to be added as the project progresses. Hence, while numbering the events continuously as 1, 2, 3, 4...and so on, the events are numbered in gaps of 5's or 10's so that other events can be inserted without causing any inconvenience to the logic of the network. The first event may be numbered 5 and subsequent events may be numbered as 10, 15, 20 and so on.

Example 10. *Let us use a simple example to illustrate the procedure we have just learnt. Listed below is the precedence chart showing the activities, their precedence (sequence), etc., for the project, 'Launching a new product' Sequencing is very important part of the construction of a network. The precedence given below must be carefully understood, as this example will be used to draw the network at a later stage.*

NOTES

Activity	*Description*	*Immediate Predecessor Activity*	*Time (Weeks)*
A	*Arranging a sales office*	–	6
B	*Hiring sales persons*	*A*	4
C	*Training sales persons*	*B*	7
D	*Selecting advertising Agency*	*A*	2
E	*Plan advertising campaign*	*D*	4
F	*Conduct advertising campaign*	*E*	10
G	*Design packaging of product*	–	2
H	*Establish packaging facility*	*G*	10
I	*Package initial stocks*	*H, J*	6
J	*Order stock from manufacturer*	–	13
K	*Select distributors*	*A*	9
L	*Sell to distributors*	*C, K*	3
M	*Transport stock to distributors*	*I, L*	5

The logic of the predecessor activities for each activity listed in the above table should be understood properly. The project 'Launching a new product' can be broken down into a number of activities. The set of activities given in the table are one perception based on simple logic. Other such logic could also be developed. The students are advised to carefully study the precedence of the activities.

Solution. Network diagram for the activities listed in the table is shown in Figure 5.8.

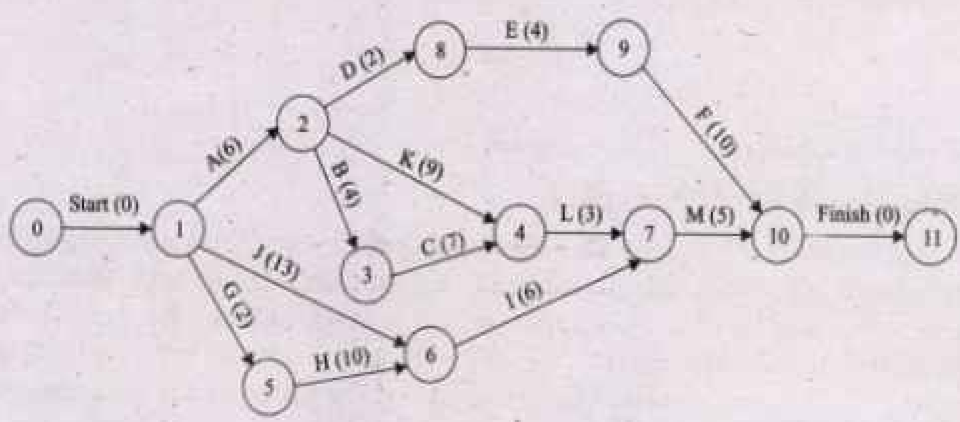

Fig. 5.8

Please see each activity carefully to understand the logic. The network is listed with 0 event and the activity has 0 time and is written as start (0). Arranging a sales office does not have any immediate predecessor activity. This is written as activity A with its time

(6 weeks) written in the brackets as A (6) on top of the arrow. From node 1 there are three activities, which do not have any immediate predecessor activity, *i.e.*, A (6), G (2) and J (13). This may be verified from the precedence table. Activity B, hiring of salespersons can only commence after arranging sales office (activity A) so activity B (4) is shown as arrow coming out of node 2. Also activity D (2) and K (9) can also start only after activity A has been completed and they are shown with arrows moving out of node 2. There is only one activity C (7), which can start after completion of B and is shown as leaving node another node 11 has been created and the finish activity moving out of node 10. Finish activity has 0 times as shown in the network diagram.

Example 11. *The characteristics of a project schedule are given below :*

S. No.	*Activity*	*Time*	*S. No.*	*Activity*	*Time*
1.	*1 – 2*	*6*	*2.*	*1 – 3*	*4*
3.	*2 – 4*	*1*	*4.*	*3 – 4*	*2*
5.	*3 – 5*	*5*	*6.*	*4 – 7*	*7*
7.	*5 – 6*	*8*	*8.*	*6 – 8*	*4*
9.	*8 – 7*	*2*	*10.*	*7 – 9*	*2*
11.	*8 – 9*	*1*			

Construct a suitable network.

Solution. The network is shown in Figure 5.9

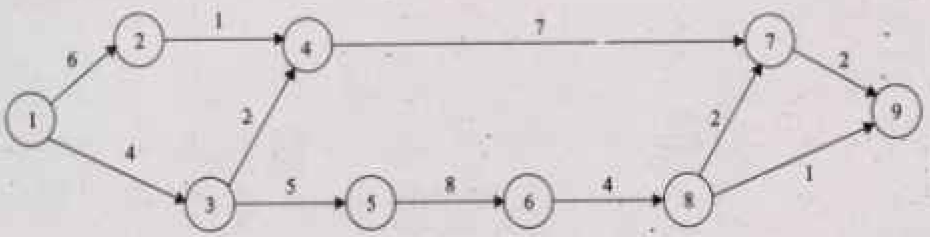

Fig. 5.9

Example 12. *Draw a network diagram based on the following project schedule information available :*

S. No.	*Activity*	*Immediate Predecessor Activity*	*Time*
1.	*A*	*–*	*2*
2.	*B*	*–*	*4*
3.	*C*	*A*	*6*
4.	*D*	*B*	*5*
5.	*E*	*C, D*	*8*
6.	*F*	*E*	*3*
7.	*G*	*F*	*2*

NOTES

Solution. The network is shown in Figure 5.10.

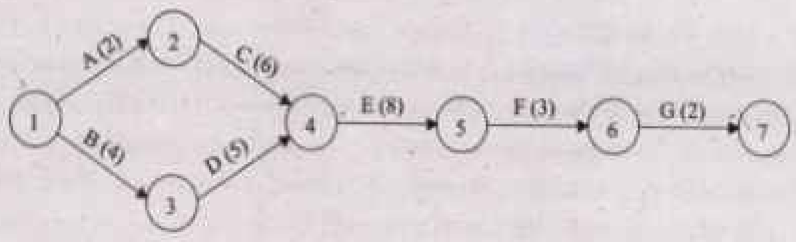

Fig. 5.10

Critical Path and Activity Times

As explained earlier *PERT is a very useful technique for planning the time and resources of any project.* It is an event-oriented approach as it is mainly concerned with various events in a project. PERT deals with probability of completion of a project in particular time, as the time of various activities involved cannot be known accurately. It is only the time an activity is expected to take for completion, which can best be calculated. Expected time of completion of each activity can be found out from the following three timings :

(*a*) Optimistic Time

(*b*) The most likely time

(*c*) Pessimistic Time.

These three timings are based on Beta Statistical Distribution. Beta distribution is used as it is extremely flexible and can take on any form of activity and times that are associated in a typical project. Four typical Beta curves are shown below.

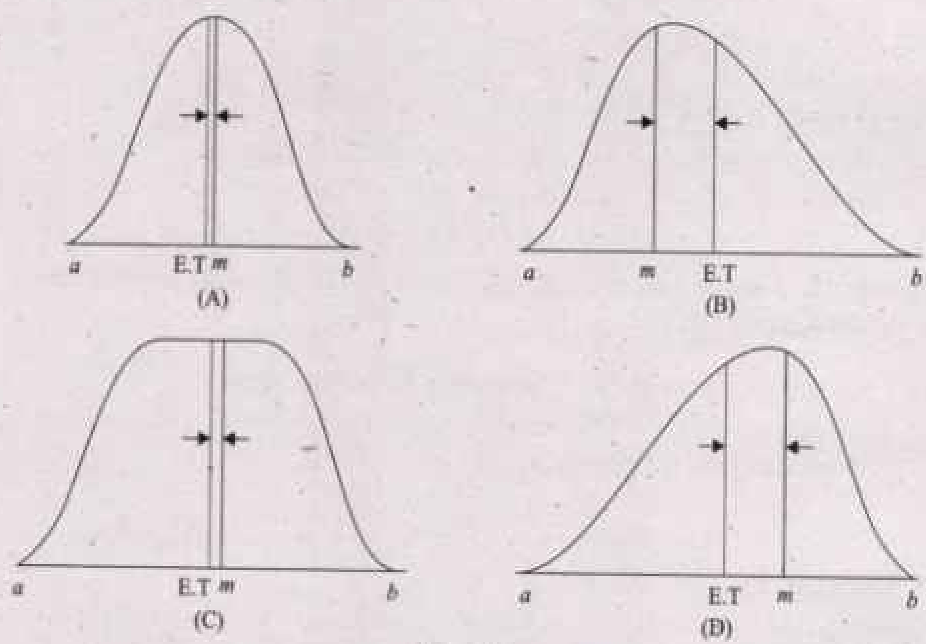

Fig. 5.11

It can be seen that the Beta distribution has finite end points like (*a*), the optimistic time and (*b*) the pessimistic time and the Expected Time (ET) of the activity is limited between these two ends. Curve (A) is a symmetrical curve and the difference between the most likely time (m) and Expected Time (ET) is very small. Had the curve been exactly symmetrical, the firm line (m) and dotted line (ET) would be exactly the same. Curve B indicates a high probability of finishing the activity m and ET indicates that if something goes wrong, the activity time can be greatly extended. Curve C is something like a rectangular distribution. Here the probability of finishing the activity early or late is almost equal. Similarly, curve D indicates very small probability of finishing the activity early but it is more probable that it will take an extended period of time. The Expected Time (ET) can be calculated from the following formula :

$$\text{ET} = \frac{a + 4m + b}{6}$$

Activity Times– Estimated Time

After constructing a network reflecting the precedence relationship, we have to ascertain the time estimate for each activity. We must calculate ET for each activity using the above formula :

i.e., $$\text{ET} = \frac{a + 4m + b}{6}$$

Now the variance of the activity time has to be calculated.

$$\text{V}^2 = \left(\frac{b - a}{6}\right)^2$$

Earliest Start and Finish Times

Let us take zero as the start time for the project, then for each activity there is an Earliest Start Time (EST) relative to the project starting time. It is the earliest possible time that activity can start, assuming that all of the predecessor activities are also started at their EST. In that case for that particular activity, its Earliest Finish Time (EFT) is EST + activity time.

Latest Start and Finish Times

If we assume that the effort is to complete the project in as soon as possible time, this is the Latest Finish Time (LFT) of the finish activity or of the project. The Latest Start Time (LST) is the latest time when an activity can start, if the project schedule is to be maintained

$$\text{LST} = \text{LFT} - \text{activity time}$$

Finish activity has zero time, hence LST = LFT

Slack. Slack of an activity can be defined as the difference between the Latest Start Time (LST) and Earliest Start Time (EST) or the difference between the Latest Finish Time (LFT) and Earliest Finish Time (EFT). This is the significance of slack or Total Slack Time (TST), that the TST for any activity must be used up.

Critical Path

If we observe the network, we can see that there are a number of paths that lead to the finish activity, *i.e.,* completion of the project. But the longest path is the most limiting path. This path is called the *Critical Path*. It can be easily determined by adding the activity times of all the activities on the largest path from start to finish of the project.

Calculation of EST and EFT

These calculations can best be described with the help of a network. Let us draw a network as shown in Figure 5.12 :

NOTES

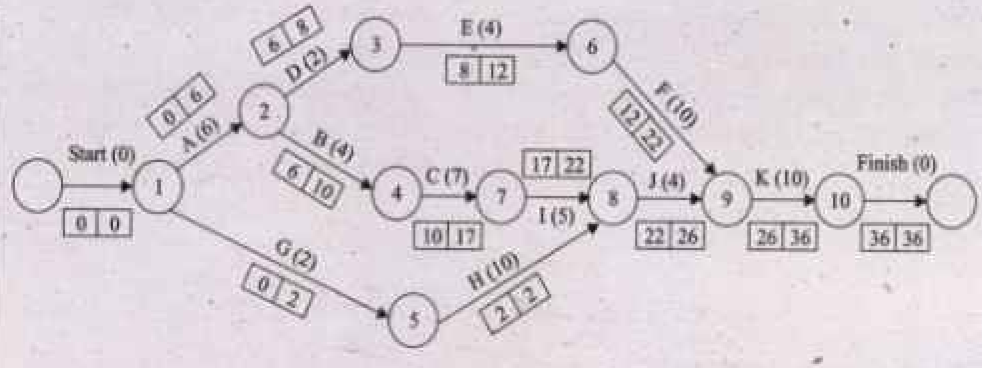

Fig. 5.12

This is the same network as was drawn in example 10.

In the above figure the name of the activities are written above the arrow and their timings are written in the brackets. The start activity and the finish activity with zero timing have only been listed for convenience.

For calculations of EST and EFT let us proceed forward through the network as follows :

(*a*) Put the value of the project start time in both EST and EFT positions near the start activity arrow. So for start activity EST and EFT is zero, which is placed under the start activity as | 0 | 0 |.

(*b*). Consider activity A with activity time of 6. For this EST is zero and EFT is 6 because that is the minimum time the activity will take. It has been placed near activity A as | 0 | 6 |.

(*c*) All activities emanating from node 2 will have EST as 6 and EFT = EST + activity time, hence for activity B it is | 6 | 10 | because activity B has a timing of 4. Similarly, near activity D has been | 6 | 8 | as it has activity time of 2. All the timings have been written in this manner.

(*d*) Continue through the entire network and mark the EST and EFT. The critical path is ABCIJK and is 36. Hence for the finish activity EST = EFT = 36.

Calculation of LST and LFT

For this purpose we work backward through the network. These timings have been listed in Figure 5.13.

For activity K, EST was 26 and EFT was 36. LST for this activity is 36 – 10 = 26 and LFT is 36 so mark it next to activity K as shown. Similarly, let us take activity F. EST was 12 and EFT 22 as activity time is 10. LST can only be 26 – 10 = 16 and LFT is 26.

NOTES

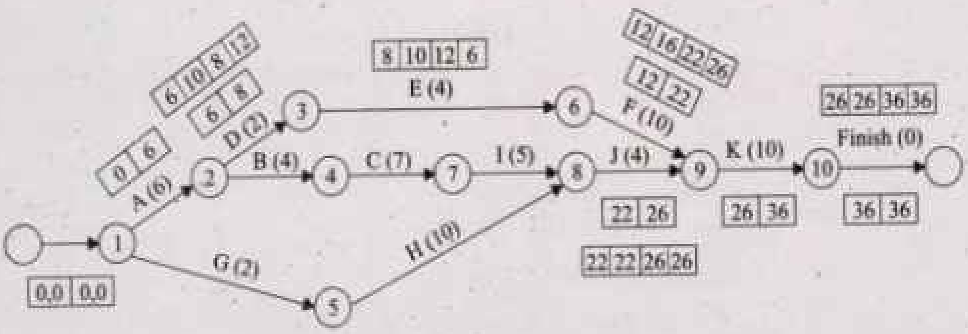

Fig. 5.13

Calculation of Float (Slack) and Crashing the Network

Example 13. *A project consists of the following activities. The Optimistic Time (OT), Pessimistic Time (PT) and Most Likely Time or the Expected Time for the activities is also listed in front of them.*

Predecessor Activity	*Successor Activity*	*OT*	*Most Likely Time*	*PT*
1 – 2	*2*	*2*	*3*	*4*
2 – 3	*3*	*3*	*6*	*9*
2 – 4	*4*	*3*	*4*	*5*
3 – 5	*5*	*2*	*4*	*6*
3 – 6	*6*	–	*0*	–
4 – 6	*6*	–	*0*	–
4 – 7	*7*	*4*	*5*	*6*
5 – 7	*7*	*4*	*6*	*8*
6 – 7	*7*	*6*	*7.5*	*12*

Draw a network diagram of the above project and calculate associate timings of the project, i.e., Earliest and Latest Occurrence times of different events, slack, identify critical events and mark the Critical Path in the diagram. What is the total project duration ?

Solution. The network diagram is as shown below.

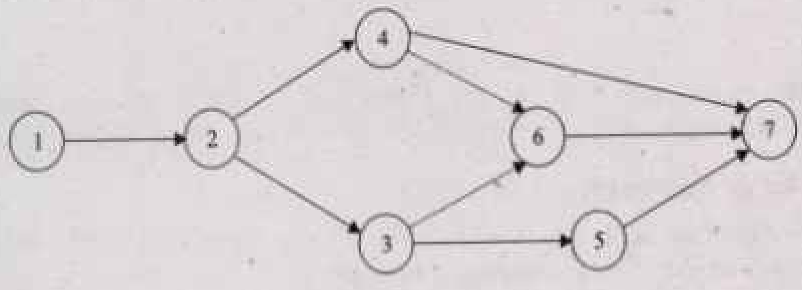

Fig. 5.14

(4, 6), (3, 6) → Dummy Activities

Critical Path 1, – 2, – 3 – 5 – 7, Longest Path

NOTES

Event		
Predecessor	Successor	$ET = \frac{9 + 4m + 6}{6}$
1	2	3
2	3	6
2	4	4
3	5	4
3	6	0
4	6	0
4	7	5
5	7	6
6	7	8

EST

Event 1 = 0

Event 2 = 0 + 3 = 3

Event 3 = 3 + 6 = 9

Event 4 = 3 + 4

Event 5 = 9 + 4 = 13

Event 6 = 9 + 0

Event 7 = 13 + 6 = 19

LST

Event 7 = 19

Event 6 = 19 − 8 = 11

Event 5 = 19 − 6 = 13

Event 4 = 11 − 0 = 11

Event 3 = 13 − 4 = 9

Event 2 = 9 − 6 = 3

Event 1 = 3 − 3 = 0

Now the slack can be calculated.

Event	EST	LST	Slack
1	0	0	0
2	3	3	0
3	9	9	0
4	7	11	4
5	13	13	0
6	9	11	2
7	19	19	0

All the events having zero slack are the Critical Events, *i.e.*, 1, 2, 3, 5 and 7. This is the Critical Path. The project duration is 19 (days/weeks).

Crashing of Network

Most of the projects result into cost overruns because of the inability of the project management team to complete the project in minimum possible time frame. *The crashing of network involves considering the cost incurred on different activities required for completing the project.* Let us understand certain terminology associated with crashing of network.

Normal Cost This is the cost of the project when all the normal activities are carried out, *i.e.*, there is no overtime or there are no special resources for which extra payment has to be made.

Normal Time It is that time in which project can be completed with the normal cost as defined above.

Crash Cost It is the minimum possible time, which is associated with the crash cost.

The relationship between these costs can be expressed as shown in Figure 5.15.

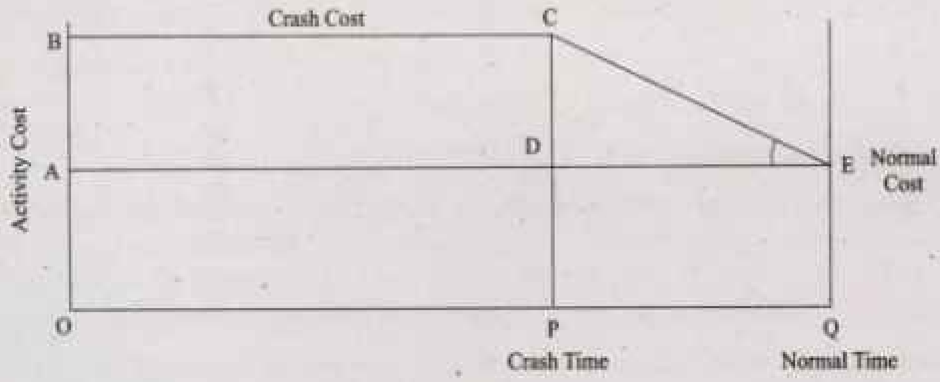

Fig. 5.15

It can be easily seen from the diagram above that the cost-time slope (angle) is $\frac{OB - OA}{OQ - OP} = \frac{CD}{DE}$.

Slack

It should be appreciated that slack can refer to an activity as well as an event. It can be defined as the difference between the Latest Time and the Earliest Time. Since normally we deal with activity time in case of activities, slack and float have the same meaning. When slack is associated with an event, then the activity can have two slacks.

Head slack (slack of the head event) = LFT – EST of head event

Tail slack (slack of the tail event) = LFT – EST of tail event.

Float

Float can be described as the free time associated with an event. It is the time available for performing an activity in addition to the duration time. Hence, really float or slack is that time by which an activity can be delayed without delaying the entire project. These activities which do not have any, float or slack are the activities, which cannot be delayed without delaying the project. These activities are called the *critical activities*. Hence, along the critical path the float or slack is zero.

Float is an important concept in project planning. It helps the project management team to :

(*i*) priorities resources for allotment;

(*ii*) transfer of resources from one area to another depending upon where these are required earlier;

(*iii*) minimization of resources;

(*iv*) smoothen the use of resources.

Total Float

Total float is that time by which any activity can be maximum delayed without delaying the entire project. If the total float is used up in an activity, that particular activity and all the subsequent activities become critical.

Total Float = Latest occurrence time of the succeeding event–Earliest occurrence time of the preceding Event - duration of the activity.

Free Float

NOTES

It is that time by which an activity can be delayed without effecting the commencement of a succeeding activity at its earliest start time. Free float results when all preceding activities occur at the earliest event times and all succeeding activities also occur at the earliest event times.

Free Float = Earliest occurrence time of the succeeding events – Earliest occurrence time of the Preceding events-duration of the activity.

Independent Float

Independent float is a measure of spare time that is available in an activity if it is started as late as possible and finished as early as possible. Hence, it is that amount of time by which an activity can be delayed, when all preceding activities are completed as late as possible and all succeeding activities are completed as early as possible.

Independent Float = Earliest occurrence time of the succeeding event–Latest occurrence time of Preceding event-duration of the activity.

Example 14. *A project consisting of eight activities is shown with the help of a network diagram below. Activity times have been marked on top of the arrow in brackets, calculate EST, LST, EFT and LFT. Also calculate the total float for each activity.*

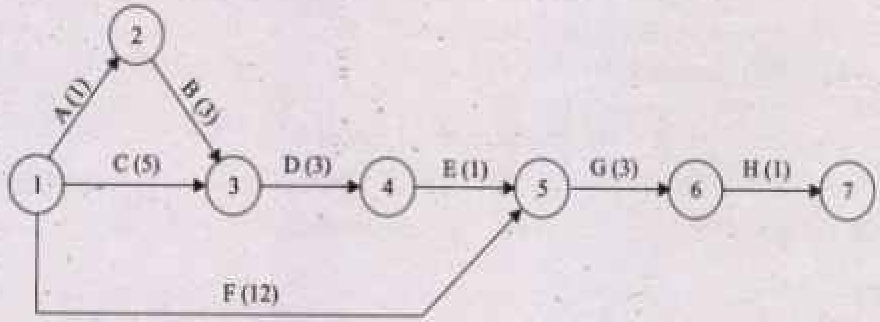

Fig. 5.16

Solution. EST = Earliest Event Time of the tail event

EET = EST (1 – 2) = 0

Earliest Event Time = Earliest occurrence time of the event preceding the event + duration of the activity.

EST (1 – 3) = 0

EST (1 – 5) = 0

EST (1 – 5) = 0

EST (2 – 3) = 1

EST (3 – 4) = 5

EST (4 – 5) = 8

EST (5 – 6) = 12

EST (6 – 7) = 15

NOTES

Also, let us calculate the Latest Finish Time from the above rework.

LFT (1 – 2) = 5
LFT (2 – 3) = 5 + 3 = 8
LFT (1 – 3) = 8
LFT (3 – 4) = 8 + 3 = 11
LFT (4 – 5) = 11 + 1 = 12
LFT (1 – 5) = 12
LFT (5 – 6) = 12 + 3 = 15
LFT (6 – 7) = 15 + 1 = 16.

LST can be calculated as

LST = LET – Duration of the activity converging on the head event

LST (1 – 2) = 5 – 1 = 4
LST (2 – 3) = 8 –3 = 5
LST (1 – 3) = 8 – 5 = 3
LST (3 – 4) = 11 – 3 = 8
LST (4 – 5) = 12 – 1 = 11
LST (1 – 5) = 12 – 12 = 0
LST (5 – 6) = 15 – 3 = 12
LST (6 – 7) = 16 – 1 = 15

EFT can be calculated as follows :

EFT = EST + duration of the activity emanating from tail event

EFT (1 – 2) = 0 + 1 = 1
EFT (2 – 3) = 3 + 1 = 4
EFT (1 – 3) = 0 + 5 = 5
EFT (3 – 4) = 5 + 3 = 8
EFT (4 – 5) = 8 + 1 = 9
EFT (1 – 5) = 0 + 12 = 12
EFT (5 – 6) = 12 + 3 = 15
EFT (6 – 7) = 15 + 1 = 16

These timing can now be entered in the network.

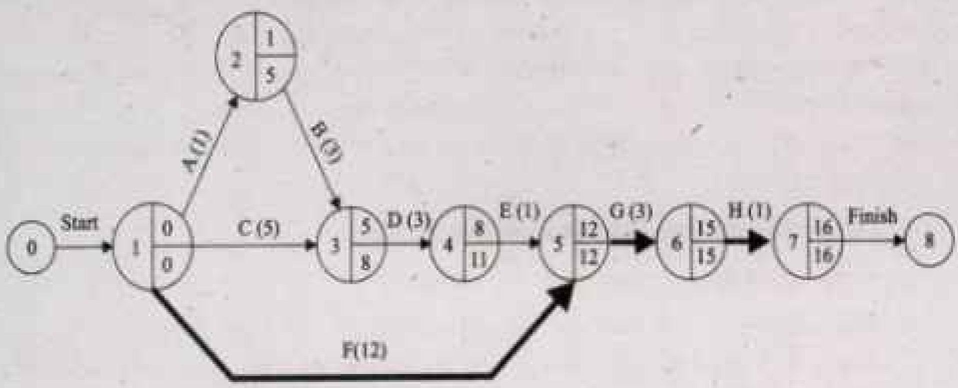

Fig. 5.17

EST have been entered on top half and LFT on the lower half.

Total Float can be calculated as follows :

NOTES

Total Float = Latest occurrence time of the succeeding event – Earliest occurrence time of preceding event – duration of the activity.

$= LST - EST$

Float $(1-2) = 4 - 0 = 4$

Float $(2-3) = 5 - 1 = 4$

Float $(1-3) = 3 - 0 = 3$

Float $(4-5) = 11 - 8 = 3$

Float $(1-5) = 0 - 0 = 0$

Float $(5-6) = 12 - 12 = 0$

Float $(6-7) = 15 - 15 = 0$

These values of EST, LST, EFT and LFT as also the total float can be put in the form of a table as shown on the next page.

Activity	Duration	EST	LST	EFT	LFT	Total float
A	1	0	4	1	5	4
B	3	1	5	4	8	4
C	5	0	3	5	8	3
D	3	5	8	8	11	3
E	1	8	11	9	12	3
F	12	0	0	12	12	0 Critical
G	3	12	12	15	15	0 activities
H	1	15	15	16	16	0

Critical path FGH has been shown with thick line $(1-5-6-7)$.

The total project duration $= 12 + 3 + 1 = 16$.

Project cost and crashing of activities

Project costs are the most vital aspects of project management, if due to any reasons, there are cost over-runs, the entire decision-making process may be affected adversely. One major advantage of Critical Path Method is that it is able to establish a relationship between time and cost. The management is always interested in cutting down the project time, since critical path measures the expected duration of the project time, through identification of the critical activities which need special attention. *The aspect of project planning in which the project duration is intended to be reduced is called project crashing.* It is desirable for the following reasons :

(*a*) Completing the project in the least possible time

(*b*) Reducing the project cost as far as possible

(*c*) Time and hence cost over-runs can be minimized as the project managers can take measures to expedite other activities if the critical activities have taken more time than planned for.

(*d*) Reduction in idle time of the facilities and smoothing the utilization of the resources.

(*e*) Plans can be made to utilize the resources and facilities in efficient manner and these can be transferred /switched over to the other more profitable / desirable projects.

(*f*) The duration of the activities can be reduced by either allocating more resources in manpower and machines as originally planned for or by working over times in different shifts.

Example 15. *Draw a network from the following activities and find a critical path and duration of the project.*

Activity	*Duration (Days)*	*Activity*	*Duration (Days)*
1 – 2	*10*	*5 – 7*	*7*
2 – 3	*8*	*6 – 8*	*9*
3 – 4	*12*	*7 – 8*	*6*
3 – 5	*13*	*8 – 9*	*15*
4 – 6	*7*	*9 – 10*	*17*
5 – 6	*11*		

Solution.

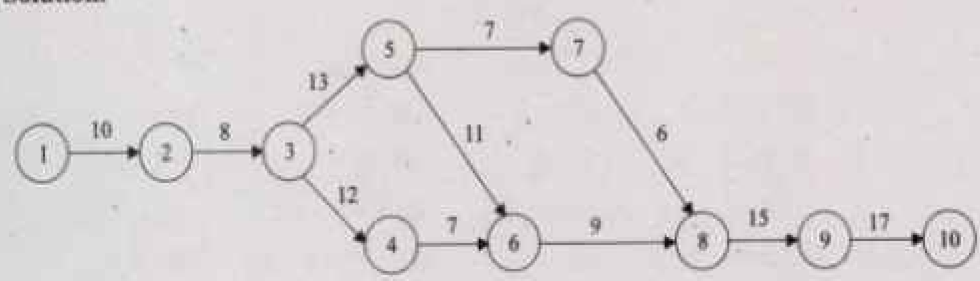

Fig. 5.18

Various paths	**Duration of paths**
1 – 2 – 3 – 5 – 7 – 8 – 9 – 10	10 + 8 + 13 + 7 + 6 + 15 + 17 = 76
1 – 2 – 3 – 5 – 6 – 8 – 9 – 10	10 + 8 + 13 + 11 + 9 + 15 + 17 = 83
1 – 2 – 3 – 4 – 6 – 8 – 9 – 10	10 + 8 + 12 + 7 + 9 + 15 + 17 = 78

Hence the critical path is 1 – 2 – 3 – 5 – 6 – 8 – 9 – 10 with total duration of 82 day. It is marked with thick lines.

Example 16. *A small project consists of the following twelve jobs whose precedence relations are identified with their node numbers as follows :*

Job	*Precedence*	*Duration (Days)*	*Job*	*Precedence*	*Duration (Days)*
A	*1 – 2*	*10*	*G*	*3 – 7*	*12*
B	*1 – 3*	*4*	*H*	*4 – 5*	*15*
C	*1 – 4*	*6*	*I*	*5 – 6*	*6*
D	*2 – 3*	*5*	*J*	*6 – 7*	*5*
E	*2 – 5*	*12*	*K*	*6 – 8*	*4*
F	*2 – 6*	*9*	*L*	*7 – 8*	*7*

(*a*) *Draw a network diagram representing the project.*

(*b*) *Find the critical path and project duration.*

(*c*) *Calculate EST, EFT, LST, LFT for all the jobs.*

(*d*) *Tabulate Total Float, Free Float, Independent Float.*

Solution. The network diagram is shown below :

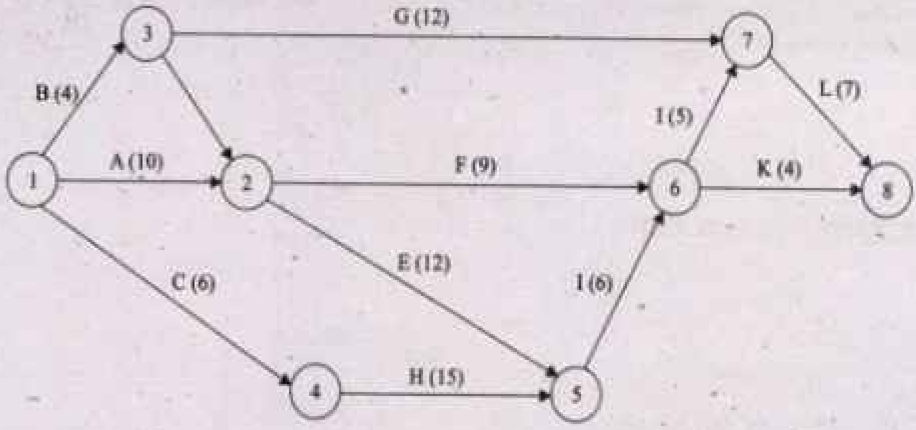

Fig. 5.19

Paths	Duration
1 – 2 – 3 – 7 – 8	(10 + 5 + 12 + 7) = 34
1 – 2 – 6 – 7 – 8	10 + 9 + 5 + 7 = 31
1 – 2 – 6 – 8	10 + 9 + 4 = 23
1 – 2 – 5 – 6 – 7 – 8	10 + 12 + 6 + 5 + 7 = 40
1 – 2 – 5 – 6 – 8	10 + 12 + 6 + 4 = 32
1 – 3 – 7 – 8	4 + 12 + 7 = 23
1 – 4 – 5 – 6 – 7 – 8	6 + 15 + 6 + 5 + 7 = 39
1 – 4 – 5 – 6 – 8	6 + 15 + 6 + 4 = 31

Critical path is 1 – 2 – 5 – 6 – 7 – 8 with duration of 40 days. It is marked with thick lines in the network diagram.

Computation of EST, EFT, LST and LFT :

Job	Duration	EST	LET	LFT	LST	Total Float	Head Event	Free Float
(1)	(2)	(3)	(4)	5 = (3 + 2)	(6 = 4 – 2)	(7 = 6 – 3)	(8)	(9 = 7 – 8)
1 – 2	10	0	10	10	0	0	0	0
1 – 3	4	0	21	4	17	6	6	11
1 – 4	6	0	7	6	1	1	1	0
2 – 3	5	10	21	15	16	6	6	0

NOTES

2 – 5	12	10	22	10	0	0	0	0
2 – 6	9	10	28	29	19	9	0	9
3 – 7	12	15	33	27	21	6	0	6
4 – 5	15	6	22	21	7	1	0	1
5 – 6	6	22	28	28	22	0	0	1
6 – 7	5	28	33	33	28	0	0	0
6 – 8	4	28	40	32	36	0	0	8
7 – 8	7	33	40	40	33	0	0	0

Terminology in Time Cost Relationship

(*a*) Normal Time of an activity (t_n)

(*b*) Crash Time of activity (t_c)

(*c*) Normal cost (C_n)

(*d*) Crash cost of the activity (C_c)

(*e*) Activity cost slope or angle $= \dfrac{\Delta C}{\Delta T} = \dfrac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \dfrac{C_c - C_n}{t_n - t_c}$

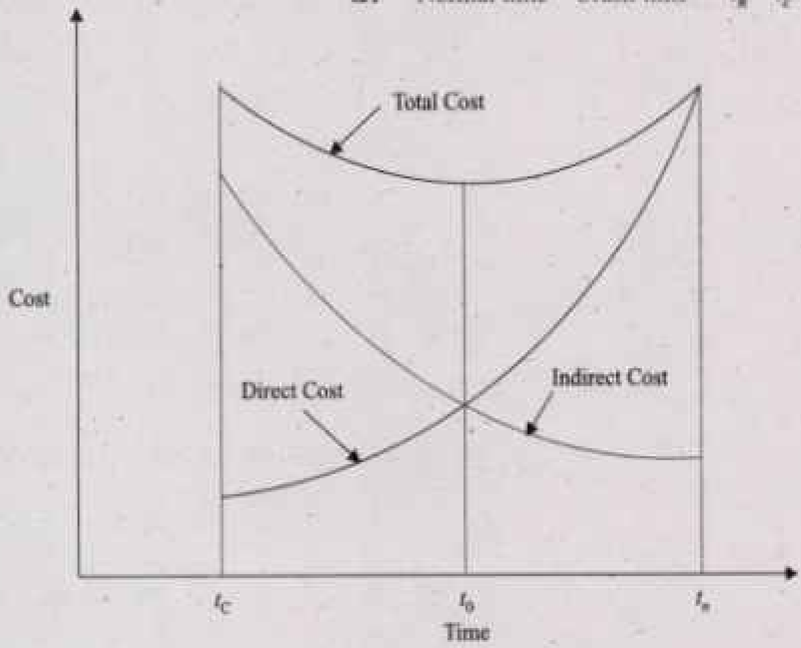

Fig. 5.20

It may be seen above, that the indirect cost of the project decreases with the increase in the duration of time, whereas direct cost increases with time and these two costs are opposite to each other. The sum of the two costs is shown as total cost. The project duration for which the total cost is minimum is called the *optimum time duration* shown as t_0.

5.6 STEPS IN PROJECT CRASHING

NOTES

The following steps are involved in project crashing :

Step I. Calculation of the cost slope

As shown earlier cost slope = $\dfrac{\Delta C}{\Delta T} = \dfrac{C_c - C_n}{t_n - t_c}$

where C_c, C_n, t_n and t_c have the usual meaning and $\dfrac{\Delta C}{\Delta T}$ denotes the cost of reducing duration of an activity by one unit of time.

Step II. Mark the critical path from which the expected duration of the project is found. Find the associated project cost for this critical path.

Step III. Select the least cost slope activity out of the critical path activities. If there happen to be more than one critical path, then select one such activity on each of the critical paths.

Step IV. Keep reducing the activity time of the selected activity unless and until either crash time is reached or the earlier non-critical parallel path becomes critical.

Step V. Step II to IV are repeated until we identify a critical path on which none of the activities can be further crashed.

Step VI. List the time and cost in the form of a matrix and select optimum duration of the project.

Example 17. *The network of a small project is shown below :*

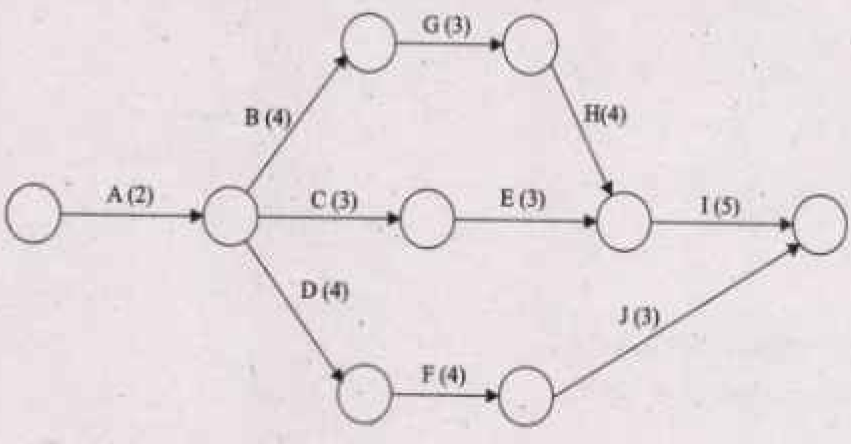

Fig. 5.21

The data for the cost and time is also given below. If the indirect cost of the project is estimated to be ₹ 100 per day of the project duration what is the optimal project duration?

Activity	*Normal Time (days)*	*Crash Time (days)*	*Normal Cost (₹)*	*Crash Cost (₹)*
A	2	1	70	80
B	4	2	80	200
C	3	1	130	230

D	4	2	130	300
E	3	3	120	120
F	4	2	80	120
H	4	2	100	280
I	5	2	80	200
J	3	2	60	00

Total Normal Cost = 1120

Solution. Step I. Calculation of the cost slopes of the each of the activities of the project.

$$A = \frac{80-60}{2-1} = 20$$

$$B = \frac{200-80}{4-2} = 60$$

$$C = \frac{100}{2} = 50$$

$$D = \frac{100}{2} = 50$$

$$E = 0$$

$$F = \frac{40}{2} = 20$$

$$G = \frac{180}{2} = 90$$

$$H = \frac{140}{2} = 70$$

$$I = \frac{120}{3} = 40$$

$$J = \frac{30}{1} = 30$$

Step II. Identify critical path and find the expected duration of the project and direct cost of the project.

CP = A – B – G – H – I, Expected normal duration = 2 + 4 + 3 + 4 + 5 = 18 days

Direct Cost = ₹ 1120

Step III. Least cost activity on the critical path is A, as it has the lowest cost slope of 20 and this can be crashed by 1 day (crash time = 1 day given in the problem, *i.e.*, 2 – 1 = 1)

New duration = 18 – 1 = 17 day

New cost = ₹ 1120 + 1 × 20 = ₹ 1140

The new network with activity A crashed (circled to show that it has been crashed) it shown below.

NOTES

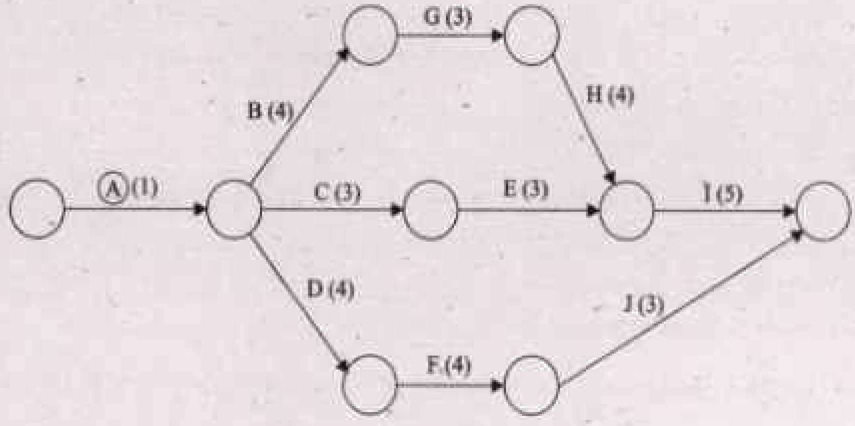

Fig. 5.22

Step IV. Repeat step II to III

Out of the remaining activities on critical path, BGHI, activity I has the lowest unit cost of crashing of 40. It can be crashed by $(5 - 2) = 3$ days.

$$\text{New Duration of the project} = 17 - 3 = 14 \text{ days}$$

$$\text{New Project cost} = ₹\ 1140 + 3 \times 40$$

$$= ₹\ 1260$$

Out of the remaining three activities on the CP, *i.e.*, BGH activity B has the lowest cost of 60 and it can be crashed by $= 4 - 2 = 2$ days

$$\text{New project duration} = 14 - 2 = 12 \text{ days}$$

$$\text{New Project Cost} = 1260 + 2 \times 60 = ₹\ 1380$$

The new network may be drawn to show the impact of crashing.

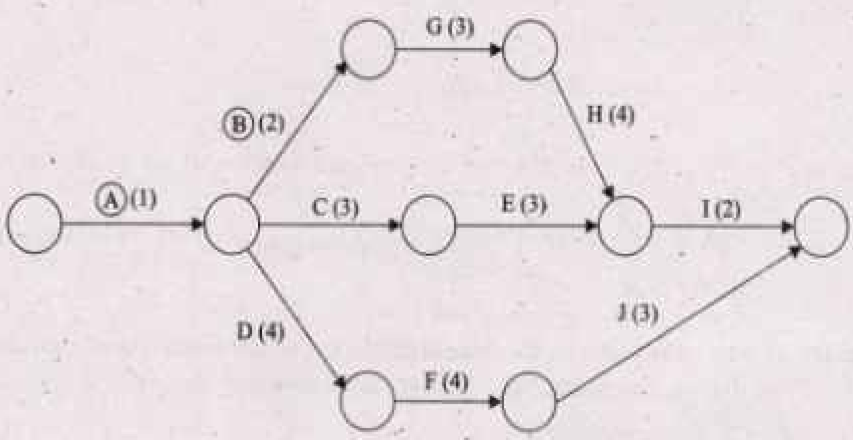

Fig. 5.23

With crashing of B, the path ADFJ has also become critical. To crash the project duration further, we select one activity from each of the two critical paths and crash each selected activity by smallest of duration by which these activities can be crashed.

On path ABGHI, G and H are left for crashing and in the path ADFJ, three activities DFJ can be crashed. Since both activities H and F can be crashed by 2 days (*i.e.*, H = 4 – 2 = 2, F = 4 – 2 = 2), it will results in

New Project duration = 12 – 2 = 10

Project Cost = 1380 + 2 × 70 + 2 ×20 as cost slope of Hand F are ₹ 70 and 20 respectively = ₹ 1560

The crashed network is shown below.

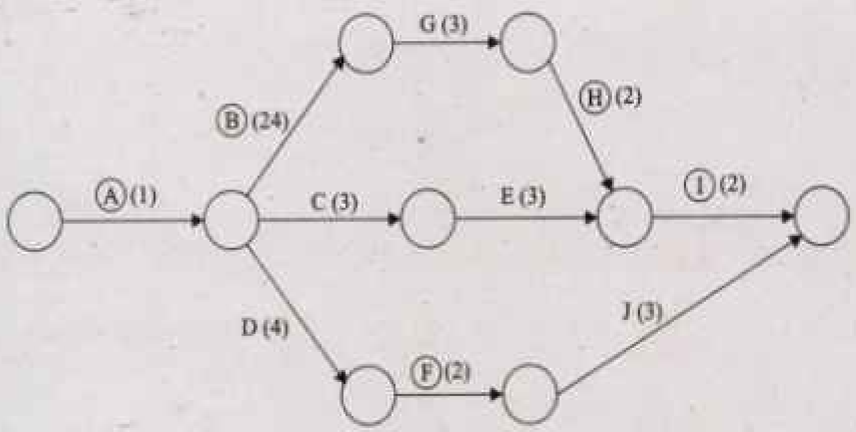

Fig. 5.24

On the original critical path only one activity G remains to be crashed which can be crashed by 2 days but costs ₹ 90/ to crash per day. On the other CP activities D and J remain to be crashed which cost ₹ 50 and ₹ 30 to crash per day. Since their total cost is less than ₹ 90 (*i.e.*, ₹ 50 + ₹ 30 < ₹ 90) activities D and J have been selected to be crashed. D can be crashed by 2 days but J can be crashed by one day, hence both will be crashed by one day.

New project duration = 10 – 1 = 9 days

New project cost = 1560 + 90 × 1 + 30 × 1

= ₹ 1680

It can be seen in the network drawn below that the crashing of activities G and J have made all the three paths critical. Now only one activity *i.e.*, G remains to be crashed on CP, A – B – G – H – I, similarly only activity D remains to be crashed on CP A – D – F – J. But on the third CP, ACEI two activities C and E remain to be crashed. We have to select one activity each from each of the CPs and crash it. From CP, A – C – E – I activity C will crash since E cannot be crashed technically. So, activities G, C and D have to crash. Out of these G could originally be crashed by 2 days but it has already been crashed by one day. All the three activities on the three CPs will be crashed by one day.

New Project duration = 9 – 1 = 8 days

New Project cost = 1680 + 90 × 1 + 50 × 1 + 50 × 1

= ₹ 1870

NOTES

The finally crashed network can be shown below.

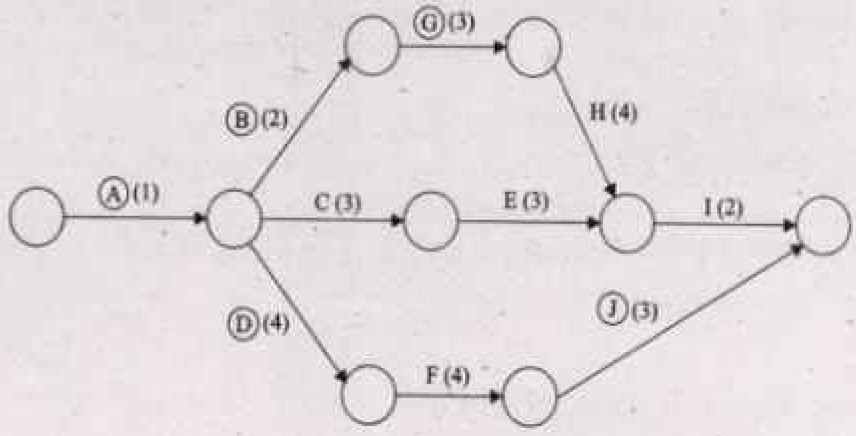

Fig. 5.25

Crashed duration of the project = 8 days on all three critical paths the total duration is 8 days only.

Step V. List the project-time cost in a table and select the optimal duration of the project.

These are drawn in the following table.

Project Duration (days)	Direct Cost (₹)	Indirect Cost (₹) @ ₹ 100 per day	Total Project Cost
18	1120	1800	2920
17	1140	1700	2840
14	1260	1400	2660
12	1380	1200	2580
10	1560	1000	2560
9	1680	900	2580
8	1870	800	2670

It may seen that the cost is minimum when the project duration is 10 days. The result of crashing exercise undertaken above can be summarized as

Normal duration of the project = 18 days

Crashed duration of the project = 8 days

Optimal duration of the project = 10 days

Minimum cost of the project = ₹ 2560

5.7 PROBABILITY AND PROJECT PLANNING

As explained earlier in this chapter, *PERT is able to provide help in decision-making under conditions of uncertainty. Uncertainty is almost always associated with the project completion time and completion of different activities in planned time.*

NOTES

Using the concept of time estimates, optimistic time, most likely time and pessimistic time and the formula associated with these,

$$T_{e_p} = \text{Expected time of completion of the project}$$

$$= \sum t_{e_1} + t_{e_2} + t_{e_3} + \ldots + t_{e_n}$$

where $t_{e_1}, t_{e_2}, \ldots, t_{e_n}$ are the expected times of the activities on critical path and $V_1, V_2, \ldots, V_n$ are the variances of the activities.

$$\text{Variance } V = \frac{b-a}{6} \text{ and Standard Deviation } s = \left(\frac{b-a}{6}\right)^2$$

then $$\sigma = \sqrt{V_1 + V_2 + V_3 + \ldots + V_n}$$

Example 18. *Activities of a small project given below. The network of this project is also drawn. What is the probability of completing the project within 26 days, within 28 days.*

Activity	*Most Optimistic Time (Days)*	*Most Likely Time (Days)*	*Most Pessimistic Time (Days)*
1 – 2	*1*	*1*	*1*
2 – 3	*1*	*4*	*7*
2 – 4	*8*	*12*	*10*
3 – 5	*3*	*5*	*7*
4 – 5	*1*	*1*	*1*
5 – 6	*3*	*6*	*9*
5 – 7	*4*	*6*	*8*
6 – 8	*4*	*8*	*12*
7 – 8	*2*	*5*	*8*

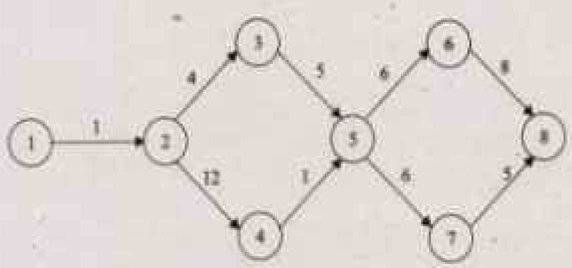

Fig. 5.26

Solution. The expected time t_e and the variance of different activities can be fond out. It is given in the following table :

NOTES

Activity	Most optimistic time (days)	Most likely time (days)	Most pessimistic time (days)	Expected variance time $t = \frac{a + 4m + b}{6}$	Standard deviation $\sigma^2 = \left(\frac{b-a}{6}\right)^2$
1 – 2	1	1	1	1	0
2 – 3	1	4	7	4	1
2 – 4	8	12	10	11	0.111
3 – 5	3	5	7	5	0.44
4 – 5	2	1	3	3	.027
5 – 6	3	6	9	6	1
5 – 7	4	6	8	6	0.44
6 – 8	4	8	12	8	1.78
7 – 8	2	5	8	5	1

EST for each activity can be calculated

Node 1 = 0

Node 2 = 0 + 8 = 8

Node 3 = 8 + 4 = 12

Node 4 = 8 + 12 = 20

Node 5 = Maximum out of [(12 + 5), (20 +1)] = 21

Node 6 = 21+ 6 = 27

Node 7 = 21 + 6 = 27

Node 8 = Maximum out of [(21 + 8) and (21 + 5)] = 29.

Similarly, LST for each activity can be calculated.

LST Node 8 = 29

Node 7 = 29 – 5 = 24

Node 6 = 29 – 8 = 21

Node 5 = Maximum out of [(21 – 6), (24 – 6)] = 15

Node 4 = 15 – 3 = 12

Node 3 = 15 – 5 = 10

Node 2 = Min out of [(12 – 11), (10 – 4)] = 1

Node 1 = 1 – 1 = 0

Let us redraw the network showing the critical path with a thick line.

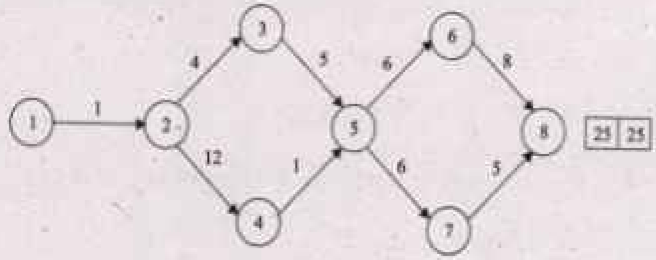

Fig. 5.27

Critical path is 1 – 2 – 4 – 5 – 7 – 8.

Expected time in completing the project = 1 + 12 + 1 + 6 + 25 days

$$\text{Project variance} = \sigma^2 = 0 + 0.111 + .027 + 0.44 + 1$$

$$= \mathbf{1.578}$$

$$\sigma = \sqrt{1.578} = 1.256$$

Probability of completing the project in 30 days,

$$Z = \frac{X - \overline{X}}{\sigma} = \frac{26 - 25}{1.256} = 0.796$$

where, $X = 30$ days (time under consideration)

$\overline{X}$ = Length of critical path = 25 days

σ = SD of critical path

The value from the cumulative normal distribution table for Z = 0.796 is 0.7852. Hence, the probability of completing the project within 28 days is 79.6 %.

Similarly, when we have to find probability of completing the project in 28 days,

$$Z = \frac{28 - 25}{1.256} = 2.388$$

The value from the table for Z = 2.388 is 991576

i.e., the probability of completing the project in 28 days is 99.15%

Example 19. *An R & D project has large number of activities but the management is interested in controlling a part of these activities 7, in number. The following data is available for these 7 activities :*

Activity	*Preceding activity*	*(a)*	*Times (m)*	*(b)*
A	*None*	*4*	*6*	*8*
B	*A*	*6*	*10*	*8*
C	*A*	*8*	*18*	*10*
D	*B*	*9*	*9*	*9*
E	*C*	*10*	*4*	*4*
F	*A*	*5*	*5*	*5*
G	*D, E, F*	*8*	*6*	*10*

(i) *Draw a PERT network for the activities shown in the table.*

(ii) *Prepare the schedule of the 7 activities.*

(iii) *Mark the critical path on the network.*

(iv) *If the management puts a deadline of 37 days for completion of this part of the project, determine the probability it will be completed in 37 days.*

(v) *When should the management start these activities to get a confidence level of 99% of completion of these activities in the scheduled time?*

Solution. The network for the above data an be drawn as shown in Figure 5.28 :

(*i*)

NOTES

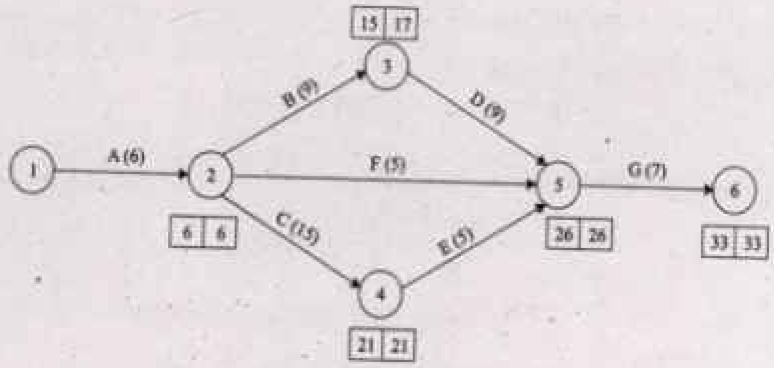

Fig. 5.28

(*ii*) Time for each activity has been determined using the formula

$t = \dfrac{a + 4m + b}{6}$ and shown is brackets along with the activities.

Activity	A	B	C	D	E	F	G
Time	6	9	15	9	5	5	7

The critical path is A – C – E – G marked with thick lines and the expected length of this part of project = 6 + 15 + 5 + 7 = 33 days.

Now let us work out the variance, *i.e.*, σ^2 using the formula $\left(\dfrac{b-a}{6}\right)^2$

Activity	A	B	C	D	E	F	G
Time	0.444	0.111	0.111	0	1	0	0.111

$$\text{Variance} = 0.444 + 0.111 + 0.111$$

$$= 1.666 \text{ or } \sigma = \sqrt{1.666} = 1.29$$

EST and LST have been shown along side each node.

(*iii*) Probability that the project will be completed in 37 days.

$$Z = \frac{37 - 39}{1.29} = \frac{-2}{1.29} = -1.55$$

For $\quad Z = -1.55$ the value from the tables is 0.93943.

i.e., the probability that this part of the project will be completed in 37 days is 93.94 %.

(*iv*) For 99% assurance the Z value from the table is 2.33.

$$Z = \frac{X - 39}{1.29} \text{ we can substitute Z value in this.}$$

$$2.33 = \frac{X - 39}{1.29} \text{ or } X - 39 = 2.33 \times 1.29 = 3$$

or $\quad X = 42$

The management has 99% assurance, that this part of project will be completed in 42 days.

Example 20. *Given below is the list of activities along with their predecessor activties. Three time estimates are also provided.*

Activity	*Predecessor Activity*	*Most optimistic (a)*	*Time (weeks) Most likely (m)*	*Most pessimistic (b)*
A	*NIL*	*1*	*2*	*9*
B	*A*	*2*	*3*	*4*
C	*A*	*2*	*4*	*6*
D	*A*	*3*	*5*	*7*
E	*C*	*5*	*7*	*9*
F	*D*	*1*	*3*	*5*
G	*B*	*1*	*4*	*7*
H	*G*	*2*	*6*	*10*
I	*E, H*	*4*	*8*	*6*
J	*F*	*2*	*6*	*10*

What is the probability of critical path being completed in (i) 23 days (ii) 21 days?

Solution. For drawing the network we need the activity time, which can be calculated using the relationship $\frac{a+4m+b}{6}$. Also for finding out the probabilities the σ^2 must be calculated. These calculations ae given below.

Activity	A	B	C	D	E	F	G	H	I	J
Activity Time	3	3	4	5	7	3	4	6	7	6
$\left(\frac{b-a}{6}\right)^2=\sigma^2$	3.16	0.11	0.44	0.44	0.44	0.44	1	1.77	0.11	1.77

Now the network can be drawn

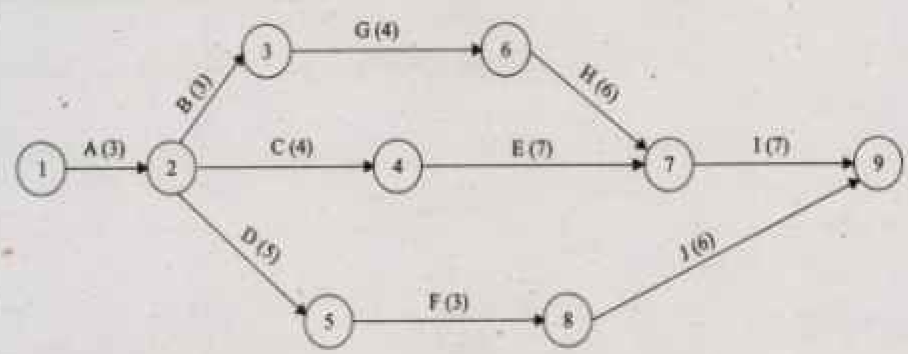

Fig. 5.29

The activity timings have been shown in brackets along with the activity on top of the arrow. The critical path, the path with longest duration is ABGHI and the total duration of the activities on critical path is 3 + 3 + 4 + 6 + 7 = 23 weeks. It is marked with thick lines in the network.

$$\text{Variance } \sigma^2 \text{ on critical path} = 3.16 + 0.11 + 1 + 1.77 + 0.11$$
$$\sigma^2 = 6.15$$
$$\sigma = 2.48$$

$$\text{Standard normal deviation } Z = \frac{\text{Scheduled time} - \text{Duration of the critical path}}{\text{Standard Deviation of critical path } (\sigma)}$$

NOTES

If the scheduled time of completion is 25 days as given in the problem,

then $$Z = \frac{25-23}{2.48} = 0.8051$$

Hence, probability of completion of the critical path of the project is 80.51%. If the scheduled time is 21 days.

$Z = \dfrac{12-23}{2.48} = -0.806$. Ighoring the negative value read the probability value, which is 0.8051.

$\therefore$ Probability of completing the project is 21 days $= 1 - 0.8051 = 0.195$

i.e., the probability completing the project in 21 days 1.95%.

5.8 SUMMARY

- Replacement of old plant and equipment and items of use like bulbs/tube-lights, refrigerators/heating, tools/gadgets, etc, is a necessity. All these items are designed for performance up to the desired level for a particular time (years/hours) or particular number of operations.
- When making replacement decisions, the management has to make certain assumptions, these are :
  - (*i*) The quality of the output remains unchanged.
  - (*ii*) There is no change in the maintenance costs.
  - (*iii*) Equipments perform to the same standards.
- Most of the machinery and equipment having moving parts deteriorate in their performance with passage of time. The cost of maintenance and repair keeps increasing with passage of time and a stage may reach when it is more economical (in overall analysis) to replace the item with a new one.
- Activity is the smallest unit of productive efforts to be planned, scheduled and controlled.
- *A network is the graphical representation of logically and sequentially connected arrows representing activities and nodes representing events of a project.*
- As explained earlier *PERT is a very useful technique for planning the time and resources of any project.*
- The longest path is the most limiting path. This path is called the *Critical Path.*
- Slack *should be appreciated that slack can refer to an activity as well as an event.* It can be defined as the difference between the Latest Time and the Earliest Time.
- One major advantage of Critical Path Method is that it is able to establish a relationship between time and cost.
- *The aspect of project planning in which the project duration is intended to be reduced is called project crashing.*
- *PERT is able to provide help in decision-making under conditions of uncertainty. Uncertainty is almost always associated with the project completion time and completion of different activities, in planned time.*

5.9 REVIEW QUESTIONS

1. What is replacement problem? When does it arise?
2. Describe various types of replacement situations.
3. Enumerate various replacement problems.
4. What are the situations which make the replacement of items necessary?
5. Give a brief account of situations of which the replacement problems arise. What does the theory of replacement establissh?
6. Discuss in brief replacement procedure for the items that deteriorate with time.
7. The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant. Show that the average annual cost will be minimized by replacing the machine when the average cost to date becomes equal to the current maintenance cost.
8. Discuss the replacement problem where items are such that maintenance costs increase with time and the value of money also changes with time.
9. Find the optimum replacement policy which minimizes the total of all future discounted costs for an equipment which costs ₹ A and which needs maintenance costs of ₹ $C_1, C_2, ..., C_n$ etc.($C_{n+1} > C_n$) during the first year, second year etc., and further D is the depreciation value per unit of money during a year.
10. State some of the simple replacement policies.
11. Construct the cost equation reflecting the discounted value of all future costs for a policy of replacing equipment after every *n* periods. Hence establish the following :
    (*i*) Replace if the next period cost is greater than the weighted average of previous costs.
    (*ii*) Do not replace if the next period's cost is less than the weighted average of previous costs.
12. What is "group replacement"? Give an example.
13. Write a short note on group replacement and individual replacement policies.
14. The cost per item for the individual replacement is C_1 and the cost per item of group replacement is C_2. If only individual replacement is more economical than the group replacement along with the individual, find relation between C_1 and C_2.
15. A truck has been purchased at a cost of ₹ 160000. The value of the truck is depreciated in the first three years by ₹ 20000 each year and ₹ 16000 per year thereafter. Its maintenance and operating costs for the first three years are ₹ 16000, ₹ 18000 and ₹ 20000 in that order and increase by ₹ 4000 every year. Assuming an interest rate of 10% find the economic life of the truck.
16. A manual stamper currently valued at ₹ 10000 is expected to last 2 years and costs ₹ 4000 per year to operate. An automatic stamper which can be purchased for ₹ 3000 will last 4 years and can be operated at an annual cost of ₹ 3000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.
17. The cost of a new machine is ₹ 5000. The maintenance cost of *n*th year is given by
    $R_n = 500\,(n-1);\ n = 1, 2, \ldots\ldots$
    Suppose that the discount rate per year is 0.05. After how many years will it be economical to replace the machine by new one?

NOTES

18. A machine costs ₹ 10000 operating costs are ₹ 500 per year for the first five years. Operating costs increase by ₹ 100 per year in the sixth and succeeding years. Assuming a 10 per cent discount rate of money per year, find the optimum length of time to hold the machine before it is replaced. State clearly the assumptions made.

19. An individual is planning to purchase a car. A new car will cost ₹ 120000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are ₹ 20000 and they increase by 15% every year. The minimum resale value of the car can be ₹ 40000.

    (*i*) When should the car be replaced to minimum average annual cost (ignore interest)?

    (*ii*) If interest of 12% is assumed, when should the car be replaced?

20. A large computer installation contains 2000 components of identical nature which are subject to failure as per probability distribution given below :

Weekend :	1	2	3	4	5
Percentage failure to date :	10	25	50	80	100

Components which fail have to be replaced for efficient functioning of the system. If they are replaced as an when failure occur, the cost of replacement per unit is ₹ 3. Alternatively, if all components are replaced in one lot at periodical intervals and individually replaced only as such failures occur between group replacement, the cost of component replaced is ₹ 1.

   (*a*) Access which policy of replacement would be economical.

   (*b*) If group replacement is economical at current costs, then assess at what cost of individual replacement would group replacement be uneconomical.

   (*c*) How high can the cost per unit in group replacement be to make a preference for individual replacement policy?

21. Let $p\,(t)$ be the probability that a machine in a group of 30 machines would breakdown in period t. The cost of repairing a broken machine is ₹ 200. Preventive maintenance is performed by servicing all the 30 machines at the end of T unit of time. Preventive maintenance cost is ₹ 15 per machine. Find optimum T which will minimize the expected total cost per period of servicing, given that

$$p(t) = \begin{cases} 0.03 & \text{for } t=1 \\ p(t-1)+0.01 & \text{for } t=2, 3, \ldots, 10 \\ 0.13 & \text{for } t=11, 12, 13, \ldots \end{cases}$$

What is the optimum replacement plan?

22. A manufacturer wants to replace a machine. The purchase price of the machine is ₹ 10000. Following other detail are available :

Year	Maintenance (₹)	Resale Price (₹)
1	1200	6000
2	1300	3000
3	1500	2000

4	2000	1000
5	2200	800
6	3000	500
7	3200	400
8	3800	300

Suggest in which year the machine may be replaced, if the supplier of the machine is prepared to prove 3 years *in situ* maintenance free of cost.

23. What is the critical path analysis? What are the areas where this technique can be applied?
24. How does PERT differ from CPM? Describe briefly the basic steps to be followed in developing PERT/CPM programmed?
25. Under what circumstances would you use PERT as opposed to CPM is project management? Name a few projects where each would be more suitable than other.
26. What is the significance of three times estimates used in PERT? How an on what basis is a single estimate derived from these estimates?
27. What is critical path? What does it signify? What are its benefits?
28. Describe with the help of a diagram, the procedure to arrive at the critical path in a PERT network.
29. What do you understand from earliest finish time and latest finish time? How are they calculated? Explain your answer with an example.
30. What do you understand from slack? What are the different types of slacks? How does knowledge of slack help better project management?
31. A management institute plans to organize a conference on the use of "Quantities techniques for-decision-making". In order to coordinate the project, it has decided to use a PERT network. The major activities and times estimate for ach activity have been completed as follows :

Activity Description	Times Estimate	Activity that must precede
a. Design conference meeting theme	1-2-3	None
b. Design front cover of conference proceedings	1-2-3	A
c. Design brochure	1-2-3	A
d. Compile list of distinguished speakers	2-4-6	A
e. Finalize brochure and print it	2-5-14	C and D
f. Make travel arrangements for distinguished speakers	1-2-3	D
g. Send broachers	1-3-5	E
h. Receive papers for conference	10-12-30	G
i. Edit papers	3-5-7	H
j. Print proceedings	5-10-15	B and I

NOTES

(a) Construct an arrow diagram called network.

(b) Calculate expected time for each activity.

(c) Identify critical path and determine the project duration.

32. Major activities involved in the develpment of an item with a vendor are as under :

Activity	Duration	(Weeks)
A	...	2
B	...	1
C	...	2
D	...	1
E	...	5
F	...	8
G	...	4
H	...	2
I	...	1
J	...	4

Constraints :

(i) A is start activity.

(ii) B can start on completion of A.

(iii) C, E and H succeed B

(iv) C controls D, E controls F and H controls I

(v) G can commence after F is over.

(vi) J can start once D and I are over.

(vii) G and J are last activities.

(a) Draw the project network and identify all the paths.

(b) How many weeks are required by the vendor to develop the item ?

(c) What suggestions do you make to reduce the development time ?

33. A company manufacturing plant and equipment plant for chemical processing is in the process of quoting a tender called by a Public Sector Undertaking. Delivery date once promised is crucial as penalty clause is applicable. The wining of tender also depends on how soon the company is able to deliver the goods. Project manager has listed down the activities in the project as under :

S. No.	Activity	Immediate Proceeding Activity	Activity Time (Weeks)
1	A	...	3
2	B	...	4
3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3

(a) Find out the delivery week from the date of acceptance of quotation.

(b) Find out total float and free float for each of the activities.

34. Calculate EST, EFT and LFT for the following network. The duration for each activity is given on upper side of arrow line.

35. Time and cost data of the activities of a small project is given below :

Activity	Normal		Crash		Cost Slope	
	Time (Days)	Cost (₹)	Time (Days)	Cost (₹)	Time (Days)	Cost (₹)
1 – 2	3	360	2	400	1	40
2 – 3	6	1,440	4	1,620	1	90
2 – 4	9	2,160	5	2,380	4	55
2 – 5	7	1,120	5	1,600	2	240
3 – 4	8	400	4	800	4	100
4 – 5	5	1,600	3	1,770	2	85
5 – 6	8	480	7	769	1	280

The overhead cost per day is ₹ 160.

(i) Find critical path.

(ii) Crash the project to achieve optimum duration and optimum cost.

36. A project consists of nine activities. Activities are identified by their beginning (i) and ending (i) node numbers. The three estimates are listed in the table below.

Activity (i – j)	Estimated Duration (Weeks)		
	Optimistic	Most Likely	Pessimistic
1 – 2	1	1	7
1 – 3	1	4	19
1 – 4	1	4	7
4 – 5	2	5	14
2 – 6	2	5	8
5 – 6	1	4	19
5 – 6	1	4	19
3 – 7	2	5	14
6 – 7	3	6	15

(a) Draw the project network and identify all the paths through it.

(b) Identify the critical path and determine the expected project duration.

(c) Calculate variance and standard deviation of the project duration.

(d) What is the probability that the project will be completed.

(i) At least 2 weeks earlier than expected?

(ii) Not more than 2 weeks later than expected?

(e) What due date has a probability of completion f 0.95?

NOTES

Given normal distribution function

Normal Deviate (z)	Probability %	Normal Deviate (z)	Probability %
−0.9	18.4	+0.9	81.6
−0.1	15.9	+1.0	84.1
−1.1	13.6	+1.1	86.4
−1.2	11.5	+1.2	88.5
−1.3	9.7	+1.3	90.3
−1.4	8.1	+1.4	91.3

37. The following table gives for each activity of a project, its duration and responding resource requirement as well as total availability of each type of resources :

Activity	Duration (Days)	Resources (Machines)	Required (Men)
1–2	7	2	20
1–3	7	2	20
2–3	8	3	30
2–4	6	4	20
3–6	9	2	20
4–5	3	2	20
5–6	5	4	40

Minimum available Resources.

(*i*) Draw the Network, compute earliest Occurrence Time and Latest Occurrence Time for each event, the total float each activity and identify the critical path assuming that there are no resource constraints.

(*ii*) Under the given resource constrains find out the minimum duration to complete the project and compare the utilization of the resources for the duration.

38. A projection consists of 10 activities, each of which requires either, or both, of the two types of resources R_1 and R_2 for its performance. The duration of the activities an their resource requirements are as follows :

Activity	Duration (Days)	R_1	R_2
1–2	3	3	2
1–3	2	6	–
1–4	6	4	–
2–6	4	–	4
3–5	2	2	2
4–5	1	4	–
4–8	4	4	–
5–7	3	3	2
6–7	2	1	3
7–8	4	4	5

Resource availability : 8 units of R_1 and 5 units of R_2

Determine the duration of the project under given resource constraint. If the resources were not a problem, how long would the project take to complete in the normal course?

39. Explain the meaning of 'crashing' in network techniques.

40. What do you understand by the term direct cost and indirect cost in PERT costing techniques? How do they behave in project cost with range of duration?

41. (*a*) What do you mean crash duration?

    (*b*) Write a short note on project crashing using network analysis. (Also give graph for cost slope).

42. What is a least-cost schedule of a project? How is it obtained?

43. How do you distinguish between resource levelling and resource allocation problems? State and explain an algorithm of resource allocation.

44. Explain how network analysis can be used for resource planning and levelling in project management.

45. Explain the use of float in levelling of resources.

46. Give a procedure of resource levelling using PERT/CPM.

47. Distinguish between 'Precedence Diagram' and 'Network Diagram'.

48. A small project consisting of 8 activities has the following characteristics :

Activity	Preceding Activity	Time Estimate (Weeks)		
		Most Optimistic	Most Likely	Most Pessimistic
A	None	2	4	12
B	None	10	12	26
C	A	8	9	10
D	A	10	15	20
E	A	7	7.5	11
F	B, C	9	9	9
G	D	3	3.5	7
H	E, F, G	5	5	5

(*a*) Draw the PERT network for the project.

(*b*) Determine the critical path.

(*c*) If a 30 weeks deadline is imposed, what is the probability that the project will be finished within the time limit?

49. The following information relates to a construction project for which your company is about to sign a contract. Seven activities are necessary and the normal duration, normal cost, crash duration and crash cost have been derived from the best available sources.

NOTES

Activity	Preceding Activity	Duration in Weeks		DirectQ Cost (₹)	
		Normal	Crash	Normal	Crash
a	–	15	12	4,500	5,250
b	–	19	14	4,000	4,500
c	–	9	5	2,500	4,500
d	A	6	5	1,700	1,940
e	A	14	9	4,300	5,350
f	b, b	9	6	2,600	3,440
g	e	8	3	1,800	3,400

Each activity may be reduced to the crash duration in weekly stages at pro rata cost. There is a fixed cost of ₹ 500 per week.

Required :

(*a*) Draw, clearly labelled, a network and indicate the notation pattern used.

(*b*) Indicate the critical path and state the normal duration and cost.

(*c*) Calculate the critical total cost, showing clearly four working, and the revised duration and cost for each activity.

BBA-501

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