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BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 4: Integration

Unit 13: Integration as Inverse of Differentiation

Structure

13.1 INTRODUCTION

Integration and differentiation are two fundamental operations in calculus. They are inverse operations of each other in a certain sense.

Differentiation is the process of finding the derivative of a function, which measures how the function's output changes with respect to its input. The derivative of a function $f(x)$ is denoted by $f'(x)$ or $\frac{dy}{dx}$, where y is the dependent variable and x is the independent variable.

Integration, on the other hand, is the process of finding the antiderivative of a function. The antiderivative $F(x)$ of a function $f(x)$ is a function whose derivative is equal to $f(x)$. This process is also known as finding the indefinite integral of a function. The symbol $\int$ represents the process of integration, and $F(x) = \int f(x)\,dx$ denotes the antiderivative of $f(x)$.

13.2 OBJECTIVES

After completing this unit, students will be able to

- Understand the concept of integration
- Rember basic integration formulae
- Apply integration formulae to solve numerical problems

13.3 Integration: Indefinite Integral

If the function $f(x)$ be the derivative of the function $g(x)$ w.r.t. x on a certain interval of the x-axis, then we say that an integral of $f(x)$ w.r.t. x is $g(x)$ and write it symbolically as

$$\int f(x)\,dx = g(x).$$

The variable x is called the variable of integration. It is customary to use the differential $'dx'$ of the variable $'x'$ in the process of integration. The differential $'dx'$ indicate that the integration is to be carried w.r.t.x. The symbol $\int$ denote the process of integration. The function whose integral is to be found out is called the integrand. If the integral of $f(x)$ w.r.t. x is $g(x)$, then we also say that $g(x)$ is an antiderivative of $f(x)$.

Examples (i) x^3 is the derivative of the function $\frac{x^4}{4}$ w.r.t. x.

Therefore, $\frac{x^4}{4}$is **an** integral of x^3w.r.t.x i.e.,

$$\int x^3\,dx = \frac{x^4}{4}.$$

(ii) Since $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x$, i.e. $\sin x$ is **an** integral of $\cos x$.

Since $\frac{d}{dx}(\sin\ x) = \cos\ x$, we have $\int \cos\ xdx = \sin\ x$.
Moreover, if C is any constant then $\frac{d}{dx}(\sin\ x + C) = \cos\ x$.
So, in general, $\int \cos\ xdx = (\sin\ x + C)$.
Clearly, different values of C will give different integrals.
Thus, a given function may have an indefinite number of integrals. Because of this property, we call these integrals indefinite integrals.

Thus, $\frac{d}{dx}[g(x)] = f(x) \Rightarrow \int f(x)dx = g(x) + C,$ where C is a constant, called the constant of integration. Any function to be integrated is known as an integrand.

It is very important to note that the integral of a function may or may not exist. In other words, it may not be possible to find a function whose derivative is equal to the given function.

On the basis of differentiation and the definition of integration, we have the following results.

1 $\frac{d}{dx}(C) = 0 \Rightarrow \int 0\,dx = C.$

2 $\frac{d}{dx}(x) = 1 \Rightarrow \int 1\,dx = x + C.$

3 $\frac{d}{dx}(kx) = k \Rightarrow \int k\,dx = kx + C.$

4 $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1 \Rightarrow \int x^n dx = \frac{x^{n+1}}{(n+1)} + C.$

5 $\frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}dx = \log|x| + C.$

6 $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C.$

7 $\frac{d}{dx}\left(\frac{a^x}{\log\ a}\right) = a^x \Rightarrow \int a^x dx = \frac{a^x}{\log a} + C.$

8 $\frac{d}{dx}(\sin\ x) = \cos\ x \Rightarrow \int \cos\ xdx = \sin\ x + C$

9 $\frac{d}{dx}(-\cos\ x) = \sin\ x \Rightarrow \int \sin\ xdx = -\cos\ x + C$

10 $\frac{d}{dx}(\tan\ x) = \sec^2\ x \Rightarrow \int \sec^2\ xdx = \tan\ x + C$

11 $\frac{d}{dx}(-\cot\ x) = \mathrm{cosec}^2\ x \Rightarrow \int \mathrm{cosec}^2\ xdx = -\cot\ x + C$

12 $\frac{d}{dx}(\sec\ x) = \sec\ x\tan\ x \Rightarrow \int \sec\ x\tan\ xdx = \sec\ x + C$

13 $\frac{d}{dx}(-\mathrm{cosec}\ x) = \mathrm{cosec}\ x\cot\ x \Rightarrow \int \mathrm{cosec}\ x\cot\ xdx = -\mathrm{cosec}\ x + C$

14 $\frac{d}{dx}(\sin^{-1}\ x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}\ x + C$

15 $\frac{d}{dx}(\tan^{-1}\ x) = \frac{1}{(1+x^2)} \Rightarrow \int \frac{1}{(1+x^2)}dx = \tan^{-1}\ x + C$

16 $\frac{d}{dx}(\sec^{-1}\ x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}\ x + C$

Some Standard Results on Integration

1. $\frac{d}{dx}\left\{\int f(x)dx\right\} = f(x)$.
2. $\int k \cdot f(x)dx = k \cdot \int f(x)dx$, where k is a constant.
   $\int \{f_1(x) + f_2(x)\}dx = \int f_1(x)dx + \int f_2(x)dx$.
3. $\int \{f_1(x) - f_2(x)\}dx = \int f_1(x)dx - \int f_2(x)dx$.
4. $\int \{k_1 \cdot f_1(x) \pm k_2 \cdot f_2(x) \pm \cdots \pm k_n \cdot f_n(x)\}dx = k_1 \cdot \int f_1(x)dx \pm k_2 \cdot \int f_2(x)dx \pm \cdots \pm k_n \cdot \int f_n(x)dx$.

Example 1: $\int x^3 dx = \frac{x^{(3+1)}}{(3+1)} + C = \frac{x^4}{4} + C$.

Example 2: $\int \sqrt[3]{x}dx = \int x^{1/3}dx = \frac{x^{\left(\frac{1}{3}+1\right)}}{\left(\frac{1}{3}+1\right)} + C = \frac{3}{4}x^{4/3} + \text{C}$.

Example 3: $\int dx = \int x^0 dx = \frac{x^{(0+1)}}{(0+1)} + C = x + C$.

Example 4: $\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{(-3+1)}}{(-3+1)} + C = -\frac{1}{2x^2} + C$.

Example 5: $\int \frac{1}{x^{1/2}}dx = \int x^{-1/2}dx = \frac{x^{\left(-\frac{1}{2}+1\right)}}{\left(-\frac{1}{2}+1\right)} + C = 2x^{1/2} + C$.

Example 6: $\int 5^x dx = \frac{5^x}{\log\ 5} + C$.

Example 7: $\int \left(5x^3 + 2x^{-5} - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}\right)dx$

$$= 5\int x^3dx + 2\int x^{-5}dx - 7\int xdx + \int x^{-1/2}dx + 5\int \frac{1}{x}dx$$
$$= 5 \cdot \frac{x^4}{4} + 2 \cdot \frac{x^{-4}}{(-4)} - 7 \cdot \frac{x^2}{2} + \frac{x^{1/2}}{(1/2)} + 5\log\ |x| + C$$
$$= \frac{5x^4}{4} - \frac{1}{2x^4} - \frac{7x^2}{2} + 2\sqrt{x} + 5\log\ |x| + C.$$

Example 8:

$$\begin{aligned}\int (1-x)(2+3x)(5-4x)dx &= \int (10 - 3x - 19x^2 + 12x^3)dx \\ &= 10\int dx - 3\int xdx - 19\int x^2dx + 12\int x^3dx \\ &= 10x - 3 \cdot \frac{x^2}{2} - 19 \cdot \frac{x^3}{3} + 12 \cdot \frac{x^4}{4} + C \\ &= 10x - \frac{3x^2}{2} - \frac{19x^3}{3} + 3x^4 + C.\end{aligned}$$

Example 9: $\int \left(\frac{3x^4-5x^3+4x^2-x+2}{x^3}\right)dx = \int \left(3x - 5 + \frac{4}{x} - \frac{1}{x^2} + \frac{2}{x^3}\right)dx$ [dividing each term by x^3]

$$= 3\int xdx - 5\int dx + 4\int \frac{1}{x}dx - \int x^{-2}dx + 2\int x^{-3}dx$$
$$= 3\cdot\frac{x^2}{2} - 5x + 4\log|x| - \left(-\frac{1}{x}\right) + 2\left(\frac{x^{-2}}{-2}\right) + C$$
$$= \frac{3x^2}{2} - 5x + 4\log|x| + \frac{1}{x} - \frac{1}{x^2} + C.$$

Example 10:

$$\int \left(\frac{x^4+1}{x^2+1}\right)dx = \int \left[x^2 - 1 + \frac{2}{(x^2+1)}\right]dx$$
$$= \int x^2dx - \int dx + 2\int \frac{1}{x^2+1}dx$$
$$= \frac{x^3}{3} - x + 2\tan^{-1} x + \text{ C}.$$

Example 11: $\int \tan^2 x\, dx = \int (sec^2\ x - 1)dx = \int sec^2\ xdx - \int dx = tan\ x - x + C.$

Example 12: $\int \cot^2 x\, dx = \int (cosec^2\ x - 1)dx = \int cosec^2\ xdx - \int dx = -cot\ x - x + C.$

Example 13: Evaluate $\int \sqrt{1-\sin 2x}dx$.

Solution:

$$\int \sqrt{1-\sin 2x}dx = \int (\cos^2 x + \sin^2 x - 2\sin x\cos x)^{1/2}dx$$
$$= \int \sqrt{(\cos x - \sin x)^2}dx$$
$$= \int (\cos x - \sin x)dx = \int \cos xdx - \int \sin xdx$$
$$= \sin x - (-\cos x) + C = \sin x + \cos x + C.$$

Example14: Evaluate $\int \frac{dx}{1+\sin x}$

Solution:

$$\int \frac{dx}{(1+\sin x)} = \int \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}dx$$
$$= \int \frac{(1-\sin x)}{(1-\sin^2 x)}dx = \int \frac{(1-\sin x)}{\cos^2 x}dx$$
$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx = \int (\sec^2 x - \sec x\tan x)dx$$
$$= \int \sec^2 xdx - \int \sec x\tan xdx = \tan x - \sec x + C.$$

Example 15: Evaluate $\int \frac{\sec x}{(\sec x+\tan x)}dx$.

Solution:

$$\int \frac{\sec x}{(\sec x + \tan x)}dx = \int \frac{\sec x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)}dx$$
$$= \int \frac{(\sec^2 x - \sec x\tan x)}{(\sec^2 x - \tan^2 x)}dx$$

$$= \int (\sec^2 x - \sec x \tan x) dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx = \tan x - \sec x + C.$$

Example 16: Evaluate $\int \left(\frac{1-\cos 2x}{1+\cos 2x}\right) dx$

Solution:

$$\int \left(\frac{1-\cos 2x}{1+\cos 2x}\right) dx = \int \frac{2\sin^2 x}{2\cos^2 x} dx = \int \tan^2 x dx$$

$$= \int (\sec^2 x - 1) dx = \int \sec^2 x dx - \int dx$$

$$= \tan x - x + C.$$

Example 17: Evaluate $\int \sin^{-1}(\cos x) dx$.

Solution: $\int \sin^{-1}(\cos x) dx = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx$

$$= \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2} \cdot \int dx - \int x dx = \frac{\pi x}{2} - \frac{x^2}{2} + C.$$

13.4 LET US SUM UP

So, in this first unit we have learned about the $\epsilon - \delta$ definition of the limit also some numerical examples studied. Finding limit of function at a point is the procedure to study the behavior of the function in the neighborhood of that point. There are some points that must be noted.

- **a)** We calculate the limit of a function to understand the behaviour of a function as it approaches a certain value.
- **b)** In the definition we seen that there exists a delta such that $0 < |x - a| < \delta$, it means that function need not be defined at the point $x = a$.
- **c)** The value of δ depends on both ϵ and the point at which we find the limit.

13.5 UNIT END EXERCISE

13.5.1 Integrate $3x + 4x^2$.

13.5.2 Integrate $\sec x \tan x - 5\, cosec^2 x$.

13.5.3 Integrate $\frac{2\text{ co}}{3\sin^2 x} + 1$.

13.5.4 Integrate $\frac{a}{x^2} + \frac{b}{x} + c$.

13.5.5 Evaluate $\int \frac{4x+6}{x} dx$.

13.5.6 Evaluate $\int (e^x + 2\sin x - 3\cos x)\, dx$.

13.5.7 Integrate $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

13.5.8 Integrate $10^x + 3e^x + x^3$.

13.6 REFERENCES AND SUGGESTED READINGS

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3. T. M. Apostol, Calculus Vol I, Wiley & Sons (Asia) Pvt. Ltd.

4. Shanti Narayan, Integral Calculus, S. Chand, New Delhi.

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