BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 2: Introduction to Matrix Theory

Unit 6: Matrix Multiplication and Determinant

Structure

6.1 INTRODUCTION

Welcome to the world of matrices, where numbers are organized into rows and columns to solve complex problems and model real-world situations. In this unit, we will embark on a journey to explore two fundamental concepts in linear algebra: matrix multiplication and determinants.

Matrices serve as powerful tools in various fields, from mathematics and computer science to engineering and economics. Understanding how to manipulate matrices through multiplication opens doors to solving systems of equations, transforming geometric shapes, and analyzing data.

Additionally, determinants provide essential information about the properties and behavior of matrices. By learning to compute determinants, we gain insights into the invertibility of matrices and their geometric significance.

6.2 OBJECTIVES

The objective of this unit is to provide students with a fundamental understanding of matrix multiplication and determinants. By the end of this unit, students should be able to perform matrix multiplication, compute determinants of matrices, and understand their significance in various mathematical and real-world applications.

6.3 MULTIPLICATION OF MATRICES

For two given matrices A and B, we say that the product AB exists only when the number of rows in A equals the number of columns in B.

When AB exists, we say that A is conformable to B for multiplication.

$$
\begin{aligned}
\text{and } c_{22} &= (2 \text{ nd row of } A) \times (2 \text{ nd column of } B) \\
&= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad AB &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.
\end{aligned}
$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So, BA exists and it is a 3×3 matrix.

Proceeding as above, we get

$$
\begin{aligned}
BA &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\
&= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}.
\end{aligned}
$$

Example 1: If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$, find AB. Does BA exist?

Solution: Here A is a 3×2 matrix and B is a 2×2 matrix.
Clearly, the number of columns in A equals the number of rows in B.
$\therefore$ AB exists and it is a 3×2 matrix.

Now, $AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$

$$
\begin{aligned}
&= \begin{bmatrix} 2 \cdot (-1) + (-1) \cdot 2 & 2 \cdot 3 + (-1) \cdot 1 \\ 3 \cdot (-1) + 4 \cdot 2 & 3 \cdot 3 + 4 \cdot 1 \\ 1 \cdot (-1) + 5 \cdot 2 & 1 \cdot 3 + 5 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} -4 & 5 \\ 5 & 13 \\ 9 & 8 \end{bmatrix}.
\end{aligned}
$$

Further, B is a 2×2 matrix and A is a 3×2 matrix. So, the number of columns in B is not equal to the number of rows in A.
So, BA does not exist.

Example 2: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix}$. Find AB and BA, and show that $AB \neq BA$.

Solution: Here A is a 2×3 matrix and B is a 3×2 matrix.
So, AB exists and it is a 2×2 matrix.
Now,

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 2 + (-2) \cdot 4 + 3 \cdot (-2) & 1 \cdot 3 + (-2) \cdot 5 + 3 \cdot 1 \\ (-4) \cdot 2 + 2 \cdot 4 + 5 \cdot (-2) & (-4) \cdot 3 + 2 \cdot 5 + 5 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} -12 & -4 \\ -10 & 3 \end{bmatrix}.
\end{aligned}
$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.
So, BA exists and it is a 3×3 matrix.
Now,

$$\begin{aligned} BA &= \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 1 + 3 \cdot (-4) & 2 \cdot (-2) + 3 \cdot 2 & 2 \cdot 3 + 3 \cdot 5 \\ 4 \cdot 1 + 5 \cdot (-4) & 4 \cdot (-2) + 5 \cdot 2 & 4 \cdot 3 + 5 \cdot 5 \\ (-2) \cdot 1 + 1 \cdot (-4) & (-2) \cdot (-2) + 1 \cdot 2 & (-2) \cdot 3 + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -6 & 6 & -1 \end{bmatrix}. \end{aligned}$$

Hence, $AB \neq BA$.

Example 3: If $[2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} = O$, find the positive value of x.

Solution: The given matrix equation is $AB = O$, where A is a (1×2) matrix and B is a (2×1) matrix.
So, AB is a (1×1) matrix.
So, O is a (1×1) matrix.

$$\begin{aligned} &\therefore \quad [2x \quad 4] \begin{bmatrix} x \\ -8 \end{bmatrix} = [0] \\ &\Rightarrow \quad 2x^2 - 32 = 0 \Rightarrow 2x^2 = 32 \\ &\Rightarrow x^2 = 16 \Rightarrow x = \sqrt{16} = 4. \end{aligned}$$

Hence, $x = 4$.

Properties of Matrix Multiplication

P 1: Matrix multiplication is not commutative in general.

Proof: Let A and B be two given matrices.

If AB exists then it is quite possible that BA may not exist.

For example, if A is a 3×2 matrix and B is a 2×2 matrix then clearly, AB exists but BA does not exist.

Similarly, if BA exists then AB may not exist.

For example, if A is a 2×3 matrix and B is a 2×2 matrix then clearly, BA exists but AB does not exist.

Further, if AB and BA both exist, they may not be comparable. For example, if A is a 2×3 matrix and B is a 3×2 matrix then clearly, AB as well as BA exists. But, AB is a 2×2 matrix while BA is a 3×3 matrix.

Again, if AB and BA both exist and they are comparable, even then they may not be equal.

For example, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then AB and BA are both defined and each one is a 2×2 matrix.

But, $AB = \begin{bmatrix} 1 & 5 \\ 1 & 8 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}$.
This shows that $AB \neq BA$.
Hence, in general, $AB \neq BA$.

REMARKS (i) When $AB = BA$, we say that A and B commute.

(ii) When $AB = -BA$, we say that A and B anticommute.

P 2: Associative

For any matrices A, B, C for which $(AB)C$ and $A(BC)$ both exist, we have

$$(AB)C = A(BC).$$

P 3: Distributive laws of multiplication over addition We have:
(i) $A \cdot (B + C) = (AB + AC)$
(ii) $(A + B) \cdot C = (AC + BC)$

P 4: The product of two nonzero matrices can be a zero matrix.
Example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
Then, $A \neq O$ and $B \neq O$.
And, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 2 + 2 \cdot 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Left zero divisor If $AB = O$ and $A \neq O$ then A is called a left zero divisor of AB.
Right zero divisor If $AB = O$ and $B \neq O$ then B is called a right zero divisor of AB.
P 5: If A is a given square matrix and I is an identity matrix of the same order as A then we have $A \cdot I = I \cdot A = A$.

P 6: If A is a given square matrix and O is the null matrix of the same order as A then $O \cdot A = A \cdot O = O$.

Positive Integral Powers of a Square Matrix

Let A be a square matrix of order n. Then, we define:

$$\begin{aligned} A^2 &= A \cdot A; \\ A^3 &= A \cdot A \cdot A = A^2 \cdot A; \\ A^4 &= A \cdot A \cdot A \cdot A = A^3 \cdot A, \text{ and so on.} \\ \therefore \quad A^n &= (A \cdot A \cdot A \cdot \ldots \cdot n \text{ times }). \end{aligned}$$

Theorem 1: If A and B are square matrices of the same order then

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Also, when $AB = BA$ then $(A + B)^2 = A^2 + 2AB + B^2$.
Proof: Let A and B be n-rowed square matrices.
Then, clearly, $(A + B)$ is a square matrix of order n.
So, $(A + B)^2$ is defined.
Now, $(A + B)^2 = (A + B) \cdot (A + B)$

$$\begin{aligned} &= A \cdot (A + B) + B \cdot (A + B) && \text{[by distributive law]} \\ &= AA + AB + BA + BB && \text{[by distributive law]} \\ &= A^2 + AB + BA + B^2. \end{aligned}$$

Hence, $(A + B)^2 = (A^2 + AB + BA + B^2)$.
Particular case When $AB = BA$
In this case, we have

$$(A + B)^2 = (A^2 + AB + AB + B^2) = (A^2 + 2AB + B^2) [\because BA = AB].$$

Theorem 2: If A and B are square matrices of the same order then

$$(A+B)(A-B) = A^2 - AB + BA - B^2.$$

Also, when $AB = BA$ then $(A+B)(A-B) = A^2 - B^2$.

Proof: We have:

$$\begin{aligned}(A+B)\cdot(A-B) &= A(A-B) + B(A-B) \quad [\text{ by distributive law }]\\ &= AA - AB + BA - BB \quad [\because A(B-C) = AB - AC]\\ &= A^2 - AB + BA - B^2.\end{aligned}$$

Hence, $\quad (A+B)(A-B) = A^2 - AB + BA - B^2$.

Particular case When $AB = BA$

In this case, $(A+B)(A-B) = (A^2 - B^2) \quad [\because BA = AB]$.

Matrix Polynomial: Let $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_{m-1}x + a_m$ be a polynomial of degree m and let A be a square matrix of order n. Then, we define

$$f(A) = a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \cdots + a_{m-1}A + a_mI,$$

where I is a unit matrix of order n.

Solved Examples

Example 1: If $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, find AB and BA whichever exists.

Solution: Here, A is a 2×2 matrix and B is a 2×3 matrix.

Clearly, the number of columns in $A =$ number of rows in B.

$\therefore$ AB exists and it is a 2×3 matrix.

$$\begin{aligned}AB &= \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}\\ &= \begin{bmatrix} 5\cdot 3 + 4\cdot 6 & 5\cdot 5 + 4\cdot 8 & 5\cdot 1 + 4\cdot 4 \\ 2\cdot 3 + 3\cdot 6 & 2\cdot 5 + 3\cdot 8 & 2\cdot 1 + 3\cdot 4 \end{bmatrix}\\ &= \begin{bmatrix} 15+24 & 25+32 & 5+16 \\ 6+18 & 10+24 & 2+12 \end{bmatrix} = \begin{bmatrix} 39 & 57 & 21 \\ 24 & 34 & 14 \end{bmatrix}.\end{aligned}$$

Again, B is a 2×3 matrix and A is a 2×2 matrix.

$\therefore$ number of columns in $B \neq$ number of rows in A.

So, BA does not exist.

Example 2: If $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$ then find AB and BA.

Show that $AB \neq BA$.

Solution: Here, A is a 2×3 matrix and B is a 3×2 matrix

So, number of columns in $A =$ number of rows in B.

$\therefore$ AB exists and it is a 2×2 matrix.

$$\begin{aligned}AB &= \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}\\ &= \begin{bmatrix} 2\cdot 2 + (-1)\cdot 4 + 3\cdot 1 & 2\cdot 3 + (-1)\cdot(-2) + 3\cdot 5 \\ -4\cdot 2 + 5\cdot 4 + 1\cdot 1 & -4\cdot 3 + 5\cdot(-2) + 1\cdot 5 \end{bmatrix}\\ &= \begin{bmatrix} 4-4+3 & 6+2+15 \\ -8+20+1 & -12-10+5 \end{bmatrix} = \begin{bmatrix} 3 & 23 \\ 13 & -17 \end{bmatrix}.\end{aligned}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix.

So, number of columns in $B =$ number of rows in A.

$\therefore$ BA exists and it is a 3×3 matrix.

$$
\begin{aligned}
BA &= \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2\cdot 2 + 3\cdot(-4) & 2\cdot(-1) + 3\cdot 5 & 2\cdot 3 + 3\cdot 1 \\ 4\cdot 2 + (-2)\cdot(-4) & 4\cdot(-1) + (-2)\cdot 5 & 4\cdot 3 + (-2)\cdot 1 \\ 1\cdot 2 + 5\cdot(-4) & 1\cdot(-1) + 5\cdot 5 & 1\cdot 3 + 5\cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 4-12 & -2+15 & 6+3 \\ 8+8 & -4-10 & 12-2 \\ 2-20 & -1+25 & 3+5 \end{bmatrix} = \begin{bmatrix} -8 & 13 & 9 \\ 16 & -14 & 10 \\ -18 & 24 & 8 \end{bmatrix}.
\end{aligned}
$$

Clearly, $AB \neq BA$.

Example 3: If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$ then verify that $(AB)\mathrm{C} = A(BC)$.

Solution: We have

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 3-0-4 & 1-2+10 \\ 9+0-0 & 3+4+0 \\ -6+0-2 & -2+0+5 \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \\
\Rightarrow \quad (AB)C &= \begin{bmatrix} -1 & 9 \\ 9 & 7 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2+27 & -1+0 & 3-9 \\ 18+21 & 9+0 & -27-7 \\ -16+9 & -8+0 & 24-3 \end{bmatrix} = \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.
\end{aligned}
$$

Also, $BC = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

$$
\begin{aligned}
&= \begin{bmatrix} 6+3 & 3+0 & -9-1 \\ 0+6 & 0+0 & 0-2 \\ -4+15 & -2+0 & 6-5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix} \\
\Rightarrow A(BC) &= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 3 & -10 \\ 6 & 0 & -2 \\ 11 & -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 9-6+22 & 3-0-4 & -10+2+2 \\ 27+12+0 & 9+0-0 & -30-4+0 \\ -18+0+11 & -6+0-2 & 20-0+1 \end{bmatrix} \\
&= \begin{bmatrix} 25 & -1 & -6 \\ 39 & 9 & -34 \\ -7 & -8 & 21 \end{bmatrix}.
\end{aligned}
$$

Hence, $(AB)\mathrm{C} = A(BC)$.

Example 4: If $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & 1 & -6 \\ 2 & -5 & 0 \end{bmatrix}$, verify that $A(B+C) = (AB + AC)$.

Solution: We have

$$\begin{aligned}A(B+C) &= \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}\cdot\left\{\begin{bmatrix}1 & -2 & 5\\0 & 7 & 3\end{bmatrix}+\begin{bmatrix}8 & 1 & -6\\2 & -5 & 0\end{bmatrix}\right\}\\ &= \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}9 & -1 & -1\\2 & 2 & 3\end{bmatrix}\\ &= \begin{bmatrix}3\cdot 9+2\cdot 2 & 3\cdot(-1)+2\cdot 2 & 3\cdot(-1)+2\cdot 3\\1\cdot 9+0\cdot 2 & 1\cdot(-1)+0\cdot 2 & 1\cdot(-1)+0\cdot 3\end{bmatrix}\\ &= \begin{bmatrix}31 & 1 & 3\\9 & -1 & -1\end{bmatrix}.\end{aligned}$$

Now, $AB = \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 5\\0 & 7 & 3\end{bmatrix}$

$$\begin{aligned}&= \begin{bmatrix}3\cdot 1+2\cdot 0 & 3\cdot(-2)+2\cdot 7 & 3\cdot 5+2\cdot 3\\1\cdot 1+0\cdot 0 & 1\cdot(-2)+0\cdot 7 & 1\cdot 5+0\cdot 3\end{bmatrix}\\ &= \begin{bmatrix}3 & 8 & 21\\1 & -2 & 5\end{bmatrix}.\end{aligned}$$

And, $AC = \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}8 & 1 & -6\\2 & -5 & 0\end{bmatrix}$

$$\begin{aligned}&= \begin{bmatrix}3\cdot 8+2\cdot 2 & 3\cdot 1+2\cdot(-5) & 3\cdot(-6)+2\cdot 0\\1\cdot 8+0\cdot 2 & 1\cdot 1+0\cdot(-5) & 1\cdot(-6)+0\cdot 0\end{bmatrix}\\ &= \begin{bmatrix}28 & -7 & -18\\8 & 1 & -6\end{bmatrix}.\end{aligned}$$

$$\begin{aligned}\therefore\quad (AB+AC) &= \begin{bmatrix}3 & 8 & 21\\1 & -2 & 5\end{bmatrix}+\begin{bmatrix}28 & -7 & -18\\8 & 1 & -6\end{bmatrix}\\ &= \begin{bmatrix}31 & 1 & 3\\9 & -1 & -1\end{bmatrix}.\end{aligned}$$

Hence, $A(B+C) = AB+AC$.

Example 5: Give an example of two matrices A and B such that $A \neq O, B \neq O$ and $AB = BA = O$.

Solution: Let $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$. Then,

$$\begin{aligned}AB &= \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1-1 & -1+1\\1-1 & -1+1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\\ \text{and } BA &= \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\\ &= \begin{bmatrix}1-1 & 1-1\\-1+1 & -1+1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.\end{aligned}$$

Thus, $A \neq O, B \neq O$.
But, $AB = BA = O$.

Example 6: If $A = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\\cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$ and $B = \begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\\cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$, show that AB is a zero matrix if θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Solution: We have

$$AB = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\\cos\theta\sin\theta & \sin^2\theta\end{bmatrix}\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\\cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta\cos^2\phi+\cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi\\\cos\theta\sin\theta\cos^2\phi+\sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi+\sin^2\theta\sin^2\phi\end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \cos\phi \cdot \cos(\theta-\phi) & \cos\theta & \sin\phi \cdot \cos(\theta-\phi) \\ \sin\theta & \cos\phi \cdot \cos(\theta-\phi) & \sin\theta & \sin\phi \cdot \cos(\theta-\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \because (\theta-\phi) \text{ being an odd multiple of } (\pi/2), \\ \text{we have } \cos(\theta-\phi) = 0 \end{bmatrix}.$$

Example 7: If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = O$.

Solution: We have

$$A^2 = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 + (-5)(-4) & 3\cdot(-5) + (-5)\cdot 2 \\ -4\cdot 3 + 2\cdot(-4) & -4\cdot(-5) + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix};$$

$$-5A = (-5)\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix};$$

$$-14I = (-14)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}.$$

$$\therefore \quad A^2 - 5A - 14I = A^2 + (-5)A + (-14I)$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 + (-15) + (-14) & -25 + 25 + 0 \\ -20 + 20 + 0 & 24 + (-10) + (-14) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $A^2 - 5A - 14I = O$.

Example 8: If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find k so that $A^2 = 8A + kI$.

Solution: We have

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix};$$

$$(8A + kI) = 8\cdot\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\therefore \quad A^2 = 8A + \Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\Rightarrow 8 + k = 1 \text{ and } 56 + k = 49 \Rightarrow k = -7.$$

Hence, $k = -7$.

Determinant of a Square Matrix

Corresponding to each square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

there is associated an expression, called the determinant of A, denoted by det A, or $|A|$, written as

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has a fixed value.

A determinant having n rows and n columns is known as a determinant of order n

The determinants of non-square matrices are not defined.

Value of a Determinant of Order 1: The value of a determinant of $a(1 \times 1)$ matrix $[a]$ is defined as $|a| = a$.

Value of a Determinant of Order 2: We define

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12}).$$

Example 1: Evaluate:

(i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix}$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 6 & -3 \\ 7 & -2 \end{vmatrix} = 6(-2) - 7(-3) = -12 + 21 = 9.$

(ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = (x^2 - x + 1)(x + 1) - (x + 1)(x - 1)$

$$= (x^3 + 1) - (x^2 - 1) = (x^3 - x^2 + 2).$$

Example 2: If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.

Solution: We have $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$$\begin{aligned} &\Leftrightarrow \quad (3x \times 4) - (-2) \times 7 = (8 \times 4) - (6 \times 7) \\ &\Leftrightarrow 12x + 14 = 32 - 42 \\ &\Leftrightarrow 12x + 14 = -10 \\ &\Leftrightarrow 12x = -24 \Leftrightarrow x = -2. \end{aligned}$$

Hence, $x = -2$.

Example 3: If $\begin{vmatrix} x + 1 & x - 1 \\ x - 3 & x + 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, find the value of x.

Solution: We have $\begin{vmatrix} x + 1 & x - 1 \\ x - 3 & x + 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$$\begin{aligned} &\Leftrightarrow \quad (x + 1)(x + 2) - (x - 3)(x - 1) = (4 \times 3) - 1 \times (-1) \\ &\Leftrightarrow \quad (x^2 + 3x + 2) - (x^2 - 4x + 3) = 12 + 1 \\ &\Leftrightarrow 7x - 1 = 13 \Leftrightarrow 7x = 14 \Leftrightarrow x = 2. \end{aligned}$$

Hence, $x = 2$.

Example 4: Show that $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1.$

Solution: We have $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$

$$\begin{aligned} &= (\sin 10^\circ)(\cos 80^\circ) - (\sin 80^\circ)(-\cos 10^\circ) \\ &= (\sin 10^\circ \cos 80^\circ + \cos 10^\circ \sin 80^\circ) \\ &= \sin (10^\circ + 80^\circ) \quad [\because \sin A \cos B + \cos A \sin B = \sin (A + B)] \\ &= \sin 90^\circ = 1. \end{aligned}$$

Value of a Determinant of Order 3 or More: For finding the value of a determinant of order 3 or more, we need the following definitions.

Minorof$\boldsymbol{a_{ij}}$ *in* $|\boldsymbol{A}|$: The minor of an element a_{ij} in $|A|$ is defined as the value of the determinant obtained by deleting the ith row and jth column of $|A|$, and it is denoted by M_{ij}.

Cofactor of $\boldsymbol{a_{ij}}$ *in* $|\boldsymbol{A}|$: The cofactor C_{ij} of an element a_{ij} in $|A|$ is defined as $C_{ij} = (-1)^{i+j} \cdot M_{ij}$.

Example 1: Find the minors and cofactors of the elements of the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Solution: Let M_{ij} denote the minor of a_{ij} in Δ.

Now, a_{11} occurs in the 1st row and 1st column. So, in order to find the minor of a_{11}, we delete the 1st row and 1st column of Δ. The minor M_{11} of a_{11} is given by $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = (a_{22}a_{33} - a_{32}a_{23})$. Similarly, we have

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = (a_{21}a_{33} - a_{31}a_{23});$$
$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = (a_{21}a_{32} - a_{31}a_{22});$$
$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{32} \end{vmatrix} = (a_{12}a_{33} - a_{32}a_{13}).$$

Similarly, we may obtain the minor of each of the remaining elements.

Now, if we denote the cofactor of a_{ij} by C_{ij} then

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = (a_{22}a_{33} - a_{32}a_{23});$$
$$C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = (a_{31}a_{23} - a_{21}a_{33});$$
$$C_{13} = (-1)^{1+3} \cdot M_{13} = M_{13} = (a_{21}a_{32} - a_{31}a_{22});$$
$$C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = (a_{32}a_{13} - a_{12}a_{33}).$$

Similarly, the cofactor of each of the remaining elements of Δ can be determined.

Example 2: Find the minor and cofactor of each element of $\Delta = \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$.

Solution: The minors of the elements of Δ are given by

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -12; M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 2; M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 23;$$
$$M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -16; M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4; M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 14;$$
$$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -4; M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6; M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = 11.$$

So, the cofactors of the corresponding elements of Δ are

$$C_{11} = (-1)^{1+1} \cdot M_{11} = M_{11} = -12; C_{12} = (-1)^{1+2} \cdot M_{12} = -M_{12} = -2;$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = M_{13} = 23; C_{21} = (-1)^{2+1} \cdot M_{21} = -M_{21} = 16;$$
$$C_{22} = (-1)^{2+2} \cdot M_{22} = M_{22} = -4; C_{23} = (-1)^{2+3} \cdot M_{23} = -M_{23} = -14;$$
$$C_{31} = (-1)^{3+1} \cdot M_{31} = M_{31} = -4; C_{32} = (-1)^{3+2} \cdot M_{32} = -M_{32} = 6;$$
$$C_{33} = (-1)^{3+3} \cdot M_{33} = M_{33} = 11.$$

Value of a Determinant

The value of a determinant is the sum of the products of elements of a row (or a column) with their

corresponding cofactors.
We may expand a determinant by any arbitrarily chosen row or column.

Expansion of a Determinant
Expanding the given determinant by 1st row, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \text{ (its cofactor) } + a_{12} \cdot \text{ (its cofactor) } + a_{13} \cdot \text{ (its cofactor)}$$
$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad [\because C_{12} = -M_{12}]$$
$$= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11} \cdot (a_{22}a_{33} - a_{32}a_{23}) - a_{12} \cdot (a_{21}a_{33} - a_{31}a_{23})$$
$$+ a_{13} \cdot (a_{21}a_{32} - a_{31}a_{22}).$$

We may expand it by any row or column.

Remark 1: If we expand a determinant by any row or column using minors, we keep in view the following symbols for a determinant of order three:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Remark 2: If a row or a column of a determinant consists of all zeros, the value of the determinant is zero.

Example 1: Evaluate

$$\Delta = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}.$$

Solution: Expanding the given determinant by 1st row, we get

$$\Delta \quad = 3 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4 \cdot \begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5 \cdot \begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix}$$
$$= 3 \cdot (14 + 3) - 4 \cdot (-42 + 24) + 5 \cdot (-6 - 16) = 13.$$

Example 2: Expand the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution: Expanding by 1 st column, we get

$$\Delta \quad = a \cdot \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \cdot \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \cdot \begin{vmatrix} h & g \\ b & f \end{vmatrix}$$
$$= a(bc - f^2) - h \cdot (ch - fg) + g \cdot (fh - bg)$$
$$= abc - af^2 - ch^2 + fgh + fgh - bg^2$$
$$= (abc + 2fgh - af^2 - bg^2 - ch^2).$$

Properties of Determinants

The properties of a determinant serve the purpose of a useful tool for finding its value. We will mention these properties and verify them for a third-order determinant.

Theorem 1: The value of a determinant remains unchanged if its rows and columns are interchanged.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging the rows and columns of Δ.

Then,

$$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

[expanded by 1st column]

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$$

$$= \Delta \quad \text{[expanded by 1st row].}$$

Note: If A is a square matrix then $|A'| = |A|$.

Theorem 2: If two rows or columns of a determinant are interchanged then the determinant retains its absolute value but its sign is changed.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by interchanging any two rows, say 1st and 3rd rows, of Δ. Then,

$$\Delta' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= a_3 \cdot \begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_1 \cdot \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix}$$

[expanded by 1st column]

$$= a_3(b_2c_1 - b_1c_2) - a_2(b_3c_1 - b_1c_3) + a_1(b_3c_2 - b_2c_3)$$

$$= -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1)$$

$$= -[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]$$

$$= -\Delta \quad \text{[expanded by 1st column].}$$

Theorem 3: If any two rows or columns of a determinant are identical then its value is zero.

Proof: If we interchange the identical rows of the given determinant Δ then clearly there is no change in Δ. But, interchanging any two rows of a determinant changes its sign.

$$\therefore \Delta = -\Delta \Leftrightarrow 2\Delta = 0, \text{ i.e., } \Delta = 0.$$

Theorem 4: If each element of a row or a column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and let Δ' be the determinant obtained by multiplying each element of a row, say the second row of Δ, by k. Then,

$$\begin{aligned}
\Delta' &= \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= a_1 \cdot \begin{vmatrix} kb_2 & kc_2 \\ b_3 & c_3 \end{vmatrix} - ka_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ kb_2 & kc_2 \end{vmatrix} \\
&= a_1(kb_2c_3 - kb_3c_2) - ka_2(b_1c_3 - b_3c_1) + a_3(kb_1c_2 - kb_2c_1) \\
&= k[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)] = k\Delta.
\end{aligned}$$

Important Note: For a square matrix A of order $n, |kA| = k^n \cdot |A|$.

Theorem 5: If any two rows or columns of a determinant are proportional then its value is zero.

Proof: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix}$

$$= k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$= k \times 0 = 0$ [by Theorem 4]

[$\because$ 1st and 3rd rows are identical].

Theorem 6: If each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants. [by Theorem 4]

$$\Delta = \begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 + \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Then, on expanding Δ by first row, we get

$$\begin{aligned}
\Delta &= (a_1 + \alpha_1)(b_2c_3 - b_3c_2) - (b_1 + \beta_1)(a_2c_3 - a_3c_2) \\
&= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + \gamma_1)(a_2b_3 - a_3b_2) - a_3b_2)] \\
&\quad +[\alpha_1(b_2c_3 - b_3c_2) - \beta_1(a_2c_3 - a_3c_2) + \gamma_1(a_2b_3 - a_3b_2)] \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.
\end{aligned}$$

Theorem 7: If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same.

Let

$$\begin{aligned}
\Delta &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \\
\Delta' &= \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.
\end{aligned}$$

Then,

$$\begin{aligned}\Delta' &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ka_3 & kb_3 & kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad [\because \text{ 2nd det. is 0, its 1st and 3rd} \\ &= \Delta. \end{aligned}$$

6.4 LET US SUM UP

So, in this first unit we have learned about

- Matrix multiplication
- Properties of matrix multiplication
- Determinant of a matrix
- Properties of determinant

6.5 UNIT END EXERCISE

6.5.1. Compute AB and BA, whenever exists when

(i) $A = \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 25 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 25 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & 2 \\ 13 & 4 & 2 \end{bmatrix}$

6.5.2. Verify that $A(B + C) = AB + AC$, when

(i) $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 6 & 5 \end{bmatrix}, \ C = \begin{bmatrix} 2 & -2 \\ 2 & 4 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 6 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

6.5.3. If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, show that $A^2 = A$.

6.5.4. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that $A^2 = I$.

6.5.5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $(A^2 - 5A + 7I) = O$.

6.5.6. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the value of x and y such that $A^2 + xA = yA$.

6.5.7. If $\begin{vmatrix} 3x + 1 & 3 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$, find the value of x.

6.5.8. Find the determinant of the following matrices

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 6 & -2 & -4 \\ -1 & 0 & 4 \\ 1 & -2 & 5 \end{bmatrix}$$

6.7 REFERENCES AND SUGGESTED READINGS

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