BACHELOR OF COMPUTER APPLICATIONS (BCA)

CSB-1113 (Mathematics - I)

Block 3: Limit, Continuity and Differentiability of Function

Unit 10: Continuous Functions

Structure

10.1 INTRODUCTION

A continuous function is a cornerstone of calculus, representing a fundamental concept where the behaviour of a function is smooth and connected without abrupt changes or breaks. In mathematics, a function$f(x)$ is considered continuous at a point$x = a$ if,its graph can be drawn without lifting the pen, indicating a seamless flow of values as x approaches a.

This notion is fundamental in understanding the behaviour of functions across their domains, enabling precise analysis of their properties and facilitating various mathematical applications. Continuous functions possess remarkable characteristics, allowing for deeper insights into calculus concepts such as limits, derivatives, integrals, and the fundamental theorem of calculus. Their significance extends beyond mathematics, finding widespread applications in fields like physics, engineering, economics, and computer science, where the concept of continuity plays a crucial role in modelling real-world phenomena. Understanding continuous functions is fundamental to grasping the fundamental principles of change, smoothness, and connectivity within mathematical landscapes.

10.2 OBJECTIVES

After completing this unit, you will be able to

- Remember the definition of continuity
- Evaluate the continuity of function
- Discuss the concept of discontinuity
- Differentiate between various types of discontinuity
- Create examples of continuous and discontinuous function

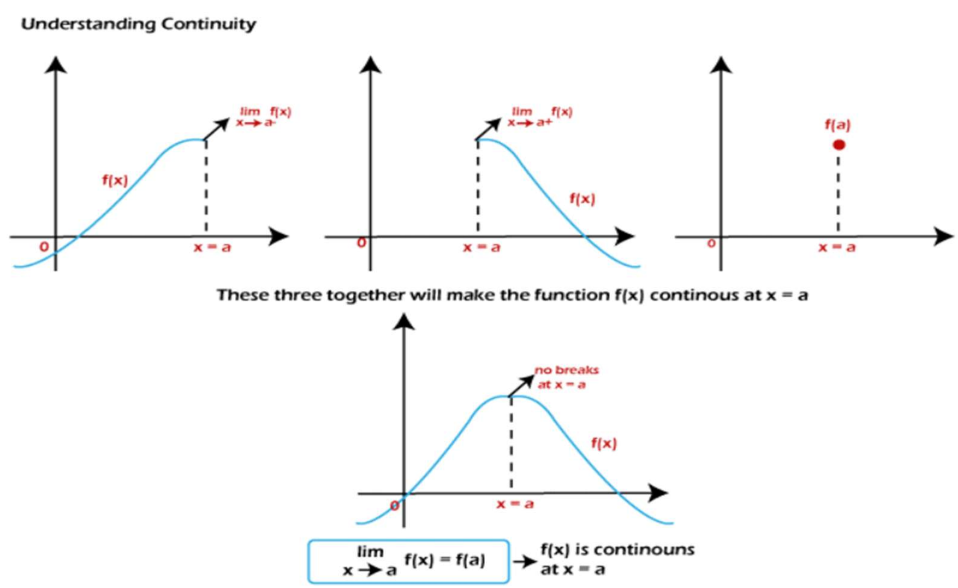
10.3 DEFINITION OF THE CONTINUOUS FUNCTION

A real valued function $f(x)$ is said to be continuous at a point $x = a$ if

1. $f(x)$ is defined at $x = a$, i.e., $f(a)$ is a finite real number.
2. Limit of $f(x)$ exists at $x = a$, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
3. $\lim_{x \to a} f(x) = f(a)$.

$\epsilon - \delta$ Definition of Continuity: A real valued function $f(x)$ is said to be continuous at a point $x = a$ if $\forall\ \epsilon > 0\ \exists\ a\ \delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Continuous Function in a Domain: Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ is a function then f is said to continuous in the domain D if f is continuous at every point of D.



10.4 SOME EXAMPLES

First, we will see some straight forward examples.

1. **Linear Functions:** Functions like $f(x) = 2x + 3$ or $g(x) = -4x$ are continuous everywhere. They represent straight lines that have a constant slope, exhibiting a continuous and unbroken graph across their entire domain.
2. **Polynomial Functions:** Functions such as $h(x) = x^2$or $k(x) = 3x^3 - 2x^2 + 5x - 1$ are continuous across their entire domain. They are smooth curves without any abrupt changes or holes.
3. **Trigonometric Functions:** Functions like $\sin(x)$ or $\cos(x)$ are continuous over their domains. They represent periodic waves without any breaks or discontinuities.

   Let $f(x) = tan\ x. f$ is not defined at the points $\frac{(2n+1)\pi}{2}$ $(n \in \mathbb{Z})$ at which the denominator $cos\ x = 0$. Let $a \in \mathbb{R}$ and $a \neq \frac{(2n+1)\pi}{2}$. Then $\lim_{x \to a} tan\ x = tan\ a$. So f is continuous at a when $a \neq \frac{(2n+1)\pi}{2}$. Similarly the functions $\cot x$, $cosec\ x$, $\sec x$ are continuous on their domains.

4. **Rational Functions:** Functions such as $\frac{1}{x}$ are continuous on their domain except at the point where the denominator becomes zero (in this case, $x = 0$). At that point, the function is not defined, leading to a discontinuity.
5. **Piecewise Functions:** Functions that are defined by different equations for different intervals can be continuous or discontinuous based on the nature of these pieces. For example, $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ is continuous at $x = 0$ because both x^2 and $-x^2$ meet smoothly at that point.

Now we will see some examples about which we cannot directly or geometrically decide about the continuity.

1. Let $f(x) = \begin{cases} x^2, & x \leq 2 \\ 3x, & x > 2 \end{cases}$, then f is not continuous at $x = 2$.

   Clearly $f(x)$ is defined at $x = 2$. Let us check the limit

$$R.H.L. = \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h) = 6$$

$$L.H.L. = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}(2-h)^2 = 4$$

$$R.H.L. \neq L.H.L.$$

   Hence limit does not exist therefore given function is not continuous at $x = 2$.

2. Let $f(x) = \begin{cases} x^2 & if \quad x \leq -1 \\ x+1 & if -1 < x \leq 1 \\ x^2 - 1 & if \quad x > 1 \end{cases}$

   The given function changes their behaviour in the neighbourhood of the points -1 and 1. So, we will check the continuity of the given function at these points.

   At $x = -1$.

   Now $L.H.L. f(-1) = \lim_{x\to -1^-} f(x) = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0}(-1-h)^2 = 1$ and

$$R.H.L. f(-1) = \lim_{x\to -1^+} f(x) = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0}(-1+h) + 1 = 0$$

$$\Rightarrow L.H.L. f(-1) \neq R.H.L. f(-1)$$

   Therefore, $f(x)$ is not continuous at $x = -1$.

   At $x = 1$.

   Here $L.H.L. f(1) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h) + 1 = 2$ and

$$R.H.L. f(1) = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}(1+h)^2 - 1 = 0$$

$$\Rightarrow L.H.L. f(-1) \neq R.H.L. f(-1)$$

   Therefore, $f(x)$ is not continuous at $x = 1$.

10.5 IMPORTANT RESULTS REGARDING CONTINUITY

Theorem 10.5.1 Let $D \subset \mathbb{R}$ and f and g are continuous function of D to $\mathbb{R}$. Let $a \in D$ and f and g are continuous at a. Then

a) $f + g$ is continuous at a.

b) kf is continuous at $a, \forall\, k \in \mathbb{R}$.

c) fg is continuous at a.

d) $\frac{f}{g}$ is continuous at a,provided $g(x) \neq 0 \; \forall \; x \in D$.

e) $f + g$ is continuous on D.

f) kf is continuous on $D, \forall \; k \in \mathbb{R}$.

g) fg is continuous on D.

h) $\frac{f}{g}$ is continuous on D, provided $g(x) \neq 0 \; \forall \; x \in D$.

Theorem 10.5.2 Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ (D). Then $|f|$ is continuous at a $(on\ D)$.

Note: If $|f|$ be on D then f may not be continuous at on D.

For example, let $f: \mathbb{R} \to \mathbb{R}$is defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{Q}^c. \end{cases}$

Then $|f|: \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = 1, x \in \mathbb{R}$, which is a constant function and clearly continuous on $\mathbb{R}$ but f is not continuous on $\mathbb{R}$.

Theorem 10.5.3 Let $A, B \subset \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous on A and let $g: B \to \mathbb{R}$ be continuous on B and $f(A) \subset B$. Then the composite function $gof: A \to \mathbb{R}$ is continuous on A.

10.6 DISCONTINUITY AND THEIR CLASSIFICATION

Understanding the concept of discontinuity in real-valued functions is fundamental in analysing their behaviour, providing insights into points where functions fail to exhibit continuity. Discontinuities highlight abrupt changes, breaks, or disruptions in the function's graph, contributing significantly to the comprehension of functions across various mathematical landscapes.

10.6.1 Definition of Discontinuity

Discontinuity in a real-valued function occurs at a point where the function fails to meet the criteria for continuity. Formally, a function $f(x)$ is discontinuous at a point$a$ if at least one of the following conditions is true:

- $f(a)$ is undefined.
- The limit of the function as x approaches a does not exist.
- The limit of the function as x approaches a exists, but it is not equal to $f(c)$.

10.6.2 Types of Discontinuity

A. Removable Discontinuity: A removable discontinuity occurs when a function has a hole or gap at a certain point, but it can be redefined at that point to make the function continuous. Mathematically, this is expressed as $\lim_{x \to a} f(x)$existing, but $f(a)$ is undefined or different.

B. Jump Discontinuity: A jump discontinuity arises when the function "jumps" from one value to another at a specific point without having a limiting value that bridges the gap. Mathematically, this is depicted as $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

C. Infinite Discontinuity: An infinite discontinuity occurs when the function approaches an infinite value (positive or negative infinity) at a particular point. This happens when the limit of the function as x approaches a results in an unbounded value.

D. **Oscillating Discontinuity:** An oscillating discontinuity, also known as essential or non-removable discontinuity, happens when the function oscillates infinitely near a point without having a limiting value. The function exhibits erratic behaviour without converging to a specific value.

10.6.3 Examples and Illustrations

Example 1: Removable Discontinuity

The function $f(x) = \frac{x^2-4}{x-2}$ has a removable discontinuity at $x = 2$, where the graph has a hole that can be filled by redefining the function at that point.

Example 2: Jump Discontinuity

The function $g(x) = \begin{cases} x, & x < 0 \\ x+1, & x \geq 0 \end{cases}$ has a jump discontinuity at $x = 0$, where there's a clear jump from $g(x) = x$ to $g(x) = x + 1$.

Example 3: Infinite Discontinuity

An example of a function exhibiting infinite discontinuity is the function $f(x) = \frac{1}{x}$.As x approaches 0 from the positive side (values slightly larger than 0),$f(x) = \frac{1}{x}$ grows larger and larger, tending towards positive infinity $(+\infty)$.as x approaches 0 from the negative side (values slightly smaller than 0), $f(x) = \frac{1}{x}$ decreases in magnitude, tending towards negative infinity $(-\infty)$.

Example 4: Oscillating Discontinuity

An example of a function exhibiting oscillating discontinuity is the Dirichlet function, also known as the indicator function of rational numbers. It's defined as:

Let $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c. \end{cases}$

The Dirichlet function demonstrates an oscillating discontinuity at every point on the real number line. This occurs because there are infinitely many rational and irrational numbers in any neighborhood of a given point, causing the function to oscillate between 0 and 1 infinitely close to that point.

10.7 LET US SUM UP

Continuous functions exhibit smooth and connected behavior without breaks, jumps, or holes in their graphs. Understanding continuity is pivotal in calculus, as it underpins the notions of limits, derivatives, integrals, and more. Continuous functions find widespread applications in modeling various phenomena in fields such as physics, engineering, economics, and beyond.

Discontinuities in functions are categorized into distinct types—removable, jump, infinite, and oscillating—based on the nature of disruptions in the function's behavior. Each type of discontinuity illustrates unique traits, highlighting instances where functions fail to meet the criteria for continuity. Discontinuities are essential in understanding the limitations of functions, providing insights into their behavior and defining points where functions exhibit abrupt changes or irregularities.

10.8 UNIT END EXERCISE

10.8.1 Let $f(x) = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$, check the continuity at $x = 0$.

10.8.2 Let $f(x) = \begin{cases} x^2, & x \leq 2 \\ 4-x, & x > 2 \end{cases}$, check the continuity at $x = 2$.

10.8.3 Let $f(x) = \begin{cases} 2x, & x < 1 \\ x^2, & x \geq 1 \end{cases}$, check the continuity at $x = 1$.

10.8.4 Let $f(x) = \begin{cases} x^2 & if \quad x < 0 \\ x + 1 & if \quad 0 \leq x \leq 2 \\ x^3 & if \quad x > 2 \end{cases}$, check the continuity at $x = 0$ and $x = 2$.

10.8.5 Classify the types of discontinuity in the above examples.

10.9 REFERENCES AND SUGGESTED READINGS

1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc., 2002.

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4. Gorakh Prasad, Differential Calculus, Pothishala Pvt. Ltd., Allahabad.

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