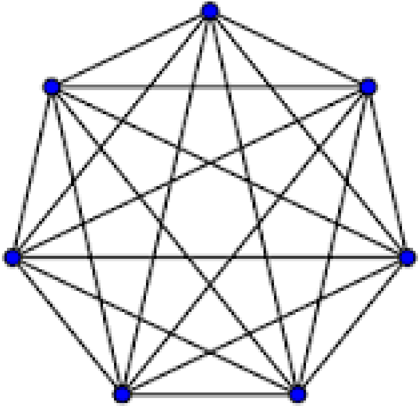
Paths on Finite Graphs

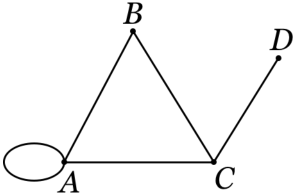
Proof School

Math Symposium • February 24, 2017



What are Graphs?

A graph is a set of vertices connected by edges
For instance:



What can we do with Graphs?

One question that readily comes to mind is how many paths are there from one vertex to another given a set amount of moves?

In this graph, we can calculate some small values of the number of paths from A to B by hand.

Moves	1	2	3	4	5
Paths	1	1	2	3	5

Generating Functions

A generating function is a way to represent or encode an infinite series of numbers by putting them in an infinite polynomial.

Example:

The generating function for the Fibonacci numbers would be: $0 + x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$

Powers of 2: $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$

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$0 + x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$
These are the Fibonacci numbers! One way we can make a generating function more compact is by putting it in a summation.

$$\sum_{i=0}^{\infty} F_i x^i$$

Moves	1	2	3	4	5
Paths	1	1	2	3	5

Generating Functions and Rational Functions

For some generating functions like the one above, the coefficients actually satisfy a recursive relationship.

The Fibonacci numbers satisfy the recursive relationship

$$F_n = F_{n-1} + F_{n-2}$$
$$F_n - F_{n-1} - F_{n-2} = 0$$

We will use this recursive relationship to manipulate the polynomial.

Moves	1	2	3	4	5
Paths	1	1	2	3	5

Generating Functions and Rational Functions

It is possible to eliminate some terms using multiplication of $P(x)$ by $-x$ and $-x^2$.

$P(x) = 0 + x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$

$-xP(x) = 0x^1 - 1x^2 - 1x^3 - 2x^4 - 3x^5 \cdots$

$-x^2P(x) = \quad 0x^2 - 1x^3 - 1x^4 - 2x^5 \cdots$

Adding these together, we see that everything after the x term crosses out leaving

$$P(x)(1 - x - x^2) = x \text{ so } P(x) = \frac{x}{1-x-x^2}$$

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This is one of the first methods we used for solving graphs and it looked promising.
Unfortunately, in more complicated graphs, this method involves solving simultaneous recursive sequences which is difficult to evaluate.
And so we looked at other ways to approach a general graph.

What is a Matrix?

A matrix is an array of numbers, symbols, or expressions. Matrices are mainly used in linear algebra but also have lots of applications to many different areas of math.

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Examples:

$$M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

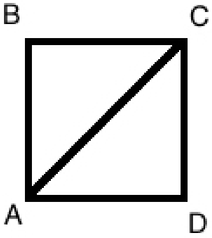
$$M_2 = \begin{bmatrix} 2x & \frac{1}{x^3} & x+3 \\ 3x^7 & \sqrt{5x} & 6x^2 \\ 7x^7 & 8x^8 & 9x^9 \end{bmatrix}$$

Cramer's Rule

In linear algebra, there is a tool called Cramer's Rule that is used to solve simultaneous systems of equations. The actual method/theorem won't be explained.

How to count paths using generating functions

Say we have the given graph:



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If we want to find the number of paths from vertex A to any other vertex we can write a generating function P_n where n is the vertex we are traversing to.

How to count paths using generating functions

- $P_A = xP_B + xP_C + xP_D$

How to count paths using generating functions

- $P_A = xP_B + xP_C + xP_D$
- $P_B = xP_A + xP_C + 1$
  Notice we add 1 because when we reach vertex B we can either continue or just stop, if we stop we must add 1.

How to count paths using generating functions

- $P_A = xP_B + xP_C + xP_D$
- $P_B = xP_A + xP_C + 1$
- $P_C = xP_A + xP_B + xP_D$

How to count paths using generating functions

- $P_A = xP_B + xP_C + xP_D$
- $P_B = xP_A + xP_C + 1$
- $P_C = xP_A + xP_B + xP_D$
- $P_D = xP_A + xP_C$

Converting a system of linear equations to matrices

To do this we must first isolate the constant terms to one side and the unknowns to another.

- $1 = P_A - xP_B - xP_C - xP_D$

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- $1 = P_A - xP_B - xP_C - xP_D$
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- $0 = -xP_A - xP_B + P_C - xP_D$
- $0 = -xP_A + 0P_B - xP_C + P_D$

Converting a system of linear equations to matrices

$$M = \begin{bmatrix} 1 & -x & -x & -x \\ -x & 1 & -x & 0 \\ -x & -x & 1 & -x \\ -x & 0 & -x & 1 \end{bmatrix}$$

Converting a system of linear equations to matrices

So if we want to solve for P_B we plug in the constants to the P_Bth row. Remember this gives the number of paths from A to B

$$M_B = \begin{bmatrix} 1 & 1 & -x & -x \\ -x & 0 & -x & 0 \\ -x & 0 & 1 & -x \\ -x & 0 & -x & 1 \end{bmatrix}$$

Now we can apply Cramer's Rule as shown previously.

We come to the conclusion that $P_B = \frac{x^2+x}{1-5x^2-4x^3}$

Solving for General Graphs

Using the same methods for a general graph to get from vertex A to a given vertex B on an N vertex graph we must follow the following steps:

- Write out equations for P_A through P_N with the equations in the form

$$P_M = \sum_{V=A}^{N} LxP_V$$

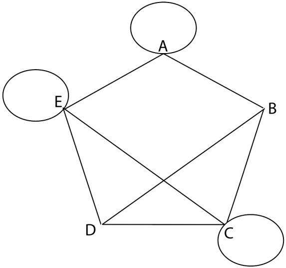
Where P_V is connected to P_M by one edge if $M = A$ We need to add 1 to the equation for the case in which you just stay.

Solving for General Graphs

- Then we apply Cramer's rule by creating a square matrix A with the coefficients of the equations as the elements.
  We also create A_i which is A but the ith column replaced by the constants

Application!

Now we can apply our method to the following graph:



We want to know how many ways are there to get from vertex A to vertex C given 30 moves.

AND THE NUMBER OF STEPS IS:

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$$3,366,748,639,878,647$$

(Even bigger than the US National Debt)