On the Numerical Solution of Black-Scholes Option Pricing Model

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Outline

- Background
- Black Scholes Equation – European Options
- Discretizing the PDE – Finite Difference Method
- Numerical Results
  - Explicit Euler, Implicit Euler and Crank Nicolson
  - Consistency, Stability and Convergence

Options: A Graphic Introduction

Fee to purchase contract:
$2.00

How to estimate this fee ??

Financial Institution

" For a $2.00 fee,
we *promise to buy your stock*
for $100 at *Expiry Time T* "

What is an Option?

- A financial derivative (contract) between two parties
- The buyer has the right to buy/sell the underlying asset
- At an agreed price (Exercise price, K)
- At a specific date (Time of maturity/expiration, T)

Time: 0 (today)

Value of the Option = ?

Value of the Option = d/dx (value of stock)

Time: Expiry of Option

Case 1:

If stock price drops at time T?

- Sell stock to Institution
- **Lose only $2.00 (contract fees)**

Case 2:

If stock price rises at time T ?

- Sell Stock in the Market at $130.00
- **Net Gain of $28.00**

The Black-Scholes Option Pricing Model

The Model:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- $S(t)$ be the value of the underlying at time t.
- $V(S(t), t)$ be the value of the derivative at time t.
- r be the zero risk interest rate.
- σ be the volatility of the underlying.

Some assumptions:

- Return on stock price follows geometric Brownian motion (constant drift and volatility)
- Risk-free interest rate and no dividend payout
- Frictionless market
- No arbitrage opportunities

The Black Scholes Formula (solution) estimates the fair price of an option (V) based on price of its' underlying stock (S) and at any given time (t).

- Fischer Black , Myron Scholes awarded 1997 Noble Prize in Economics

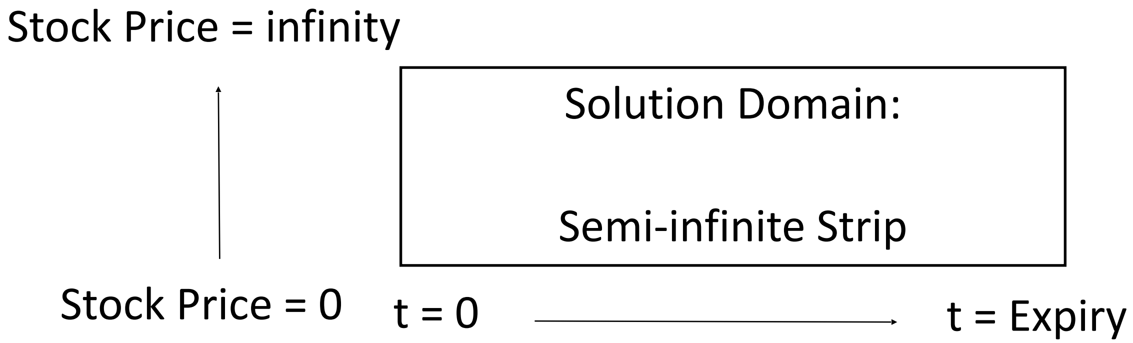
The Continuous Problem: Black-Scholes PDE for European Call Options

Terminal Boundary Value Problem:

- PDE: 2nd order, linear, (backwards-time) parabolic, constant coefficient

$$rV = V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S$$

$$\begin{cases} V(0,t) = 0, \text{ for all } t \\ V(S,t) \sim S \text{ as } S \to \infty \\ V(S,T) = max(S-K,0) \end{cases} .$$



Transforming:

(with time-dependent change of variables)

Initial Boundary Value Problem: with Dirchlet Boundary Conditions

- (forward-time) 1-D Diffusion Equation

Discretizing the PDE – Finite Difference Method

Taylors’ Series Expansion at xo:

$$U(x_o+h)=U(x_o)+hU_x(x_o)+h^2\frac{U_{xx}(x_o)}{2!}+\ldots+h^{n-1}\frac{U_{(n-1)}(x_o)}{(n-1)!}+O(h^n)$$

Truncating after first derivative:

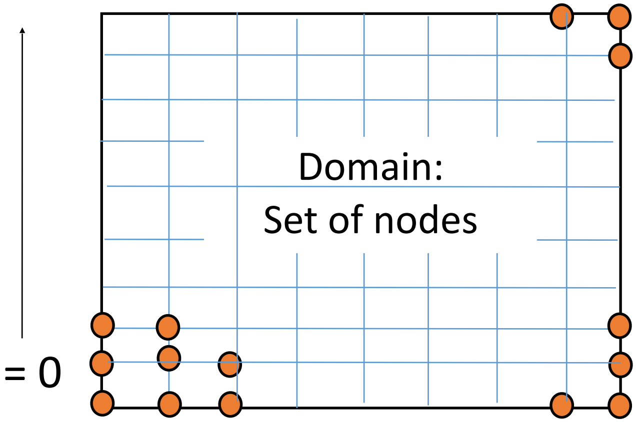
$$U(x_o+h)=U(x_o)+hU_x(x_o)+O(h^2)$$

Rearranging terms to get First Order Approximation to First Partial Derivative:

$$U_x=\frac{U(x_o+h)-U(x_o)}{h}+O(h)$$

‘Big O’: Truncation Error

Stock Price = S_Max

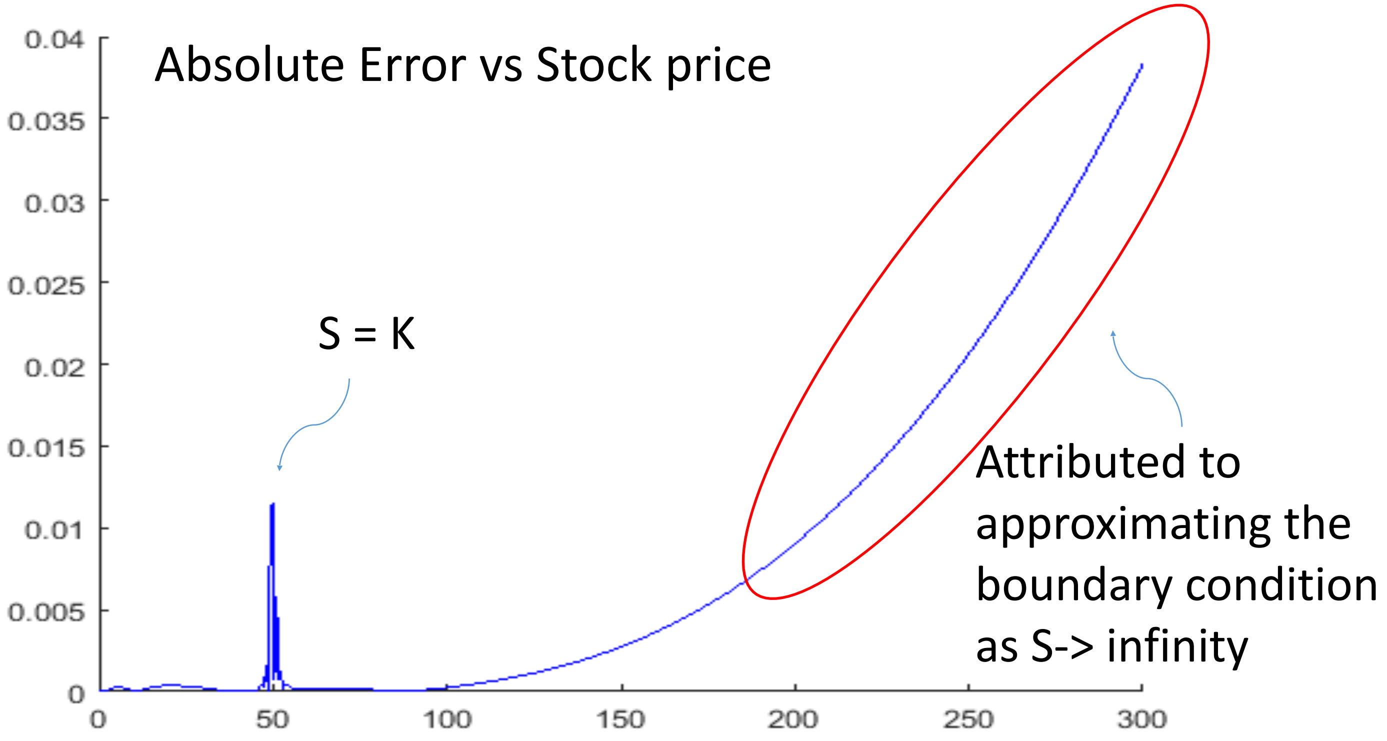


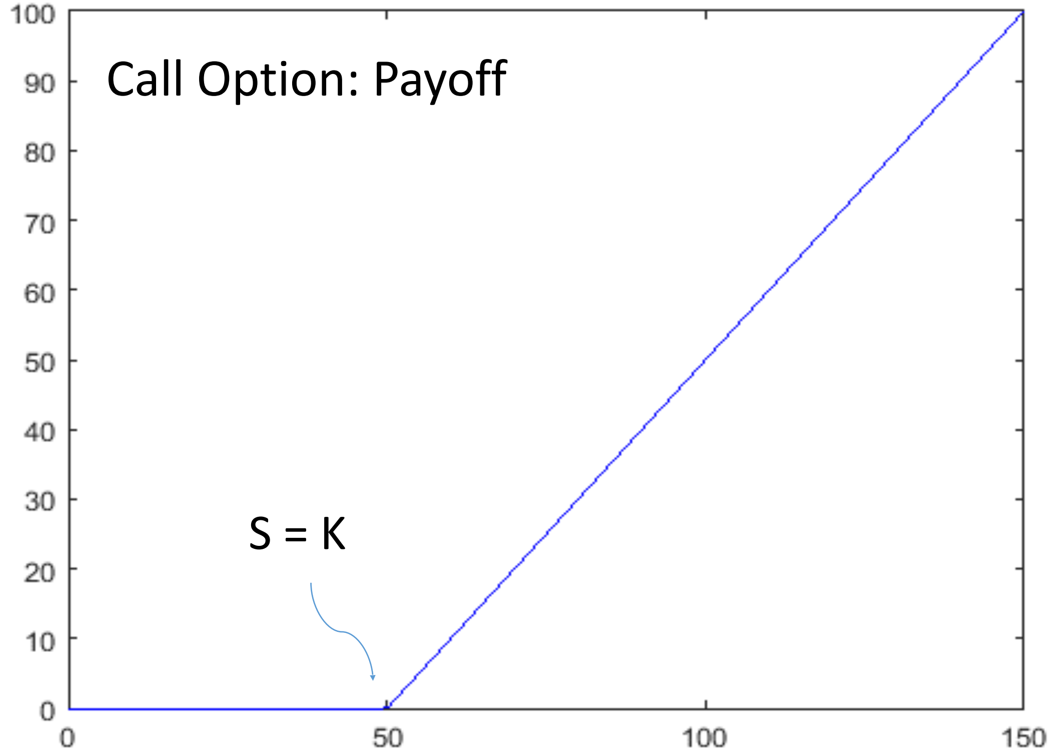
Stock Price = 0

t = 0 t = Expiry

At each node, replace PDE with Difference Equation

- Accumulation of truncation errors at Stock Price (S) = Exercise Price (K)





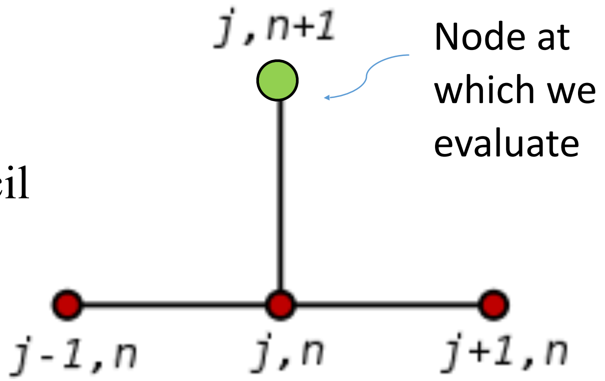
- Finally, evaluating PDE at different combinations of nodes => Different "Schemes"

- Analyzing 3 standard schemes: <u>Explicit Euler</u>, <u>Implicit Euler</u> and <u>Crank Nicolson</u>

Overview of the 3 Schemes

Explicit Euler

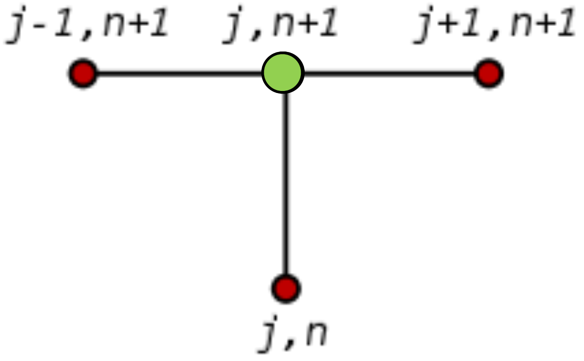
Stencil



- Replacing d/dt with forward differences
- Replacing d/dS and d^2/dS^2 with central differences
- Order of accuracy: O(dt^2)+O(dS^4)
- Conditionally Stable

Implicit Euler

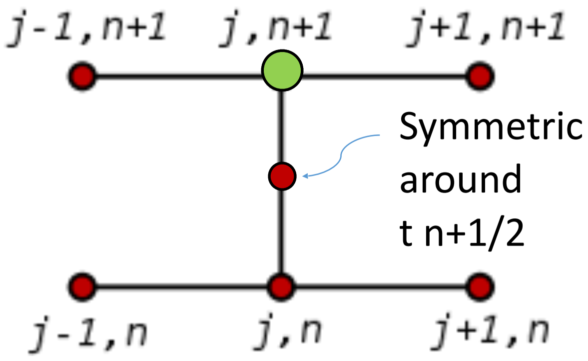
Stencil



- Replacing d/dt with backward differences
- Replacing d/dS and d^2/dS^2 with central differences
- Order of accuracy: O(dt^2)+O(dS^4)
- Unconditionally Stable

Crank-Nicolson

Stencil



- Replacing d/dt, d/dS and d^2/dS^2 with averages of Explicit Euler and Implicit Euler
- (Highest) Order of accuracy: O(dt^4)+O(dS^4)
- Unconditionally Stable

Numerical Analysis of Finite Difference Schemes

Consistency, Stability, Convergence

- Consistency – How well the exact solution of the PDE satisfies the finite difference scheme as mesh becomes finer (dt -> 0)

Truncation Error: A measure of consistency and order of accuracy of scheme

$$p \approx \frac{\log(E(h_1)/E(h_2))}{\log(h_1/h_2)}$$

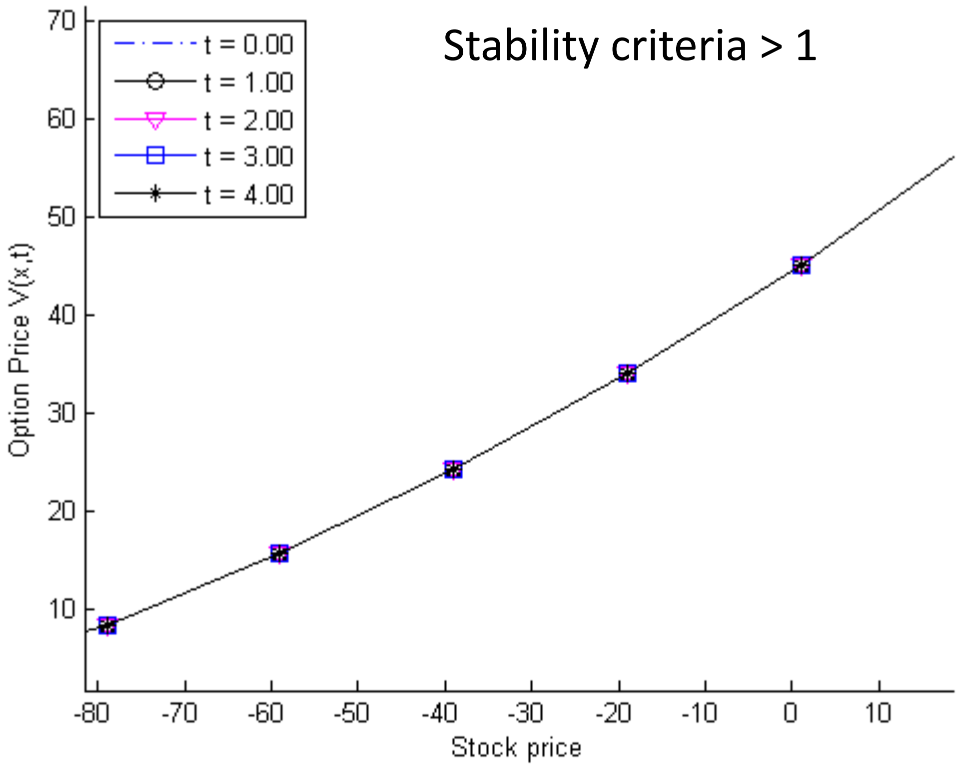
Explicit Euler : L-infinity Error		Implicit Euler : L-infinity Error		Crank Nicolson : L-infinity Error	
dS fixed, dt halved	**dt fixed**, dS halved	**dS fixed,** dt halved	**dt fixed**, dS halved	**dS fixed**, dt halved	**dt fixed**, dS halved

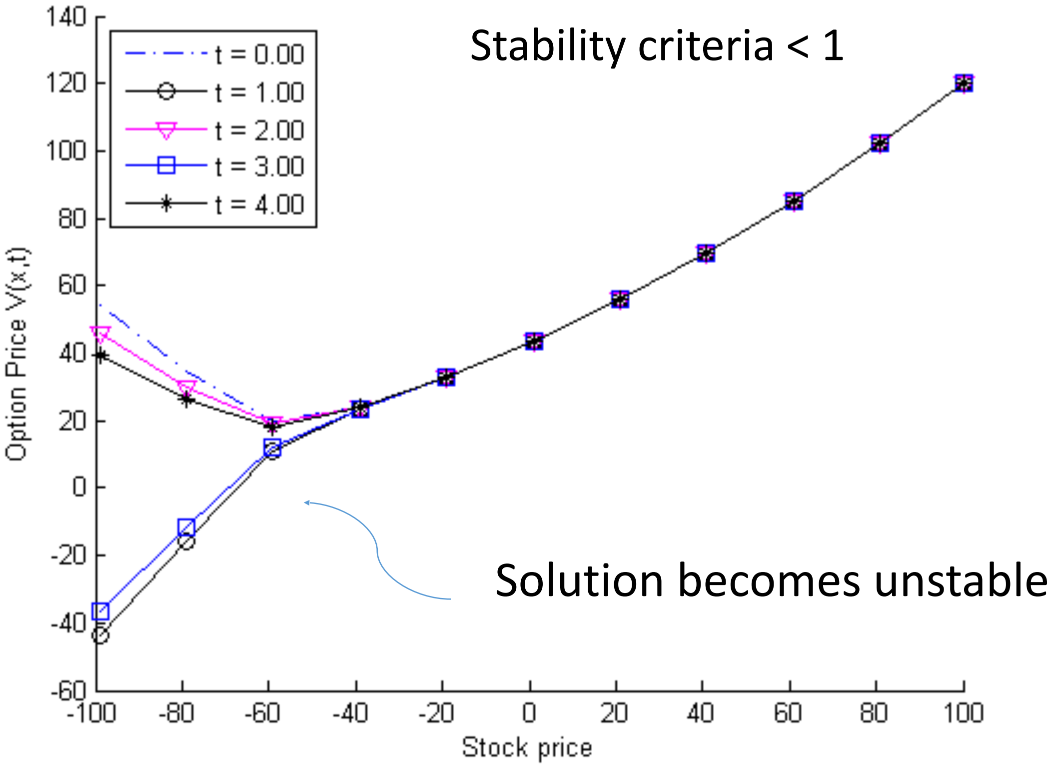
p = 2 (in time) p = 4 (in space) p = 2 (in time) p = 4 (in space) p = 4 (in time) p = 4 (in space)

Highest order of accuracy in both domains

- Stability – Is the approximate solution at each node bounded, as mesh size becomes smaller or larger ?
  - Bounded-ness of Point-wise error: A measure of stability

- Explicit Euler – Conditionally Stable





- Stability of Implicit Euler and Crank Nicolson - Unconditionally Stable

measured by error in L-infinity norm: $\mathrm{err}^0 := \max_{1 \le j \le N_x - 1} |V_j^{N_t} - V(jh, \tau = T)|$

dS	dt	Implicit Euler : L-infinity Error	Crank Nicolson : L-infinity Error
1.5	0.003	0.019544293	0.006853
0.15	0.003	0.019544293	0.006858
0.075	0.0015	0.019544293	0.006859
0.05	0.001	0.019544293	0.00686

- The error at time = 0 is bounded by a constant as mesh becomes finer

- Convergence: Does the approximate solution approach exact solution as mesh becomes finer?

  - Lax Equivalence Theorem: for a consistent finite difference method for a well-posed linear initial value problem, the method is convergent if and only if it is stable.

  - Rate of Convergence (Truncation Error): Measure of how quickly the approximate solution approaches the exact solution

dt	dS	Explicit Euler: Approx Solution	Implicit Euler: Approx Solution	Crank Nicolson: Approx Solution	Exact Solution
1.5	0.0025		2.188576726	2.191644589	2.1780901
0.75	0.00125		2.179979553	2.181531976	2.1780901
0.5	0.000833333		2.175066446	2.176108048	2.1780901
0.3	0.0005		2.17801576	2.178638629	2.1780901
0.15	0.00025		2.177915247	2.178226783	2.1780901

Fastest convergence to exact solution

Conclusion

Application of finite difference schemes to solve the linear Black-Scholes Equation for European Call and Put Options:

- Explicit Euler
  - Fastest computationally, but conditional stability causes it to take longer as mesh becomes finer
  - Transforming the PDE to get rid of dependence of S as a co-efficient leads to high round-off errors

- Crank Nicolson
  - Computationally expensive but fastest convergence, highest order of accuracy in both domains, and unconditionally stable

Future Work

- Application of finite difference schemes to:
  - Non-linear Black-Scholes PDE
  - Linear Black-Scholes PDE for pricing American Options (analytical solution does not exist)

References

Acknowledgement

- Dr. Narayan Thapa,
  Research Mentor, Minot State University

- MAA/NSF: Student Travel Award

- Thank you! :)