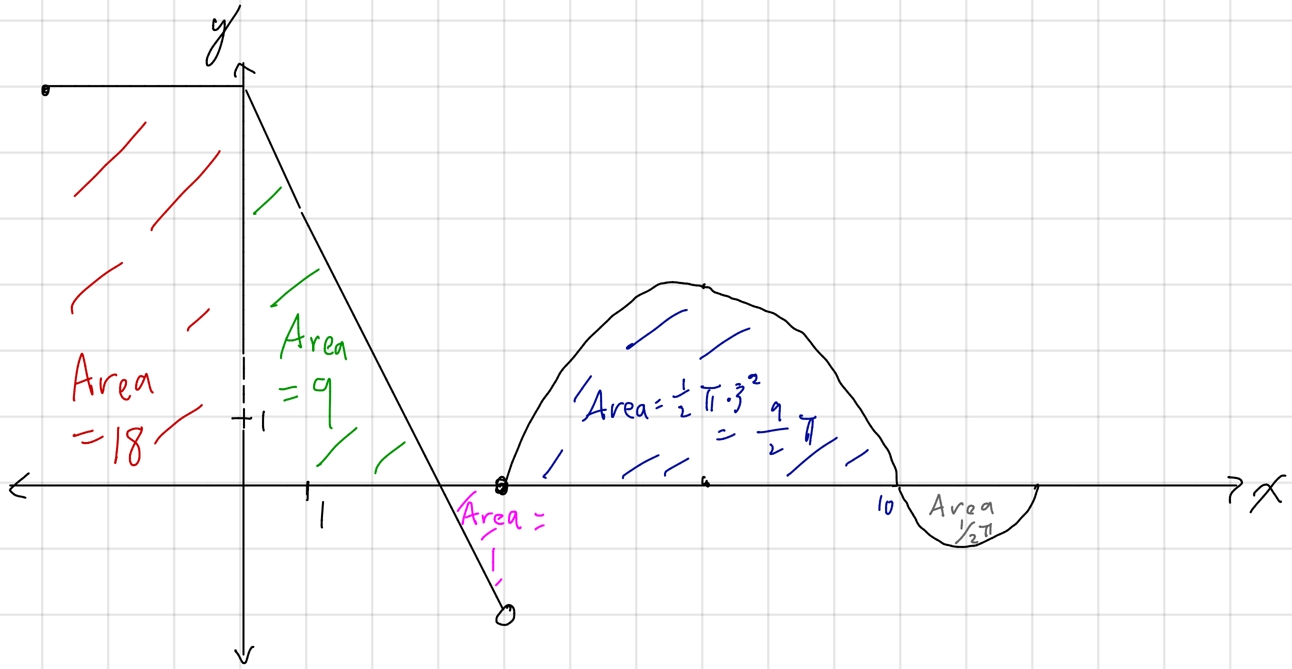
(1) Consider the following function,

$$f(x) = \begin{cases} 6 & -3 \le x < 0 \\ 6 - 2x & 0 \le x < 4 \\ \sqrt{9 - (x-7)^2} & 4 \le x < 10 \\ -\sqrt{1 - (x-11)^2} & 10 \le x \le 12 \end{cases}$$

(a) Sketch a graph of the above function. Graph paper is recomended.



(b) Use your sketch to find the following values

(i) $\int_{-3}^{0} f(x)\,dx$ $= 18$

(ii) $\int_{0}^{-3} f(x)\,dx = -\int_{-3}^{0} f(x)dx = -18$

(iii) $\int_{0}^{3} f(x)\,dx = 9$

(iv) $\int_{3}^{4} f(x)\,dx = -1$

(v) $\int_{0}^{4} f(x)\,dx = 9 - 1 = 8$

(vi) $\int_{0}^{4} |f(x)\,dx| = 9 + 1 = 10$

(vii) $\int_{4}^{10} f(x)\,dx = \frac{9}{2}\pi$

(viii) $\int_{11}^{10} f(x)\,dx = -\int_{10}^{11} f(x)dx = \frac{1}{4}\pi$

(ix) $\int_{4}^{11} f(x)\,dx = \frac{9}{2}\pi - \frac{1}{4}\pi = \frac{17}{4}\pi$

(x) $\int_{4}^{11} |f(x)\,dx| = \frac{9}{2}\pi + \frac{1}{4}\pi = \frac{19}{4}\pi$

(2) Calculate $\int_1^{-3} |2(x+2)|\, dx$. Show explicitely how you use the laws of integrals to calculate this, splitting up the integral into two pieces appropriately.

$$|2(x+2)| = \begin{cases} 2(x+2), & \text{if } 2(x+2) \geq 0 \\ -2(x+2), & \text{if } 2(x+2) \leq 0 \end{cases}$$

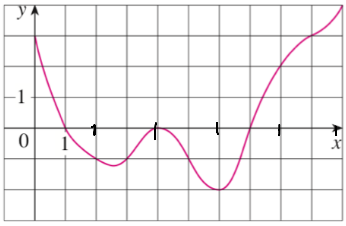
$$2(x+2) \geq 0$$
$$x+2 \geq 0$$
$$x \geq -2$$

$$|2(x+2)| = \begin{cases} 2(x+2), & \text{if } x \geq -2 \\ -2(x+2), & \text{if } x \leq -2 \end{cases}$$

$$\int_1^{-3} 2|x+2|\,dx = -\int_{-3}^{1} 2|x+2|\,dx = \int_{-3}^{-2} 2(x+2)\,dx - \int_{-2}^{1} 2(x+2)\,dx$$

$$= \frac{-2+0}{2} \cdot 1 - \frac{0+6}{2} \cdot 3 = -1 - 9 = -10$$

(3) Consider the function graphed below: in each of the following,



$y = f(x)$

(a) Estimate the value of $\int_0^{10} f(x)\,dx$ using a Riemann sum with 5 sub-intervals taking the sam points to be left endpoints.

$$(3 - 1 + 0 - 2 + 2)\,2 = 4$$

(b) Estimate the value of $\int_0^{10} f(x)\,dx$ using a Riemann sum with 5 sub-intervals taking the sample points to be right endpoints.

$$(-1 + 0 - 2 + 2 + 4)\,2 = 6$$

(c) Estimate the value of $\int_0^{10} f(x)\,dx$ using a Riemann sum with 5 sub-intervals taking the sample points to be midpoints.

$$(0 - 1 - 1 + 0 + 3)\,2 = 2$$

(d) Estimate the value of $\int_0^{10} |f(x)|\,dx$ using a Riemann sum with 5 sub-intervals taking the sample points to be left endpoints.

$$(3 + 1 + 0 + 2 + 2)\,2 = 16$$

(e) Estimate the value of $\int_0^{10} |f(x)|\,dx$ using a Riemann sum with 5 sub-intervals taking the sample points to be right endpoints.

$$(1 + 0 + 2 + 2 + 4)\,2 = 18$$

(f) Estimate the value of $\int_0^{10} |f(x)|\,dx$ using a Riemann sum with 5 sub-intervals taking the sample points to be midpoints.

$$(0 + 1 + 1 + 0 + 3)\,2 = 10$$

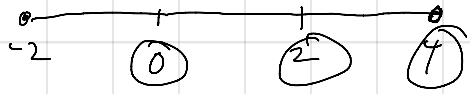
(4) Consider $\int_{-2}^{4} x^2 dx$

(a) Estimate the value of the above integral using a Riemann sum with 3 sub-intervals taking the sample points to be left endpoints.

$$\left((-2)^2 + 0^2 + 2^2\right) 2 = 16$$

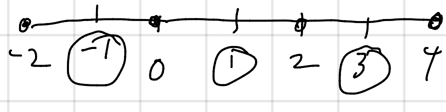
(b) Estimate the value of the above integral using a Riemann sum with 3 sub-intervals taking the sample points to be right endpoints.

$$\left(0^2 + 2^2 + 4^2\right) \cdot 2 = 40$$



(c) Estimate the value of the above integral using a Riemann sum with 3 sub-intervals taking the sample points to be midpoints.

$$\left((-1)^2 + (1)^2 + 3^2\right) \cdot 2 = 22$$



(d) Write the value of the above integral as a limit of Riemann sums.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-2 + \frac{6}{n} i\right)^2 \cdot \frac{6}{n}$$

$$\Delta x = \frac{4-(-2)}{n} = \frac{6}{n}$$

$$x_i = -2 + \frac{6}{n} i$$

RH:

$$x_i^* = -2 + \frac{6}{n} i$$

(5) What definite integral is equal to $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+(5+\frac{2k}{n})^2} \cdot \frac{2}{n}$?

$$\int_5^7 \frac{1}{1+x^2}\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+(5+\frac{2k}{n})^2} \cdot \frac{2}{n}$$

$$f(x) = \frac{1}{1+x^2} \qquad \Delta x = \frac{7-5}{n} = \frac{2}{n} \qquad x_k^* = x_k = 5 + k\Delta x$$

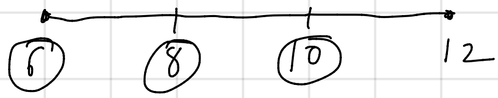
$$= 5 + \frac{2k}{n}$$

(6) Suppose the values of a function are given by the following table

x	6	7	8	9	10	11	12
$f(x)$	12	11	9	7	3	-2	-5

(a) Write and evaluate a Riemann sum with 3 subdivisions using left endpoints as sample points to estimate $\int_6^{12} f(x)\, dx$

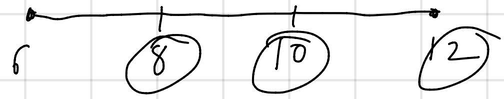
$$(f(6) + f(8) + f(10)) \cdot 2$$



$$= (12 + 9 + 3) \cdot 2 = 48$$

(b) Write and evaluate a Riemann sum with 3 subdivisions using right endpoints as sample points to estimate $\int_6^{12} f(x)\, dx$

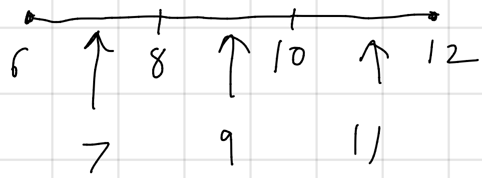
$$(f(8) + f(10) + f(12)) \cdot 2$$



$$= (9 + 3 + (-5)) \cdot 2 = 14$$

(c) Write and evaluate a Riemann sum with 3 subdivisions using midpoints as sample points to estimate $\int_6^{12} f(x)\, dx$

$$(f(7) + f(9) + f(11)) \cdot 2$$



$$= (11 + 7 + (-2)) \cdot 2$$

$$= 32$$

(7) Suppose $\int_2^5 f(x)\, dx = 7$ and $\int_2^9 f(x)\, dx = 12$. Find $\int_5^9 f(x)\, dx$. Show explicitely how you use the properties of integrals to find this.

$$\int_5^9 f(x)\,dx = \int_5^2 f(x)\,dx + \int_2^9 f(x)\,dx = -\int_2^5 f(x)\,dx + \int_2^9 f(x)\,dx$$

$$= -7 + 12 = 5$$

(8) True or false?

False (a) $\int_0^7 xe^x\,dx = x\int_0^7 e^x\,dx$

True (b) $\int_2^5 3\sin x\,dx = -3\int_5^2 \sin x\,dx$

True (c) $\int_a^b (f(x) - 3g(x))\,dx = \int_a^b f(x)\,dx - 3\int_a^b g(x)\,dx$

True (d) x is a dummy variable in $\int_0^5 \cos^2(x)\,dx$

False (e) $\int_2^4 \cos(x^2)\,e^x dx = \left(\int_2^4 \cos(x^2)\,dx\right)\left(\int_2^4 e^x\,dx\right)$

False (f) $\int_0^5 \cos(x^2)\,dx = \int_0^5 \cos(t^2)\,dx$

(a) False - Only constants may be factored out of an integral

(b) True - we may factor the constant 3 out of the integral, and then switching the limits of integration multiplies the integral by -1

(c) True - From the sum rule and constant multiply rule for integrals

(d) True - The variable of integration in a definite integral is a dummy variable

(e) False - The integral of a product is not in general the product of the integrals

(f)

$$\int_0^5 \cos x^2\,dx = \int_0^5 \cos t^2\,\underline{dt} \text{ is true}$$

In substituting a new dummy variable, we must substitute all instances of the variable.