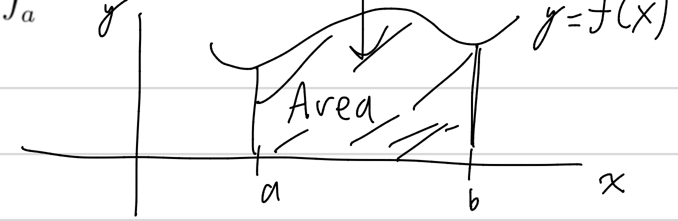
Math 16 7 Jan 2019

Interpreting the definite integral as an area under a curve

Property #1 of the definite integral: Suppose f is a function on a closed interval $[a, b]$ (where $a < b$). Suppose $f(x) \geq 0$ for all x in $[a, b]$. then

$$\int_a^b f(x)\,dx = \text{The area of the region given by } a \leq x \leq b \text{ and } 0 \leq y \leq f(x)$$



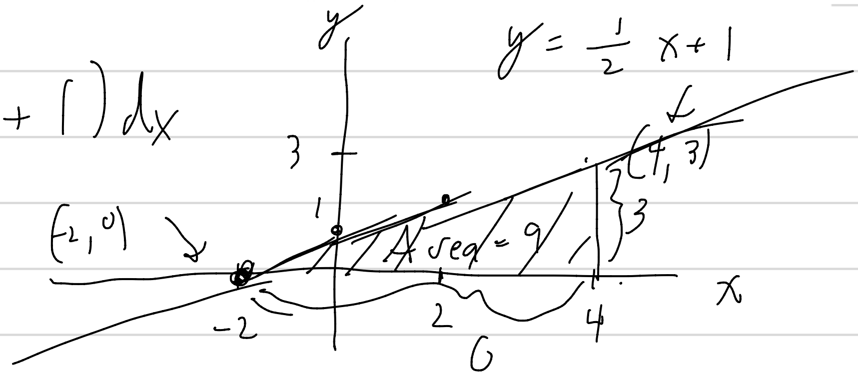
$a < b$

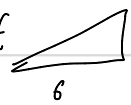
Example. Let's find

$$\int_{-2}^{4} \left(\frac{1}{2}x + 1\right) dx = \int_{-2}^{4} \left(\frac{1}{2}t + 1\right) dt$$

We call $\int_a^b f(x)\,dx$ a "definite integral." a and b are the "limits" of integration. a is the "lower limit" of integration and b is the "upper limit" of integration. x is the "variable of integration" (the factor dx tells us that x is the variable of integration). $f(x)\,dx$ is the "integrand" (the thing we are integrating).

$$\int_{-2}^{4} \left(\frac{1}{2}x + 1\right) dx$$



$\int_{-2}^{4} \left(\frac{1}{2}x + 1\right) dx =$ Area of  $= \frac{1}{2}\,\text{base} \cdot \text{height}$

$= \frac{1}{2} \cdot 6 \cdot 3 = 9$

$$\int_{-2}^{4} \left(\frac{1}{2}x + 1\right) dx = 9$$

Example. Let

$$f(x) = \begin{cases} 2x & x < 2 \\ 4 & x \geq 2 \end{cases}$$

$f(0) = 2\cdot 0 = 0$

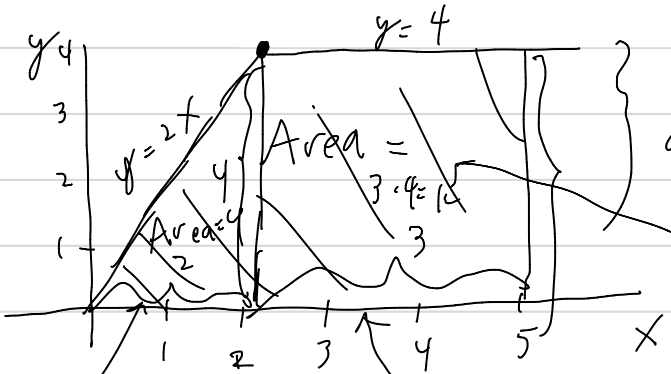
$f(1) = 2\cdot 1 = 2$

$f(2) = 4$

$f(3) = 4$

Find

$$\int_0^5 f(x)\, dx$$

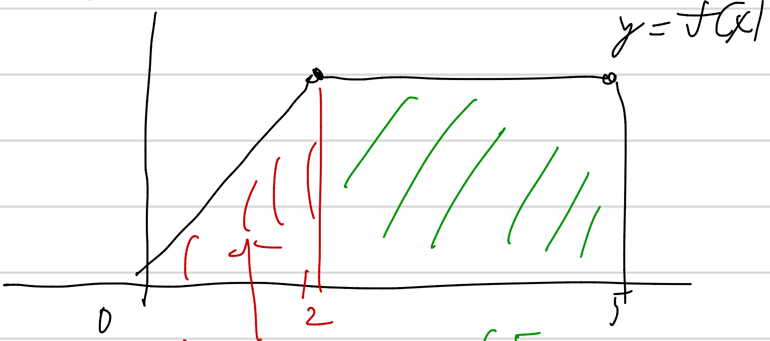


$y = f(x)$

$\int_0^2 f(x)\,dx = \frac{1}{2}(2\cdot 4) = 4$

$\int_2^5 f(x)\,dx$

$\int_0^5 f(x)\,dx = \text{Area} = 4 + 12 = 16$

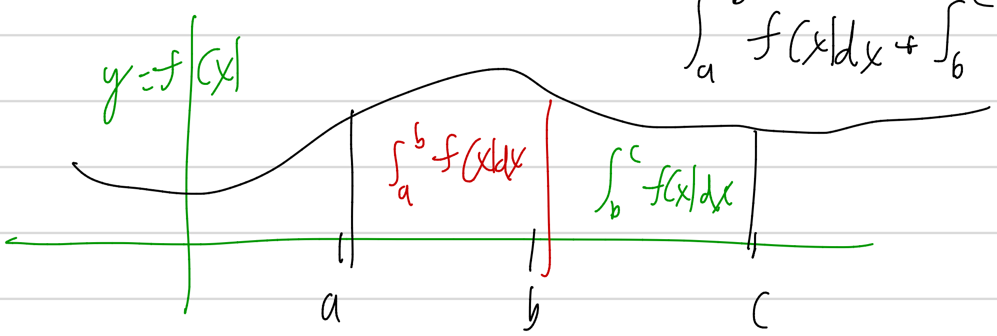


$$\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx$$

Property #2 of the definite integral:

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



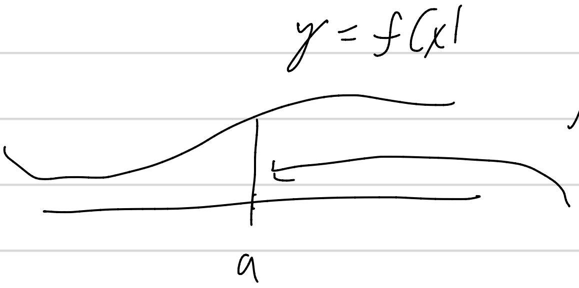
$$\int_a^a f(x)\,dx + \int_a^a f(x)\,dx = \int_a^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

Property #3 of the definite integral.

$$\int_a^a f(x)\ dx = 0$$

$$\int_3^3 e^x\, dx = 0 \qquad \int_{-4}^{-4} \cos x\, dx = 0 \qquad \int_0^0 \sqrt{x^2+1}\, dx = 0$$



$\int_a^a f(x)\,dx$ = Area of this line segment = 0

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

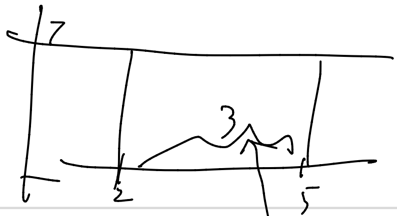
Property #4 of the definite integral:

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

Changing the order of the limits of integration negates the integral.

Example: Find

$$\int_5^2 7\, dx$$



$$\int_5^2 7\,dx = ???$$

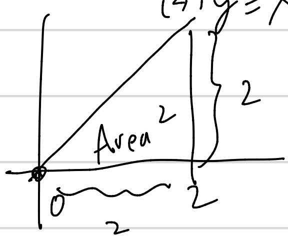
$$\int_2^5 7\,dx = \text{Area} = 3\cdot 7 = 21$$

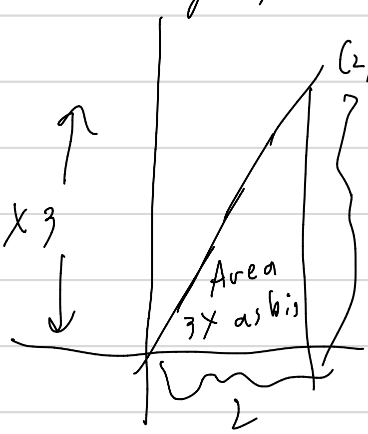
$$\int_5^2 7\,dx = -21$$

Property #5 of the definite integral: If c is a constant, then

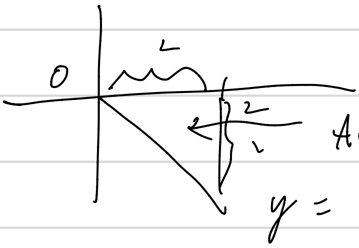
$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \qquad \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_0^2 x\,dx = \frac{1}{2}(2)\cdot 2 = 2 \qquad \int_0^2 3x\,dx = 3\cdot\int_0^2 x\,dx = 3\cdot 2 = 6$$





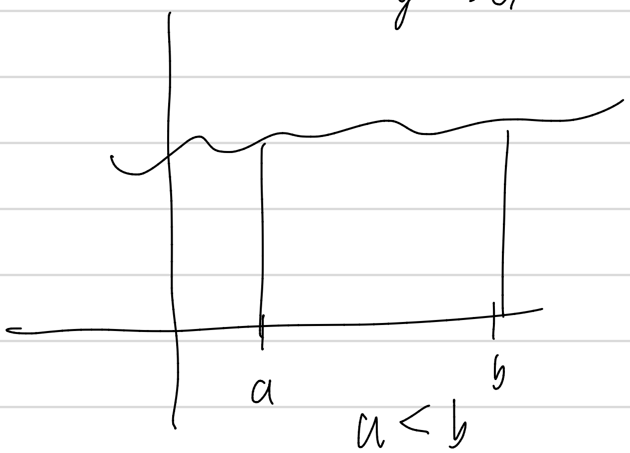
$$\int_0^2 x\,dx = 2 \qquad \int_0^2 (-x)\,dx = -\int_0^2 x\,dx = -2$$



Area of triangle = 2

$y = -x$ $\qquad \int_0^2 (-x)\,dx = -\text{Area} = -2$

$y = f(x) \ge 0$



$a < b$

$\int_a^b f(x)dx \ge 0$

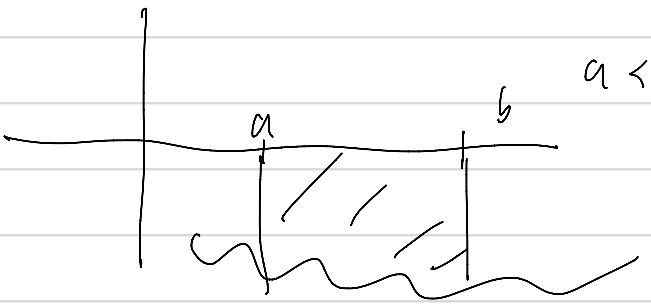
$\int_b^a f(x)dx < 0$

$y = g(x) \le 0$

$\int_a^b g(x)dx < 0$

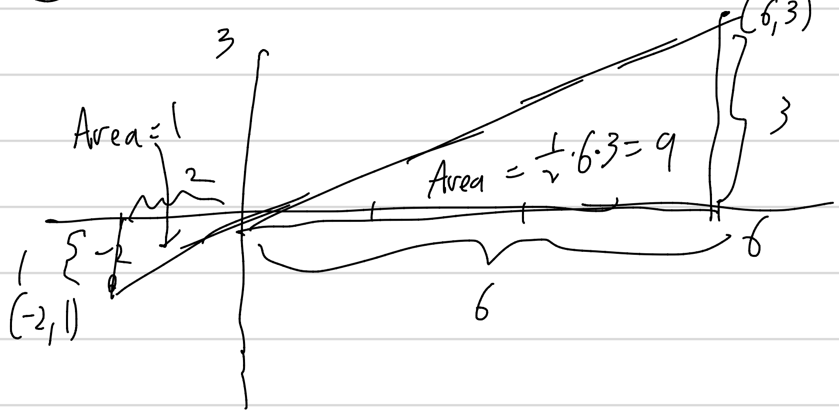
$a < b$

$\int_b^a g(x)dx \ge 0$



$y = g(x)$

$\int_{-2}^{6} \frac{1}{2}x\, dx$



$$\int_{-2}^{6} \frac{1}{2} x\,dx = \int_{-2}^{0} \frac{1}{2} x\,dx + \int_{0}^{6} \frac{1}{2} x\,dx = -1 + 9 = 8$$