Math 1b (8:30AM) 5 Feb 2020

Integration by parts:

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx$$

Integration by parts for definite integrals

$$\int_a^b f(x)g(x)dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)dx$$

Example

$$\int_1^e \ln x \, dx = x\ln x\Big|_1^e - \int_1^e \frac{1}{x} x \, dx$$

	diff	int
+	$\ln x$	1
−	$1/x$	x

$$= \underbrace{(e\ln e - 1\ln 1)}_{e} - \int_1^e 1 \, dx$$

$$= e - [x]_1^e$$

$$= e - (e-1) = 1$$

Section 7.2

(7.2 is needed for 7.3)

7.2 integration combinations of sin and cos
of sec and tan
of csc and cot

$$\int \sin^m x \cos^n x \, dx$$

Example [If n is odd, $\int \sin^m x \cos^{odd} x \, dx$

$$\int \sin^2 x \cos^3 x \, dx$$

Let $u = \sin x$ $\quad du = \underbrace{\cos x \, dx}$

$$= \int \overbrace{\sin^2}^{u^2} \overbrace{\cos^2 x}^{} \overbrace{\cos x \, dx}^{du}$$

$$= \int \sin^2 (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{1}{3} u^3 - \frac{1}{5} u^5 + C$$

$$= \boxed{\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C}$$

$$\int \sin^m x \cos^{odd} x \, dx \qquad u = \sin x$$

$$du = \cos x \, dx$$

$$= \int \sin^m x \overbrace{\cos^{odd-1}}^{even} x \overbrace{\cos x \, dx}^{du}$$

$$= \int \sin^m x (1 - \sin^2 x)^{\frac{odd-1}{2}} \cos x \, du = \int \overbrace{u^m (1-u^2)^{\frac{odd-1}{2}}}^{polynomial} du$$

This same trick works if sin is to an odd power

$$\int \sin^{odd} \cos^n x \, dx$$

This same trick works if sin is to an odd power

$$\int \sin^{odd} \cos^n x \, dx$$

$$\int \sin^5 x \cos^4 x \, dx \qquad u = \cos x$$

$$-du = \sin x \, dx$$

$$= \int \underset{\text{even}}{\sin^4 x} \cos^4 \underbrace{\sin x \, dx}$$

$$= \int (1-\cos^2 x)^2 \cos^4 x \sin x \, dx$$

$$= -\int (1-u^2)^2 u^4 \, du = \ldots$$

To summarize $\int \sin^m x \cos^n x \, dx$

If m is odd

$u = \cos x \quad -du = \sin x \, dx$

$$= \int \sin^{m-1} x \cos^n x \sin x \, dx$$

$$= \int (1-u^2)^{\frac{m-1}{2}} u^n \, du$$

If n is odd

$u = \sin x \quad du = \cos x \, dx$

If both m and n are odds, either of these tricks will work.

The hard case is integrating $\int \sin^m x \cos^n x \, dx$ when m and n are both even.

$$\int \cos^2 x \, dx \qquad \int \sin^2 x \, dx \qquad \int \cos^2 x \sin^2 x \, dx$$

$$\int \cos^4 x \, dx$$

$\int \sin^m x \cos^n x\, dx$ m, n both even

Step #1

Since m and n are both even, we can convert all our sines to cosines, or all are cosines to sines. Pick one, so we only have even powers of sine (or even powers of cosine)

$$\int \sin^2 x \cos^4 x\, dx$$

$$\int (1-\cos^2 x) \cos^4 x\, dx$$

$$= \int \cos^4 x\, dx - \int \cos^6 x\, dx$$

$$\int \sin^2 x\,(1-\sin^2 x)^2\, dx$$

$$\int \sin^6 x\, dx - 2\int \sin^4 x\, dx + \int \sin^2 x\, dx$$

Step #2: Two options (you choose which you like best)

#1 Use Trig identities

#2 Use integration by parts

Trig Identities

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2} \qquad \sin^2 x = -\frac{1}{2}\cos 2x + \frac{1}{2}$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

$$\sin 2x = 2\sin x \cos x$$

Trig Identities

$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$ $\quad\sin^2 x = -\frac{1}{2}\cos 2x + \frac{1}{2}$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin 2x = 2\sin x \cos x$

$\int \cos^2 x \, dx = ???$

we should use trig identities to continue

$= \int \left(\frac{1}{2}\cos 2x + \frac{1}{2}\right) dx = \frac{1}{2}\frac{1}{2}\sin 2x + \frac{1}{2}x + C$

$= \frac{1}{2}\frac{1}{2}\, 2\sin x \cos x + \frac{1}{2} + C = \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C$

$$\int \cos^4 x \, dx$$

$$= \int (\cos^2 x)^2 dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \int \frac{1}{4} dx + \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos^2 2x \, dx$$

$$= \frac{1}{4}x + \frac{1}{2}\frac{1}{2}\sin 2x + \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}X + \frac{1}{4}2\sin x\cos x + \frac{1}{8}X + \frac{1}{4}\cdot\frac{1}{8}\sin 4x + C$$

$$= \frac{1}{4}X + \frac{1}{2}\sin x\cos x + \frac{1}{8}X + \frac{1}{32}2\sin 2x\cos 2x + C$$

$$= \frac{3}{8}X + \frac{1}{2}\sin x\cos x + \frac{1}{32}\left(2\cdot 2\sin x\cos x\,(1-2\sin^2 x)\right) + c$$