Math 1b (8:30 AM) 11 Feb 2020

How to integrate $\int \tan^n \theta \, d\theta$

$$\int \tan x \, dx = \int -\frac{-\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$$

$$= -\ln|\cos x| + c = \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$$

For any higher power, pull off $\tan^2\theta$ and convert to $\sec^2\theta - 1$ repeatedly until done.

$$\int \tan^7 \theta \, d\theta = \int \tan^5\theta \tan^2\theta \, d\theta = \int \tan^5\theta (\sec^2\theta - 1) \, d\theta$$

$u = \tan\theta \quad du = \sec^2\theta \, d\theta$

$$= \int \underbrace{\tan^5\theta}_{u^5} \underbrace{\sec^2\theta \, d\theta}_{du} - \int \tan^5\theta \, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta \, d\theta$$

$$= \frac{1}{6}\tan^6\theta - \int \tan^3\theta (\sec^2\theta - 1) \, d\theta =$$

$$= \frac{1}{6}\tan^6\theta - \frac{1}{4}\tan^4\theta + \int \tan^3\theta \, d\theta$$

$$= \frac{1}{6}\tan^6\theta - \frac{1}{4}\tan^4\theta + \int \tan\theta \sec^2\theta \, d\theta - \int \tan\theta$$

$$= \frac{1}{6}\tan^6\theta - \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C$$

Integrating $\int \csc^m x \cot^n x \, dx$

This is almost the same as $\int \sec^m x \tan^n x \, dx$

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$1 + \tan^2\theta = \sec^2\theta$

$d \tan\theta = \sec^2\theta \, d\theta$

$d \sec\theta = \sec\theta \tan\theta \, d\theta$

$1 + \cot^2\theta = \csc^2\theta$

$d \cot\theta = -\csc^2\theta \, d\theta$

$d \csc\theta = -\csc\theta \cot\theta \, d\theta$

even power

$\int \csc^2\theta \cot^2\theta \, d\theta$

$\int \overset{u^2}{\cot^2\theta} \overbrace{\csc^2\theta \, d\theta}^{-du}$

$u = \cot\theta$

$du = -\csc^2\theta \, d\theta$

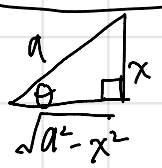
$-du = \csc^2\theta \, d\theta$

$= -\int u^2 du = -\frac{1}{3}u^3 + C$

$= -\frac{1}{3}\cot^3\theta + C$

Three types of trig substitutions

$$\int f(x, \sqrt{a^2-x^2})\,dx =$$
$$= \int f(a\sin\theta, a\cos\theta)\cos\theta\, d\theta$$



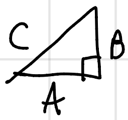
$\theta = \arcsin\frac{x}{a}$

$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

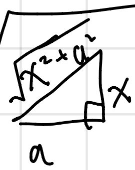
$dx = \cos\theta\, d\theta$



$A^2+B^2=C^2$

$1-\sin^2\theta = \cos^2\theta$, $\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

$$\int f(x, \sqrt{a^2+x^2})\,dx$$
$$= \int f(a\tan\theta, a\sec\theta)\, a\sec^2\theta\, d\theta$$



$\theta = \arctan(x/a)$

$\tan\theta = x/a$

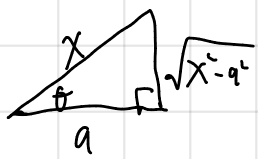
$x = a\tan\theta$

$\sqrt{a^2+x^2} = a\sec\theta$ $\quad dx = a\sec^2\theta\, d\theta$

$1+\tan^2\theta = \sec^2\theta$ $\quad \sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$

$$\int f(x, \sqrt{x^2-a^2})\,dx$$

$\theta = \operatorname{arcsec}\left|\frac{x}{a}\right|$



$\sec\theta = \frac{x}{a}$

$$= \int f(a\sec\theta, a\tan\theta)\, a\sec\theta\tan\theta\, d\theta$$

$x = a\sec\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$\sec^2\theta - 1 = \tan^2\theta$

Example #1 $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$$\sqrt{9-x^2} = \sqrt{3^2-x^2}$$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{(3\sin\theta)^2}\,3\cos\theta\,d\theta$$

$$= \int \frac{9\cos^2\theta}{9\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta = \int(\csc^2\theta-1)\,d\theta$$

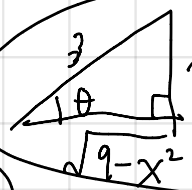
$$\theta = \arcsin\frac{x}{3}$$

$$\sin\theta = \frac{x}{3}$$

$$x = 3\sin\theta$$

$$\sqrt{9-x^2} = 3\cos\theta$$

$$dx = 3\cos\theta\,d\theta$$



$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \arcsin\frac{x}{3} + C$$

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

Example #3

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

$\tan\theta = \frac{x}{2}$

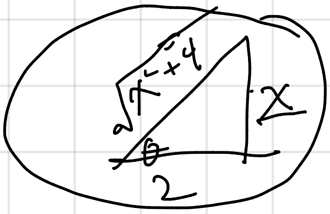
$x = 2\tan\theta$

$\sqrt{x^2+4} = 2\sec\theta$

$dx = 2\sec^2\theta\,d\theta$

$\sqrt{x^2+2^2}$

tan substitution



$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{(2\tan\theta)^2\,2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\cdot\frac{-1}{u} + C$$

$u = \sin\theta \quad du = \cos\theta\,d\theta$

$$\frac{-1}{4u} + C = \frac{-1}{4\sin\theta} + C = \frac{-1}{4}\csc\theta + C$$

$$\boxed{= \frac{-1}{4}\left(\frac{\sqrt{x^2+4}}{x}\right) + C}$$

Example #7

$$\int \frac{dx}{\sqrt{x^2-4}}$$

$x = 2\sec\theta \qquad \sec\theta = \frac{x}{2}$

$\sqrt{x^2-2^2} = 2\tan\theta \qquad \tan\theta = \frac{\sqrt{x^2-4}}{2}$

$\sqrt{x^2-2^2}$

$dx = 2\sec\theta\tan\theta \, d\theta$

$$\int \frac{dx}{\sqrt{x^2-4}} dx = \int \frac{\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{2}\,\cancel{\tan\theta}} = \int \sec\theta \, d\theta$$

$$\ln|\sec\theta + \tan\theta| + c = \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + c$$

simplify

$= \ln$ ______ (to be continued)