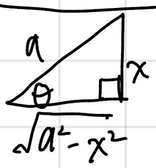
Math 1b (8:30AM) 12 Feb 2020

Three types of trig Substitions

$$\int f(x, \sqrt{a^2-x^2})\,dx =$$

$$= \int f(a\sin\theta, a\cos\theta)\cos\theta\,d\theta$$



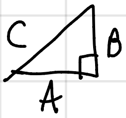
$\theta = \arcsin \frac{x}{a}$

$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$\sqrt{a^2-x^2} = a\cos\theta$

$dx = a\cos\theta\,d\theta$

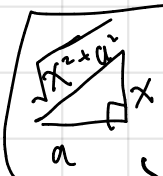


$A^2+B^2=C^2$

$1-\sin^2\theta = \cos^2\theta$, $\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

$$\int f(x, \sqrt{a^2+x^2})\,dx$$

$$= \int f(a\tan\theta, a\sec\theta)\,a\sec^2\theta\,d\theta$$



$\theta = \arctan(x/a)$

$\tan\theta = x/a$

$x = a\tan\theta$

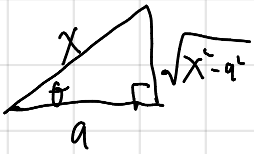
$\sqrt{a^2+x^2} = a\,\underline{\sec\theta}$ $dx = a\sec^2\theta\,d\theta$

$1+\tan^2\theta = \sec^2\theta$ $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$

$$\int f(x, \sqrt{x^2-a^2})\,dx$$

$$= \int f(a\sec\theta, a\tan\theta)\,a\sec\theta\tan\theta\,d\theta$$

$\theta = \operatorname{arcsec}\left|\frac{x}{a}\right|$



$\sec\theta = \frac{x}{a}$

$x = a\sec\theta$

$\sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\,d\theta$

$\sec^2\theta - 1 = \tan^2\theta$

Find $\int \frac{1}{\sqrt{x^2-\boxed{4x}}}\,dx$

We need to get rid of the annoying middle term using complete the square

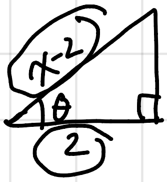
$\sqrt{a^2-x^2}$ $x=a\sin\theta$

$\sqrt{a^2+x^2}$ $x=a\tan\theta$

$\boxed{\sqrt{x^2-a^2} \quad x=a\sec\theta}$

$x^2-4x=\underbrace{x^2-4x+4}-4$

$= (x-2)^2-4 = (x-2)^2-2^2$



$\sqrt{(x-2)^2-2^2}$

$=\sqrt{x^2-4x}$

$\sec\theta=\frac{x-2}{2}$

$x-2=2\sec\theta$

$> \sqrt{x^2-4x}=2\tan\theta$

$dx=2\sec\theta\tan\theta\,d\theta$

$\int \frac{1}{\sqrt{x^2-4x}}\,dx$

$=\int \frac{2\sec\theta\tan\theta}{2\tan\theta}\,d\theta$

Memorize

$=\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|+c$

$\ln\left|\frac{A}{B}\right| = \ln|A|-\ln|B|$

$= \ln\left|\frac{x-2}{2}+\frac{\sqrt{x^2-4x}}{2}\right|+c = \ln\left|\frac{x-2+\sqrt{x^2-4x}}{2}\right|+c$

$= \ln|x-2+\sqrt{x^2-4}| \underbrace{-\ln 2+c} = \boxed{\ln|x-2+\sqrt{x^2-4x}|+c}$

$\sqrt{x^2-4x} = \sqrt{(x-2)^2-4} = \sqrt{(2\sec\theta)^2-4} = 2\sqrt{\sec^2\theta-1} = 2\tan\theta$

$$\int \frac{1}{(bx+K)^2+a^2}\,dx$$

where a, b, k are constants

$$\theta = \arctan\left(\frac{bx+K}{a}\right)$$

$$\tan\theta = \frac{bx+K}{a}$$

$$bx+K = a\tan\theta$$

$$(bx+K)^2+a^2 = (a\tan\theta)^2+a^2 = a^2\sec^2\theta$$

$$d(bx+K) = d(a\tan\theta)$$

$$b\,dx = a\sec^2\theta\,d\theta$$

$$dx = \frac{a}{b}\sec^2\theta\,d\theta$$

$$= \int \frac{\frac{a}{b}\sec^2\theta}{a^2\sec^2\theta}\,d\theta$$

$$= \int \frac{1}{ab}\,d\theta$$

$$= \frac{1}{ab}\theta + c$$

$$= \frac{1}{ab}\arctan\left(\frac{bx+K}{a}\right) + c$$

Memorize

$$\int \frac{1}{(bx+K)^2+a^2}\,dx = \frac{1}{ab}\arctan\frac{bx+k}{a}$$

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{x^2+3^2}\,dx = \frac{1}{1\cdot 3}\arctan\frac{x}{3}$$

$$\int \frac{1}{5x^2+2}\,dx = \int \frac{1}{(\sqrt{5}\,x)^2+(\sqrt{2})^2} = \frac{1}{\sqrt{5}\cdot\sqrt{2}}\arctan\frac{\sqrt{5}\,x}{\sqrt{2}}$$

7.4

Integrating Rational Functions.

A rational function is a $\frac{\text{polynomial}}{\text{polynomial}}$

Examples of rational functions:

$$\frac{x^3-x^2+4x+2 \leftarrow 3}{x^5+1 \leftarrow 5}$$

$$\frac{1 \leftarrow 0}{x \leftarrow 1}$$

$$\frac{x \leftarrow 1}{x^4+4x+1 \leftarrow 4}$$

Proper Rational functions

degree of numerator < degree of denominator

$$\frac{4 \rightarrow x^4-3x^3+x-7}{2 \rightarrow x^2+2x+1}$$

$$\frac{2 \rightarrow x^2+2x+5}{2 \rightarrow x^2-x+1}$$

$$\frac{x-3 \leftarrow 1}{x+2 \leftarrow 1}$$

Improper Rational function

degree of numerator $\geq$ degree of denominator

Any improper rational function can and should be rewritten as a sum of a polynomial and a proper rational function using long division

4 →

2 →

$$\overset{\text{Improper}}{\frac{x^4-3x^3+x-7}{x^2+2x+1}} = \overbrace{x^2-5x+9}^{\text{Polynomial}} + \frac{-12x-16}{x^2+2x+1}$$

$$\begin{array}{r l}
 & x^2-5x+9 \quad r \; -12x-16 \\
x^2+2x+1 \,\big) & x^4-3x^3+0x^2+x-7 \\
 & -(x^4+2x^3+x^2) \\ \hline
 & -5x^3-x^2+x-7 \\
 & -(-5x^3-10x^2-5x) \\ \hline
 & 9x^2+6x-7 \\
 & -(9x^2+18x+9) \\ \hline
 & \boxed{-12x-16}
\end{array}$$

When integrating a rational function, <u>before anything else</u>, check to see if the integrand is a proper rational function. If it is not, then before anything else, rewrite the integrand as the sum of a polynomial with a <u>proper</u> rational function

All the techniques we are about to learn assume we are dealing with a proper rational function and fail otherwise.

Example:

2 →

2 ↗

$$\int \overset{\text{improper}}{\frac{x^2-1}{x^2+1}}\,dx = \int\left(1+\frac{-2}{x^2+1}\right)dx$$

$$= \boxed{x-2\arctan x + C}$$

$$\begin{array}{r l}
 & 1 \\
x^2+1 \,\big) & x^2-1 \\
 & -(x^2+1) \\ \hline
 & -2
\end{array}$$

$$\frac{x^2-1}{x^2+1} = \frac{x^2+1-2}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{2}{x^2+1} = 1 - \frac{2}{x^2+1}$$

$$\int \frac{dx+e}{ax^2+bx+c}$$

ax^2+bx+c can be factored (over real numbers)

Partial Fraction

ax^2+bx+c can not be factored (over real numbers)

Completing the square, get ln and arctan