Math 1b (8:30AM) 13 Feb 2020

Midterm 3 will cover 7.1–7.3

$$\frac{\text{polynomial}}{\text{Quadratic } ax^2+bx+c}$$

if improper use long division to make it proper

$$\frac{dx+e}{ax^2+bx+c}$$

ax^2+bx+c is irreducible over the real numbers

$b^2-4ac<0$

ax^2+bx+c can be factored

$b^2-4ac \geq 0$

$$\frac{dx+e}{ax^2+bx+c}$$

ax^2+bx+c is irreducible over the real numbers

$$b^2-4ac<0$$

in this case, complete the square

Example $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+10}\,dx$

$$\begin{array}{r|l} & 1 \\ 4x^2-4x+10 & 4x^2-3x+2 \\ & -(4x^2-4x+10) \\ \hline & x-8 \end{array}$$

$$\int \frac{4x^2-3x+2}{4x^2-4x+10}\,dx = \int 1\,dx + \int \frac{x-8}{4x^2-4x+10}\,dx$$

$$= x + ?$$

$$\int \frac{x-8}{4x^2-4x+10}\,dx$$

b^2-4ac

$(-4)^2-4(4)(10)$

<0

complete the square

$$4x^2-4x+10 = 4(x^2-x)+10$$

$$= 4\Big(x^2-x+\tfrac{1}{4}-\tfrac{1}{4}\Big)+10$$

$$= 4\underbrace{\Big(x^2-x+\tfrac{1}{4}\Big)}-1+10$$

$$= 4\Big(x-\tfrac{1}{2}\Big)^2+9 = \Big(2\Big(x-\tfrac{1}{2}\Big)\Big)^2+9$$

$$\int \frac{x-8}{4x^2-4x+10}\,dx$$

$b^2 - 4ac$

$(-4)^2 - 4(4)(10)$

< 0

complete the square

$$4x^2-4x+10 = 4(x^2-x)+10$$

$$= 4\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) + 10 \quad \left(-\frac{1}{2}\right) \to \left(\frac{1}{2}\right)^2$$

$$= 4\underbrace{\left(x^2 - x + \frac{1}{4}\right)} - 1 + 10$$

$$= 4\left(x-\frac{1}{2}\right)^2 + 9 = \left(2\left(x-\frac{1}{2}\right)\right)^2 + 9$$

$$= (2x-1)^2 + 3^2$$

$$\int \frac{x-8}{4x^2-4x+10}\,dx = \int \frac{x-8}{(2x-1)^2+3^2}\,dx$$

$x = \frac{u+1}{2}$

$2x = u+1$

$u = 2x-1$

$(2x-1)^2+3^2 = u^2+3^2$

$du = 2dx$

$dx = \frac{1}{2}du$

$$= \int \frac{\frac{u+1}{2} - 8}{u^2+3^2}\,\frac{1}{2}du \qquad \frac{1}{2}u + \frac{1}{2} - 8$$

$$= \int \frac{\frac{1}{2}u - \frac{15}{2}}{u^2+3^2}\,\frac{1}{2}du$$

Memorize

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}}$$

$$= \frac{1}{4}\int \frac{u}{u^2+3^2} + \frac{-15}{4}\int \frac{1}{u^2+3^2}\,du$$

$$= \frac{1}{4\cdot 2}\int \frac{2u}{u^2+3^2}\,du \qquad \int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + C$$

$$= \frac{1}{8}\ln(u^2+3^2) - \frac{15}{4}\,\frac{1}{3}\arctan\frac{u}{3}$$

$$= \boxed{\frac{1}{8}\ln(4x^2-4x+10) - \frac{5}{4}\arctan\frac{2x-1}{3} + C}$$

$$\frac{1}{8}\ln(4x^2-4x+10)-\frac{5}{4}\arctan\frac{2x-1}{3}+C$$

$$\int\frac{4x^2-3x+2}{4x^2-4x+10}\,dx=\int 1\,dx+\int\frac{x-8}{4x^2-4x+10}\,dx$$

$$=x+\frac{1}{8}\ln(4x^2-4x+10)-\frac{5}{4}\arctan\frac{2x-1}{3}+C$$

$$\int \frac{2x+5}{x^2-x-6}\,dx$$

x^2-x-6

b^2-4ac

$(-1)^2-4(1)(-6)$

$1+24=25>0$

reducible

$x^2-x-6=(x-3)(x+2)$

$$\int \frac{2x+5}{(x-3)(x+2)}\,dx$$

Use the method of Partial Fractions

$$(x-3)(x+2)\left(\frac{2x+5}{(x-3)(x+2)}\right)=\left(\frac{A}{x-3}+\frac{B}{x+2}\right)(x-3)(x+2)$$

Solve for A and B

$$\boxed{2x+5=A(x+2)+B(x-3)}$$

shortcut

$x=-2$

$2(-2)+5=A(0)+B(-5)$

$B=-1/5$

$2x+5=Ax+2A+Bx-3B$

$x=3$

$2x+5=(A+B)x+(2A-3B)$

$11=A5$, $A=\frac{11}{5}$

$\frac{10}{5}$

$A+B=2$ $\quad$ $2A-3B=5$

$A+B=2$

$\times 3$

$3A+3B=6$

$5A=11$

$A=\frac{11}{5}, \quad B=\frac{-1}{5}$

$$\frac{2x+5}{(x-3)(x+2)}=\frac{11/5}{x-3}+\frac{-1/5}{x+2}$$

check

$$=\frac{\frac{11}{5}(x+2)-\frac{1}{5}(x-3)}{(x-3)(x+2)}=\frac{2x+5}{(x-3)(x+2)}$$ ✓

$$\int \frac{2x+5}{x^2-x-6}\,dx=\frac{11}{5}\ln(x-3)-\frac{1}{5}\ln|x+2|+C$$

Theorem

We can always rewrite a proper rational function as a sum of simpler partial fractions, using the template I'm about to give.

$$\frac{\text{blah blah blah}}{(2x-3)^3(5x+4)\,x^2\,(x^2+4)(2x^2+5)} = \boxed{\frac{A}{2x-3} + \frac{B}{(2x-3)^2} + \frac{C}{(2x-3)^3} + \frac{D}{5x+4} + \frac{E}{x} + \frac{F}{x^2} + \frac{Gx+H}{x^2+4} + \frac{Ix+J}{2x^2+5}}$$

linear factors

irreducible quadratic factors