Math 1b (8:30AM) 24 Feb 2020

5.3 The Fundamental Theorem of Calculus (FTC) I

Suppose f is a continuous function on an interval

a is a point in the interval

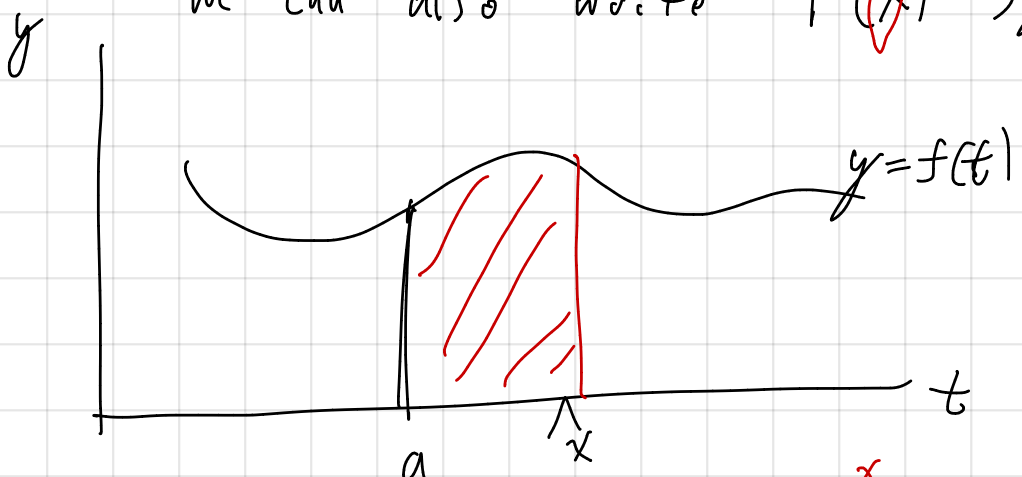
then if we define

$$F(x) = \int_a^x f(t)\,dt$$

then

$$F(a) = 0, \quad F'(x) = f(x)$$

we can also write $F(x) = \int_a^x f(x)\,dx$

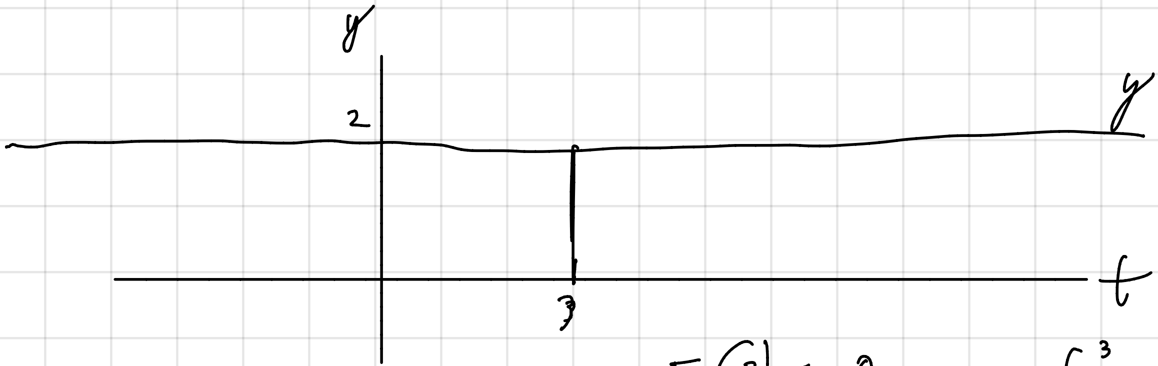


$$F(x) = \int_a^x f(t)\,dt$$

$F(x)$ is intuitively the accumulated area under $y=f(t)$ from $t=a$ to $t=x$.

Define $f(t) = 2$, $F(x) = \int_3^x \underbrace{2}_{f(t)}\,dt$

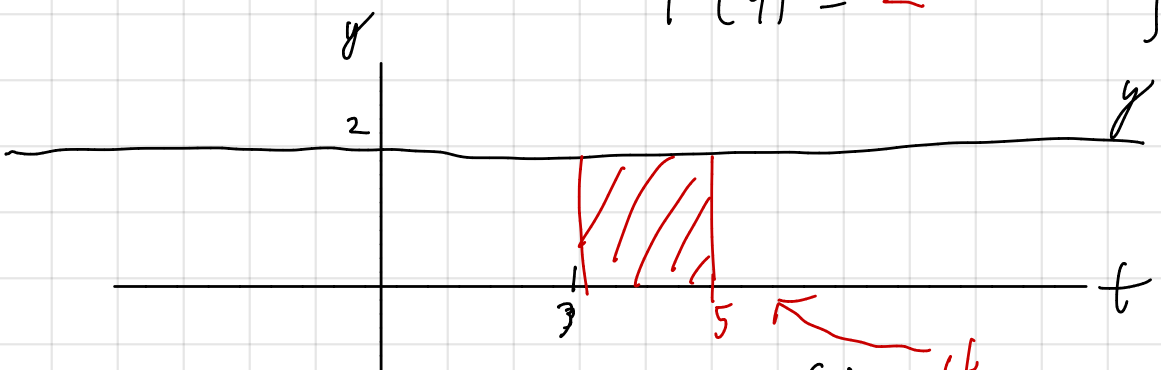
Define $f(t)=2$, $F(x)=\int_3^x \overset{f(t)}{2}\,dt$



$F(3)=0$ $\int_3^3 2\,dt=0$

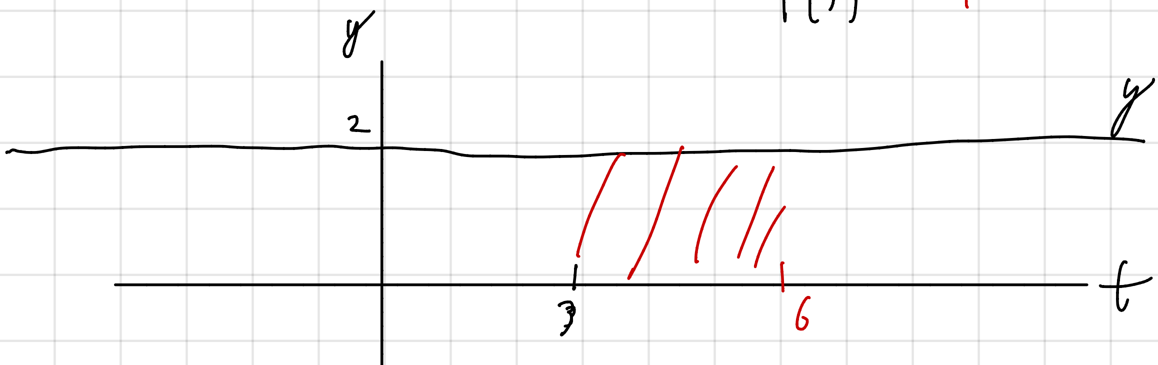
$F(4)=2$

$\int_3^4 2\,dt=2$



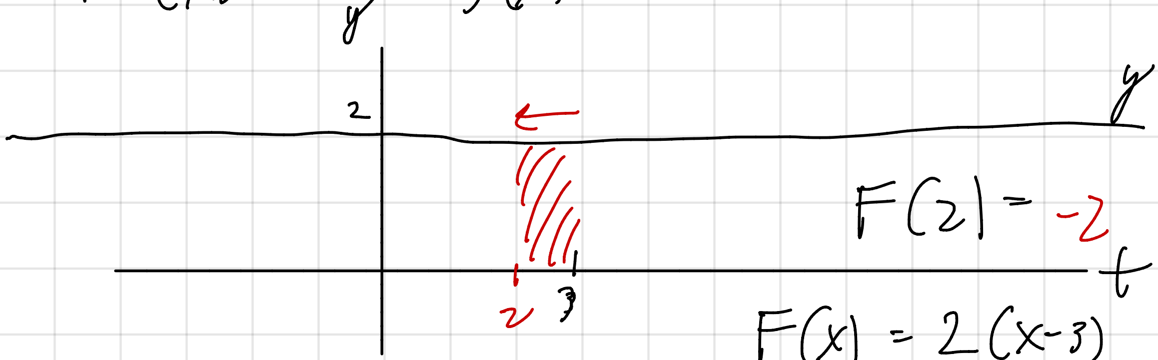
$F(5)=4$

$\int_3^5 2\,dt=$



$F'(x)=2=f(x)$

$F(6)=6$



$F(2)=-2=\int_3^2 2\,dt$

$F(x)=2(x-3)$

Intuitive Proof of FTC I

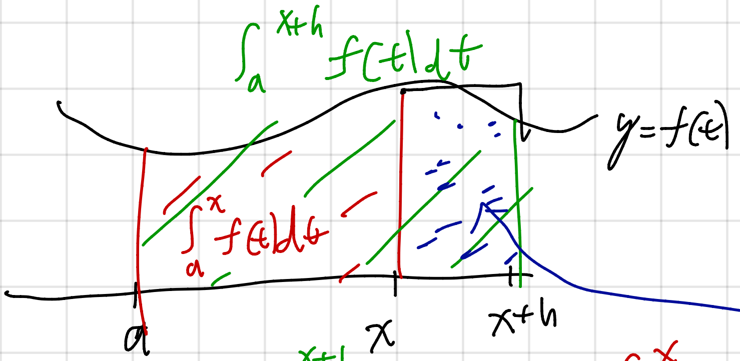
Let f be continuous on an interval that contains a, and let

$$F(x) = \int_a^x f(t)\,dt$$

(I need to show $F'(x) = f(x)$)

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$F'(x) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt + \int_x^a f(t)dt}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h}$$



$$\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

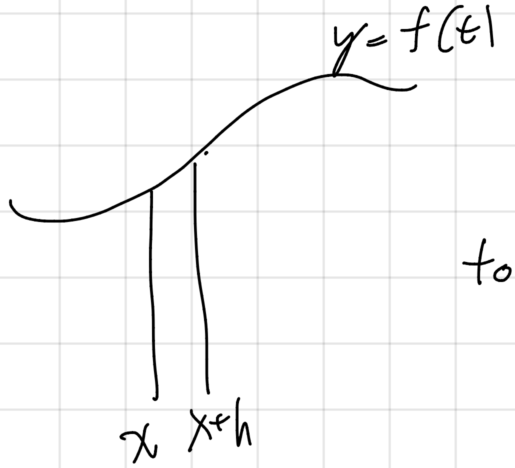
$$F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h}$$

Since f is continuous, when when t gets close to x, $f(t)$ gets close to $f(x)$

$$f(t) \approx f(x)$$

$$\int_{\cancel{a}\,x}^{x+h} f(t)dt \approx f(x)\,h$$

therefore $\dfrac{\int_x^{x+h} f(t)dt}{h} \to f(x)$



More formal argument

to show $$\underbrace{\lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}}_{F'(x)} = f(x)$$

Let M be the maximum value of $f(t)$ on the interval $[x, x+h]$ (assume $h>0$)

Let m be the minimum value of $f(t)$ on the same interval

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$

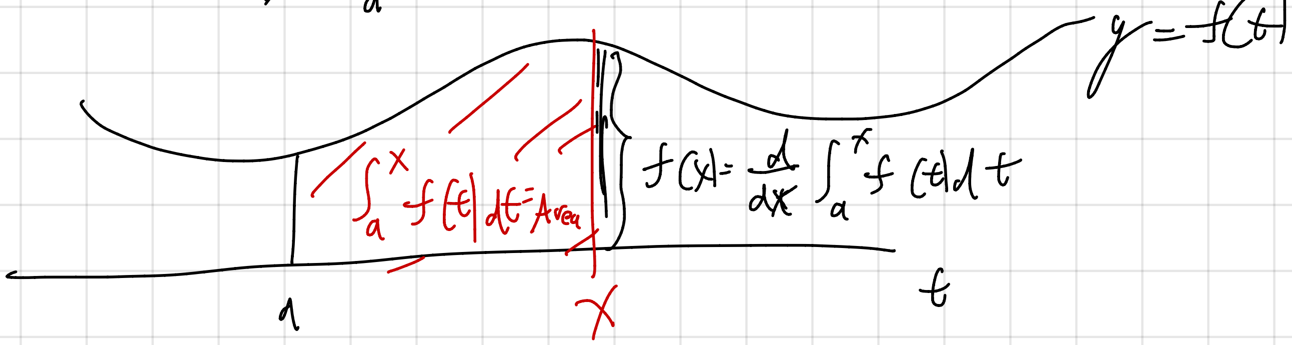
$$m \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le M$$

As $h \to 0$, $m \to f(x)$ and $M \to f(x)$ (because f is continuous)

so by the squeeze theorem, as $h \to 0$

$$\frac{\int_x^{x+h} f(t)\,dt}{h} \to f(x)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



$$\frac{d}{dx}\int_{7}^{x}\frac{\sin\sqrt[3]{1+t^2}}{\cos t+e^t}\,dt$$

$$=\frac{\sin\sqrt[3]{1+x^2}}{\cos x+e^x}$$

$$\frac{d}{dx}\int_{5}^{x}\frac{\sin\sqrt[3]{1+t^2}}{\cos t+e^t}\,dt=\frac{d}{dx}\left[\int_{5}^{7}\frac{\sin\sqrt[3]{1+t^2}}{\cos t+e^t}+\int_{7}^{x}\frac{\sin\sqrt[3]{1+t^2}}{\cos t+e^t}\right]$$

constant

$$=0+\frac{d}{dx}\int_{7}^{x}\frac{\sin\sqrt[3]{1+t^2}}{\cos t+e^t}$$

$$\frac{d}{dx}\int_{3}^{x^2}\sqrt{1+t^3}\,dt$$

Let $F(x)=\int_{3}^{x}\sqrt{1+t^3}\,dt$

$F(x^2)=\int_{3}^{x^2}\sqrt{1+t^3}\,dt$

$$\frac{d}{dx}F(x)=F'(x)=\sqrt{1+x^3}$$

$$\frac{d}{dx}\int_{3}^{x^2}\sqrt{1+t^3}\,dt=\frac{d}{dx}F(x^2)=F'(x^2)\cdot 2x$$

$$=\sqrt{1+(x^2)^3}\;2x$$

$$\frac{d}{dx}\int_{3}^{x^2}\sqrt{1+t^3}\,dt=\frac{dx^2}{dx}\frac{d}{dx^2}\int_{3}^{x^2}\sqrt{1+t^3}=2x\cdot\sqrt{1+(x^2)^3}$$

Find an antiderivative $F(x)$ of e^{-x^2} satisfying $F(0) = 0$.

$$\int e^{-x^2}\,dx =$$ This is impossible to integrate using the techniques of 7.1–7.4 of the text.

Instead we use FTC I.

$$\boxed{F(x) = \int_0^x e^{-t^2}\,dt}$$

$$F(0) = \int_0^0 e^{-t^2}\,dt = 0 \quad \checkmark$$

$$F'(x) = e^{-x^2} \quad \checkmark$$

We can now use a computer that is able to calculate approximations of definite integrals to compute $F(x)$ like any other function.

Find a function $F(x)$ satisfying

$$F(3) = 7, \quad F'(x) = 2x$$

use FTC I to do this.

$$F(3) = 7 + 0 = 7$$

$$F(x) = 7 + \int_3^x 2t\,dt$$

$$F'(x) = 0 + 2x = 2x$$

$$F(x) = 7 + \int_3^x 2t\,dt = 7 + \left[t^2\right]_3^x$$

$$= 7 + x^2 - 9 \qquad \underline{F(x) = x^2 - 2}$$

$$F(3) = 3^2 - 2 = 7 \checkmark$$

$$F'(x) = 2x \checkmark$$