Math 1b (8:30AM) 26 Feb 2020

Type I improper integrals (integrating over a horizontal asymptote as $x \to \infty$ or $x \to -\infty$)

Type II improper integrals (integrating over a vertical asymptote as $t \to a^-$ or $t \to a^+$!)

Example of type I

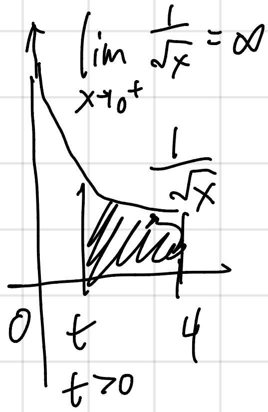
$$\int_4^\infty \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty} \int_4^t \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to\infty} \left[2x^{\frac{1}{2}}\right]_4^t = \lim_{t\to\infty} \underbrace{2\sqrt{t}}_{\to\infty} - \underbrace{2\sqrt{4}}_{=4}$$

$$= \infty$$

$\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (to ∞)



~~Type II~~

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \int_t^4 \frac{1}{\sqrt{x}}\,dx$$

$$\overset{?}{=} \lim_{t\to 0^+} \left[2x^{\frac{1}{2}}\right]_t^4$$

$$= \lim_{t\to 0^+} \underbrace{2\sqrt{4}}_{4} - \underbrace{2\sqrt{t}}_{\to 0}$$

$$= 4$$

$\int_0^4 \frac{1}{\sqrt{x}}\,dx$ converges

$$\int_0^9 \frac{1}{\sqrt{x}}\,dx = 4$$

$p = \frac{1}{2} < 1$

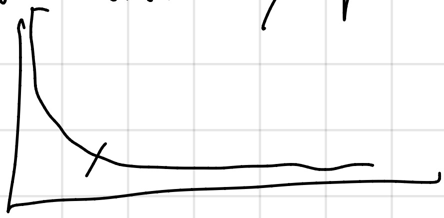
$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges but $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

Yesterday we established that — Memorize

$$\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p < 1 \\ \text{diverges if } p \geq 1 \end{cases} \qquad \int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges if } p \leq 1 \\ \text{converges if } p > 1 \end{cases}$$

$$\int_0^\infty \frac{1}{x^2}\,dx$$

has both a vertical and a horizontal asymptote



Axiom If $a < b < c$ (a might be $-\infty$ and c might be ∞) then

If $\int_a^b f(x)\,dx$ converges and $\int_b^c f(x)\,dx$ converges

then $\int_a^c f(x)\,dx$ converges

and $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

If $\int_a^b f(x)\,dx$ diverges or $\int_b^c f(x)\,dx$ diverges

then $\int_a^c f(x)\,dx$ diverges

If both of these converge, $\int_a^c f(x)\,dx$ converges

If even one of these diverges, so does $\int_a^c f(x)\,dx$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

If both of these converge, $\int_a^c f(x)\,dx$ converges

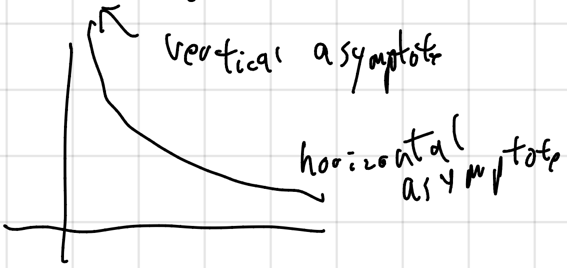
If even one of these diverges, so does $\int_a^c f(x)\,dx$

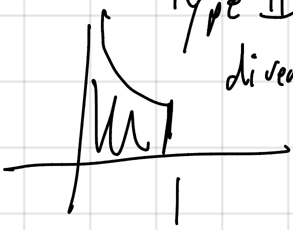
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

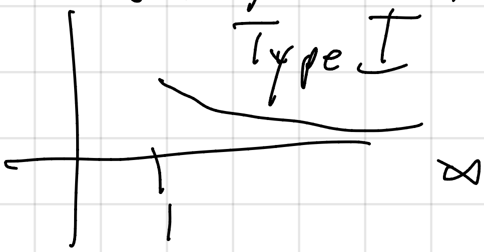
$$\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^\infty \frac{1}{x^2}\,dx$$

Type II diverges

converges (to 1) Type I

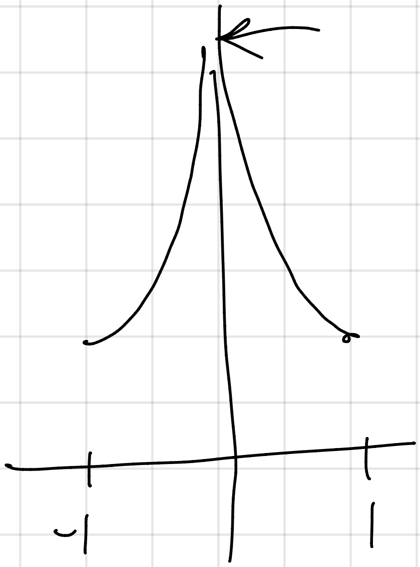






$\int_0^\infty \frac{1}{x^2}\,dx$ diverges because $\int_0^1 \frac{1}{x^2}\,dx$ diverges.

"Diverge" means <u>does not converge.</u>



$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

$F(x) = \frac{-1}{x}$ $f(x) = \frac{1}{x^2}$

$F'(x) = f(x)$ except at $x=0$

WRONG ANSWER

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -2$$

FTC $\int_a^b f(x)\,dx = F(b) - F(a)$

provided that $F'(x) = f(x)$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \underbrace{\int_{-1}^{0} \frac{1}{x^2}\,dx}_{x \to 0^-} + \underbrace{\int_{0}^{1} \frac{1}{x^2}\,dx}_{x \to 0^+}$$

Type II Type II

diverges

$\int_{-1}^{1} \frac{1}{x^2}\,dx$ diverges because

$\int_{0}^{1} \frac{1}{x^2}\,dx$ diverges

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

two horizontal asymptotes, $x \to -\infty$, $x \to \infty$

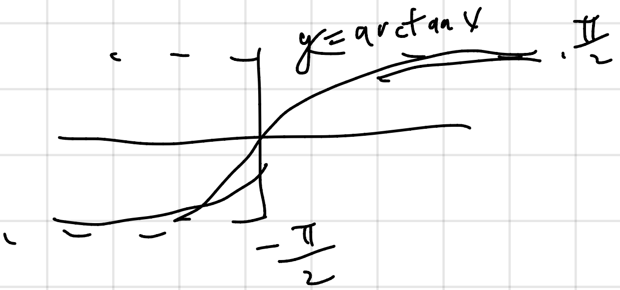
$1+x^2$ is never zero
No vertical asymptotes

(I could have written $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = \int_{-\infty}^{42} \frac{1}{1+x^2}dx + \int_{42}^{\infty} \frac{1}{1+x^2}dx$)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t \to -\infty} \Big[\arctan x\Big]_t^0$$

$$= \lim_{t \to -\infty} = \arctan(0) - \arctan(t) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$



$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \ldots = \lim_{t \to \infty} \arctan t - \arctan 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges

It turns out often that we are more interested in whether an integral converges or diverges than what the integral converges to.

If the integral of a function over an interval converges, the integral of that function over any sub-interval also converges.

$$\int_1^{\infty} \frac{1}{x^3}\,dx \text{ converges,}$$

Therefore $\int_7^{\infty} \frac{1}{x^3}\,dx$ also converges

so does $\int_{100}^{\infty} \frac{1}{x^3}\,dx$

If the integral over a sub-interval diverges so does the integral over the interval.

$$\int_0^1 \frac{1}{x^2}\,dx \text{ diverges}$$

Therefore

$$\int_0^{10} \frac{1}{x^2}\,dx \text{ diverges.}$$

If the integral over a sub-interval diverges so does the integral over the interval.

$$\int_0^1 \frac{1}{x^2}\,dx \text{ diverges}$$

Therefore

$$\int_0^{10} \frac{1}{x^2}\,dx \text{ diverges.}$$

If $a < b < c$ then

$\int_a^c f(x)\,dx$ converges if and only if

$\int_a^b f(x)\,dx$ converges and $\int_b^c f(x)\,dx$ converges

$\int_0^1 \frac{1}{x^2}\,dx$ diverges, $\int_{\frac{1}{2}}^1 \frac{1}{x^2}\,dx$ converges

Therefore $\int_0^{\frac{1}{2}} \frac{1}{x^2}\,dx$ diverges

If $p \geq 1$

$\int_0^1 \frac{1}{x^p}\,dx$ diverges, $\int_0^{0.1} \frac{1}{x^p}\,dx$ diverges

$\int_0^{100} \frac{1}{x^p}\,dx$ diverges

If $p < 1$ $\int_0^1 \frac{1}{x^p}\,dx$ converges, $\int_0^{0.1} \frac{1}{x^p}\,dx$ converges

$\int_0^{100} \frac{1}{x^p}\,dx$ converges

If $p > 1$ then $\int_1^{\infty} \frac{1}{x^p}\,dx$ Converges

$\int_{100}^{\infty} \frac{1}{x^p}\,dx$ converge

$\int_{\frac{1}{10}}^{\infty} \frac{1}{x^p}\,dx$ converges

If K is a constant, $K \neq 0$, then

$\int_a^b f(x)\,dx$ converges if and only if $\int_a^b kf(x)\,dx$ converges

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

$\int_1^{\infty} \frac{1}{x^2}\,dx$ Converges

so $\int_{1000}^{\infty} \frac{17}{x^2}\,dx$ converges

$\int_{0.1}^{\infty} \frac{-2}{x^2}\,dx$ converges