Math 1B (8:30 AM) 19 March 2020

(1) A random variable X has a ~~CDF~~ (cumulative distribution fucntion)

$$F(x) = \begin{cases} 0 & x \le 1 \\ 1 - e^{1-x} & x > 1 \end{cases}$$

$1 - e^{1-1} = 1 - 1 = 0$

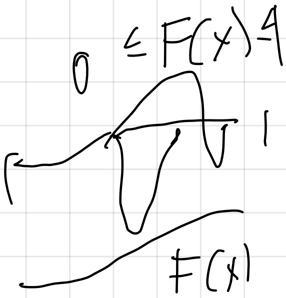
(a) Verify that $F(x)$ is a well defined CDF.

(b) Find the PDF (probability distribution funciton) for X

(c) Find $P(2 < X < 3)$

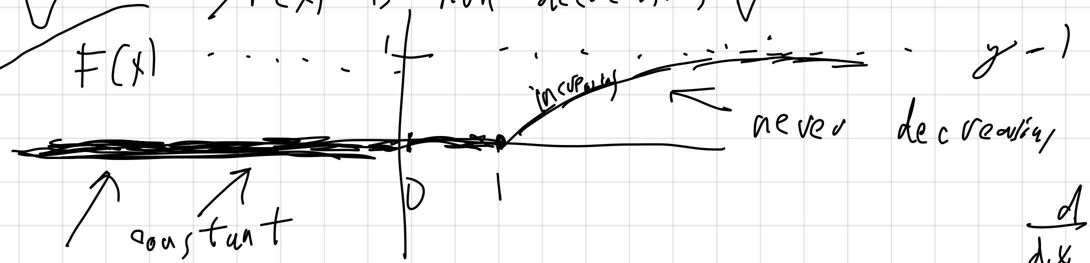
(a) > $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} 0 = 0$ ✓ $= 0$

$0 \le F(x) \le 1$



> $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 - \underbrace{e^{1-x}}_{0} = 1 - 0 = 1$ ✓ $= 1$

> $F(x)$ is non-decreasing ✓



$\frac{d}{dx}(1 - e^{1-x}) = e^{1-x}$

(b) $f(x) = F'(x) = \begin{cases} 0 & , x \le 1 \\ e^{1-x} & , x > 1 \end{cases}$

$$\boxed{\begin{array}{l} \text{CDF} \to \text{PDF} \\ f(x) = F'(x) \\ \text{PDF} \to \text{CDF} \\ F(x) = \int_{-\infty}^{x} f(x)\,dx \end{array}}$$

(c) $P(2 < X < 3)$

method #1 CDF

$P(2 < X < 3) = F(3) - F(2)$

$= (1 - e^{1-3}) - (1 - e^{1-2}) = e^{-1} - e^{-2}$

PDF

$P(a \le X \le b) = \int_a^b f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

for continuous random variable

method #2 PDF

$P(2 < X < 3) = \int_2^3 f(x)\,dx$

$= \int_2^3 e^{1-x}\,dx = -e^{1-x}\Big|_2^3 = -e^{1-2} + e^{1-3} = e^{-1} - e^{-2}$

(2) A random variable X has a PDF (probability distribution function)

$$f(x) = \begin{cases} \frac{k}{x^3} & x \leq -1 \\ 0 & x > -1 \end{cases}$$ $\int_{-1}^{\infty} f(x)dx = 0$

where k is a constant.

(a) Find the value k must have for $f(x)$ to be a valid PDF.

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{-\infty}^{-1} \frac{k}{x^3} dx = 1$$

$$\lim_{t \to -\infty} \int_{t}^{-1} \frac{k}{x^3} dx = 1$$

$$\lim_{t \to -\infty} \left. \frac{-k}{2x^2} \right|_{t}^{-1} = 1$$

$$= \lim_{t \to -\infty} \frac{-k}{2 \cdot 1^2} + \frac{k}{2t^2} = 1$$

$\frac{k}{2t^2} \to 0$

$$\frac{-k}{2} = 1$$

$$k = -2$$

$f(x) \geq 0$ for all x

$$\boxed{f(x) = \begin{cases} \frac{-2}{x^3}, & x \leq -1 \\ 0, & x > -1 \end{cases}}$$

(b) Find the CDF (cumulative distribution function) for X.

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

If $x \geq -1$, then $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{x} f(x)dx$

$$1 + \int_{-1}^{x} 0\,dx = 1$$

If $x < -1$, then $F(x) = \int_{-\infty}^{x} f(x)dx = \lim_{t \to -\infty} \int_{t}^{x} \frac{-2}{x^3} dx$

$$= \lim_{t \to -\infty} \left. \frac{+2}{2x^2} \right|_{t}^{x} = \lim_{t \to -\infty} \left(\frac{1}{x^2} - \frac{1}{t^2} \right) = \boxed{\frac{1}{x^2}}$$

$$F(x) = \begin{cases} \frac{1}{x^2}, & x < -1 \\ 1, & x \geq -1 \end{cases}$$

$F(x)$ $y=1$ -1

$\frac{d}{dx} \frac{1}{x^2} = \frac{-2}{x^3}$

$$f(x) = \begin{cases} \frac{-2}{x^3}, & x \leq -1 \\ 0, & x > -1 \end{cases}$$

$\frac{d}{dx} 1 = 0$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-1}{2x^3}$$

(c) Find $P(X > -2)$

(d) Find the expected value of X.

(e) Find the median value of X (the value m that satisfies $P(X \leq m) = 0.5$

(c) $P(X > -2)$

method #1 CDF:

$$P(X > -2) = 1 - P(X \le -2)$$
$$= 1 - F(-2) = 1 - \frac{1}{(-2)^2} = \frac{3}{4} = 75\%$$

method #2 CDF

$$P(X > -2)$$
$$= \int_{-2}^{\infty} f(x)\,dx$$
$$= \int_{-2}^{-1} \frac{-2}{x^3}\,dx$$
$$= \frac{1}{x^2}\Big|_{-2}^{-1}$$
$$= \frac{1}{(-1)^2} - \frac{1}{(-2)^2} = \frac{3}{4} = 75\%$$

$$f(x) = \begin{cases} \frac{-2}{x^3}, & x \le -1 \\ 0, & x > -1 \end{cases}$$

(d) Find the expected value of X.

(e) Find the median value of X (the value m that satisfies $P(X \le m) = 0.5$

(d) $\cancel{\int_{-\infty}^{\infty} x f(x)\,dx} = \int_{-\infty}^{-1} x \cdot \frac{-2}{x^3}\,dx$

$$= \lim_{t \to -\infty} \int_{t}^{-1} \frac{-2}{x^2}\,dx = \lim_{t \to -\infty} \left[\frac{2}{x}\right]_t^{-1} = \lim_{t \to -\infty} \frac{2}{-1} - \frac{2}{t} = -2$$

(e) $P(X \le m) = 0.5$

method #1 CDF

$$\underbrace{P(X \le m)}_{F(m)} = 0.5$$
$$F(m) = 0.5$$
$$\frac{1}{m^2} = 0.5$$
$$m^2 = 2$$
$$m = \pm\sqrt{2}$$

$m = -\sqrt{2}$

$$F(x) = \begin{cases} 1/x^2, & x \le -1 \\ 1, & x > -1 \end{cases}$$

$\cancel{F(\sqrt{2}) = 1 \text{ not } .5}$

$F(-\sqrt{2}) = \frac{1}{2}$ ✓

method #2 PDF

$$\int_{-\infty}^{m} f(x)\,dx$$

$m = -\sqrt{2}$