Questions/Thoughts – Teaching Fractions in Arithmetic and Rational Expressions in Algebra
Study the examples below and notice the difference in the rules/procedures required to solve the different types of fraction problems.

Questions:

1. Recent information regarding Reading and Math test scores across the country have continued the trend 0f the past several years, students' scores in both areas continue to go down. If you are the person in charge of math education for the country, the state, or your school what would you do to change this trend?
2. Since our area of education is mathematics, why do you think this national trend regarding math scores seems to be occurring?
3. Do elementary arithmetic teachers **focus on the same rules/procedures** to teach students how to simplify/solve fraction problems the same way that secondary math teachers teach students to simplify/solve rational expressions in algebra???
4. What rules/procedures are emphasized in arithmetic to work with fractions in elementary school and what rules/procedures are emphasized to work with algebra rational expressions one year later or 4 years later?

A. What are the present teaching methods to teach students to work with fractions?

1. Focus your thoughts on what you believe are the present methods teachers use to teach students to solve more challenging fractions in arithmetic? Let's start by looking at adding and subtracting fraction problems in arithmetic. For years the method used to teach elementary students to simplify/solve all types of fraction problems involving reducing, adding, subtracting, comparing, multiplying, and dividing slightly challenging fraction problems, relies so much on the students' mental math skills regarding the multiplication and division facts for the numbers from 0-10. I am not talking about fraction problems such as - "$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$" or even "$\frac{5}{9}+\frac{7}{9}-\frac{1}{9}=\frac{11}{9}=1\frac{2}{9}$", but I am talking about more challenging fractions. What about these problems, are they simple problems or more challenging addition and subtraction problems, "$\frac{5}{6}-\frac{3}{8}=\frac{20}{24}-\frac{9}{24}=\frac{11}{24}$" or "$\frac{7}{10}-\frac{1}{4}+\frac{3}{5}=\frac{14}{20}-\frac{5}{20}+\frac{12}{20}=\frac{21}{20}=1\frac{1}{20}$". To simplify the first pair of problems there was no need to use any multiplication and division facts but in the second pair of problems, multiplication facts were used several times. The first time in each of the second pair of fraction problems, students had to mentally find a least common denominator based on their mental math knowledge regarding multiplication and division facts. Then once the least common denominator was mentally determined, the students mentally changed each of the given fractions into equivalent fractions with the common denominator. The last step was to add and/subtract the numerators of the fractions with the common denominator.
2. Now, let's consider how students are taught to solve/simplify fraction problems in arithmetic involving reducing, multiplying, and dividing fractions. Again, the main procedures taught in elementary school when solving/simplifying these types of operations, requires students to have a strong understanding of the multiplication and division facts. Consider the following simple

problems and how students are taught to simplify these problems "$\frac{10}{15} = \frac{2}{3}$" or "$\frac{8}{16} = \frac{1}{2}$". Slightly more challenging problems to reduce but again students are taught to use their multiplication and division fact knowledge "$\frac{22}{36} = \frac{11}{18}$" or $\frac{55}{60} = \frac{11}{12}$". The same procedures are required when students are taught to simplify/solve multiplying and division (once the division problems are converted into a fraction multiplication problems) fraction problems. Think about how a strong knowledge of the multiplication and division facts are necessary to find the final simplified solution for these simple and slightly more challenging problems: "$two\ simple\ ex.\ \frac{4}{15} \times \frac{5}{8} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$" or "$\frac{9}{16} \div \frac{27}{32} = \frac{9}{16} \times \frac{32}{27} = \frac{1}{1} \times \frac{2}{3} = \frac{2}{3}$" or $slightly\ more\ difficult\ ex.\ \frac{8}{15} \times \frac{2}{3} \times \frac{27}{32} = \frac{1}{5} \times \frac{2}{3} \times \frac{9}{4} = \frac{1}{5} \times \frac{1}{1} \times \frac{3}{2} = \frac{3}{10}$" or "$2\frac{3}{5} \div 1\frac{11}{15} = \frac{13}{5} \div \frac{26}{15} = \frac{13}{5} \times \frac{15}{26} = \frac{1}{1} \times \frac{3}{2} = 1\frac{1}{2}$". Again, the main teaching point to simplify these problems is a strong mental understanding of the multiplication and division facts.

B. Just a simple teaching suggestion: Research indicates that 2-3 months after students are taught all the rules to work with these sample problems over 50% of students get the rules confused regarding comparing, adding, and subtracting fractions versus reducing, multiplying and dividing fractions. Here is a simple step to add to all the example problems just indicated above to help students not confuse the different rules and procedure 2-3 months after they are taught how to solve fraction problems. Prior to computing the final answer, the students may write each problem as ONE BIG FRACTION. When adding and subtracting fractions, the one big fraction will be written after equivalent fractions are determined with the common denominator, but when multiplying fractions the one big fraction can be written early in the process.

1. Adding and subtracting fraction examples one big fraction. 1. "$\frac{3}{5} + \frac{1}{5} = \frac{\mathbf{3+1}}{\mathbf{5}} = \frac{4}{5}$"
   2. "$\frac{5}{9} + \frac{7}{9} - \frac{1}{9} = \frac{\mathbf{5+7-1}}{\mathbf{9}} = \frac{11}{9} = 1\frac{2}{9}$" 3. "$\frac{5}{6} - \frac{3}{8} = \frac{20}{24} - \frac{9}{24} = \frac{\mathbf{20-9}}{\mathbf{24}} = \frac{11}{24}$"
   4. "$\frac{7}{10} - \frac{1}{4} + \frac{3}{5} = \frac{14}{20} - \frac{5}{20} + \frac{12}{20} = \frac{\mathbf{14-5+12}}{\mathbf{20}} = \frac{21}{20} = 1\frac{1}{20}$
2. Multiplying and dividing fraction example problems using one big fraction.
   1. "$\frac{4}{15} \times \frac{5}{8} = \frac{4 \times 5}{15 \times 8} = \frac{1 \times 1}{3 \times 2} = \frac{1}{6}$" 2. "$\frac{9}{16} \div \frac{27}{32} = \frac{9}{16} \times \frac{32}{27} = \frac{9 \times 32}{16 \times 27} = \frac{1 \times 2}{1 \times 3} = \frac{2}{3}$"

C. When students are working with rational expressions whether they are comparing, adding, subtracting, reducing, multiplying, or dividing fractions, if the problem is even slightly challenging the first step in the process to determine all the solutions is to factor. If someone is comparing, adding, or subtracting rational expressions then the first step is usually to factor the denominators and find equivalent rational expressions with the least common denominator. One can only find the least common denominator after they analyze the factors in the denominators. If someone is reducing, multiplying, or dividing (after the expression is written as a multiplication problem) rational expressions, then the first step is to factor all the numerators and all the denominators. Once the numerators and denominators are factored and the problem is written as one big fraction, it is easy to determine identical factors in both the numerator and denominator and divide the identical factors which become ONE (1).

1. Reduce 2 basic rational expressions and then 2 challenging arithmetic fractions.
“$a.\ \frac{x^2-9}{x^2-8x+15} = \frac{(x+3)(x-3)}{(x-5)(x-3))} = \frac{(x+3)}{(x-5)} = \frac{x+3}{x-5}$” or “b. $\frac{x^3+2x^2-15x}{x^3+5x^2+10x} = \frac{x(x+5)(x-3)}{x((x+5)(x+2)} = \frac{(x-3)}{(x+2)} = \frac{x-3}{x+2}$"
$Use\ Factor\ Trees$ "$c.\ \frac{108}{288} = \frac{2\times2\times3\times3\times3}{2\times2\times2\times2\times2\times3\times3} = \frac{3}{2\times2\times2} = \frac{3}{8}$” or “$d.\ \frac{77}{363} = \frac{7\times11}{3\times11\times11} = \frac{7}{33}$”

2. Multiply and divide rational expressions and then 2 challenging arithmetic fractions.
“$a.\ \frac{x^2+6x+9}{x^2-16} \times \frac{x^2+12x+7}{x^2+8x+15} = \frac{(x+3)(x+3)}{(x-4)(x+4)} \times \frac{(x+4)(x+3)}{(x+5)(x+3)} = \frac{(x+3)(x+3)}{(x-4)(x+5)} = \frac{(x+3)^2}{(x-4)\_(x+5)}$” or
“$b.\ \frac{x^2+10x+25}{x^2-4} \times \frac{x^2+x-6}{x^2+8x+15} \div \frac{x^2-25}{x^2-3x-10} = \frac{x^2+10x+25}{x^2-4} \times \frac{x^2+x-6}{x^2+8x+15} \times \frac{x^2-3x-10}{x^2-25} =$
$\frac{(x+5)(x+5)}{(x-2)(x+2)} \times \frac{(x+3)(x-2)}{(x+3)(x+5)} \times \frac{(x-5)(x+2)}{(x-5)(x+5)} = \frac{(x+5)(x+5)\cdot(x+3)(x-2)\cdot(x-5)(x+2)}{(x-2)(x+2)\cdot(x+3)(x+5)\cdot(x-5)(x+5)} = 1$
$Use\ the\ Fundamental\ Theorem\ of\ Arithmetic\ and\ Factor\ Trees,$ $\boldsymbol{very\ important}$!
“$c.\ \frac{98}{105} \div \frac{56}{95} = \frac{98}{105} \times \frac{95}{56} = \frac{2\times7\times7\ \cdot\ 5\times19}{3\times5\times7\ \cdot\ 2\times2\times2\times7} = \frac{19}{2\times2\times3} = \frac{19}{12} = 1\frac{7}{12}$
“d. $\frac{42}{105} \times \frac{16}{81} \times \frac{60}{75} = \frac{2\times3\times7\ \cdot\ 2\times2\times2\ \cdot\ 2\times2\times3\times5}{3\times5\times7\ \cdot\ 3\times3\times3\times3\ \cdot\ 3\times5\times5} = \frac{2\times2\times2\times2\times2}{3\times3\times3\times3\times5\times5} = \frac{32}{2125}$ Two number sense questions to think about. In problems, “c” and “d”, what do you notice about the magnitudes of the answers compared with the first fraction in each problem? Factoring all the terms in these problems is the wisest way to simplify all these problems.

D. Finally, it is time to discuss how to teach students the rules and procedures to simplify/solve rational expressions and challenging arithmetic fraction problems involving addition and subtraction. Adding and subtracting fractions in arithmetic especially challenging fractions which are never even mentioned in arithmetic textbooks in the United States is the most difficult topic to teach in arithmetic and the topic which presents the most problems for many students. If students do not know the rules to add and subtract fractions in arithmetic maybe that could be the reason that adding and subtracting rational expressions is a very challenging topic for many students to master in algebra. Notice, the first thing one must concentrate on is factoring the denominators with the goal of finding the least common denominator. The second challenging step is to convert all the given fractions into equivalent fractions with the LCD.
LCD rules: Step 1 - list all the factors as a multiplication problem
Step 2 – determine if any of the individual factors ever occurs more than once in either set of factors and then always list the factor which might occur more than once in the initial multiplication problems in step 1 the most times that the factor occurs in anyone set of factors to find the LCD.
Step 3 – Set up a multiplication problem starting with the multiplication problem in step 1 with all the factors and now multiply by the factors which occurred more than once in one of the initial sets of factors.

1. Add and/or subtract these two rational expression problems and then the 2 challenging arithmetic fractions.
“$a.\ \frac{7x}{x^2+5x+6} + \frac{8x}{x^2+4x+4}$" Factor the denominators first. x^2+5x+6 x^2+4x+4
$(x+2)(x+3)$ $(x+2)(x+2)$
Step 1: Different factors multiplied: $(x+2)(x+3)$ Not Necessarily the LCD yet.
Step 2: The factor $(x+2)$, occurs the most in the second set of factors so in this case go back to step 1 and multiply the factors in step 1 by an additional factor of $(x+2)$.

Step 3 - LCD = $(x+2)(x+2)(x+3)$

Next step - divide the LCD by the denominators of the initial fractions, the quotient will represent what you must multiply the numerator by to get the equivalent fraction. Actually, you should multiply each of the original fractions by another fraction which equals ONE (1).

$\frac{LCD}{First\ Denominator} = \frac{(x+2)(x+2)(x+3)}{(x+2)(x+3)} = (x+2)\ \therefore multiply\ the\ first\ expression\ by\ \frac{(x+2)}{(x+2)} = 1$

$\frac{LCD}{Second\ Denominator} = \frac{(x+2)(x+2)(x+3)}{(x+2)(x+2)} = (x+3)\ \therefore\ multiply\ the\ second\ expression\ by\ \frac{(x+3)}{(x+3)} = 1$

$\frac{7x}{x^2+5x+6} + \frac{8x}{x^2+4x+4} = \frac{7x}{x^2+5x+6} \times \frac{(x+2)}{(x+2)} + \frac{8x}{x^2+4x+4} \times \frac{(x+3)}{(x+3)} = \frac{7x^2+14x\ +\ 8x^2+24x}{(x+2)(x+2)(x+3)} = \frac{15x^2+38x}{(x+2)(x+2)(x+3)} =$

$\frac{x(15x+38)}{(x+2)(x+2)(x+3)}$ Wow, now that was easy!

"b. $\frac{(x+1)}{x^2+6x+8} - \frac{(x-1)}{x^2-16}$" Denominator factors - $x^2+6x+8 = (x+2)(x+4)$ & $x^2-16 = (x+4)(x-4)$

Maybe the LCD Step 1 - $(x+2)(x+4)(x-4)$ Apply Step 2 - ∴ LCD $= (x+2)(x+4)(x-4)$

Divide to determine the value of the number ONE (1) to multiply each fraction by to find equivalent fractions.

$\frac{LCD}{1st\ denomin} = \frac{(x+2)(x+4)(x-4)}{(x+2)(x+4)} = (x-4)\ \therefore\ 1 = \frac{(x-4)}{(x-4)}$ & $\frac{LCD}{2nd\ denomin} = \frac{(x+2)(x+4)(x-4}{(x+4)(x-4)} = (x+2)\ \therefore\ 1 = \frac{(x+2)}{(x+2)}$

Equivalent fractions Subtracted -

$\frac{(x+1)}{x^2+6x+8} - \frac{(x-1)}{x^2-16} = \frac{(x+1)\ (x-4)}{(x^2+6x+8)(x-4)} - \frac{(x-1)\ (x+2)}{(x^2-16)(x+2)} = \frac{x^2-3x-4-(x^2+x-2)}{(x+2)(x+4)(x-4)} = \frac{-4x-2}{(x+2)(x+4)(x-4)} = \frac{-2(2x+1)}{(x+2)(x+4)(x-4)}$

2. Now, it is time to use the rules used in the previous 2 examples to teach arithmetic fraction addition and subtraction problems as an enrichment topic prior to students moving on, so students are prepared to solve algebra rational expressions.

"$c.\ \frac{5}{28} + \frac{9}{49}$" Denominator factors - $28 = 2 \times 2 \times 7$ & $49 = 7 \times 7$

Maybe the LCD Step 1 - 2×7 Apply Step 2 - $\therefore LCD = 2 \times 2 \times 7 \times 7 = 196$

Find values of "1" to multiply the original given fractions to find equivalent fractions to add.

$\frac{LCD}{1st\ Denominator} = \frac{196}{28} = 7 \therefore 1 = \frac{7}{7}$ & $\frac{LCD}{2nd\ Denominator} = \frac{196}{49} = 4\ \therefore 1 = \frac{4}{4}$

Add Equivalent Fractions - $\frac{5}{28} \times \frac{7}{7} + \frac{9}{49} \times \frac{4}{4} = \frac{35}{196} + \frac{36}{196} = \frac{71}{196}$ Can it reduce? NOT in this case!

"$d.\ \frac{3}{75} + 1 - \frac{7}{60}$" Denominator factors – $75 = 3 \times 5 \times 5$ & $60 = 2 \times 2 \times 3 \times 5$

Note: The denominator of 1 is "1" ∴ *irrelevant*

Maybe the LCD Step 1 - $2 \times 3 \times 5$ Apply Step 2 - $\therefore LCD = 2 \times 2 \times 3 \times 5 \times 5 = 300$

Find values of "1" to multiply the original given fractions and "1" to find equivalent fractions to "+" & " - ".

$\frac{LCD}{1st\ Den} = \frac{300}{75} = 4\ \therefore 1 = \frac{4}{4}$ $\frac{LCD}{Den\ of\ \#1} = \frac{300}{1} = 300\ \therefore 1 = \frac{300}{300}$ $\frac{LCD}{3rd\ den} = \frac{300}{60} = 5\ \therefore 1 = \frac{5}{5}$

Simplify Problem - $\frac{3}{75} + 1 - \frac{7}{60} = \frac{3}{75} \times \frac{4}{4} + \frac{1}{1} \times \frac{300}{300} - \frac{7}{60} \times \frac{4}{4} = \frac{12}{300} + \frac{300}{300} - \frac{28}{300} = \frac{284}{300}$

$\frac{284}{300}$ It will reduce ∴ use factor trees - $284 = 2 \times 2 \times 71$ & $300 = 2 \times 2 \times 3 \times 5 \times 5$

Final simplified solution - $\frac{284}{300} = \frac{2\times2\times71}{2\times2\times3\times5\times5} = \frac{71}{75}$

Explain to me why these rules/procedures should not be taught in arithmetic class starting in 5th or 6th grade but mastered before a student enters an algebra 1 class?