MAP Interpolation of an Ising Image Block

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Abstract. This paper considers the problem of finding the set of MAP reconstructions of an $N \times N$ block conditioned on a boundary configuration consisting of 1 or 2 alternating runs of black and white within a uniform Ising model with no external field. It shows that when the boundary contains a single run, the set of minimum odd bond reconstructions are described by *simple paths* connecting the endpoints of either the black or white run. When the boundary consists of 2 runs, the set of minimum odd bond reconstructions are formed in one or more of the following ways: by simple paths connecting the endpoints of the two black runs; by simple paths connecting the two white runs; or by three simple paths connecting one of the boundary odd bonds to each of the other three. The paper provides a closed form solution for determining all minimum odd bond reconstructions for a 2-run boundary.

Keywords: inpainting, MAP interpolation, Ising model, odd bonds

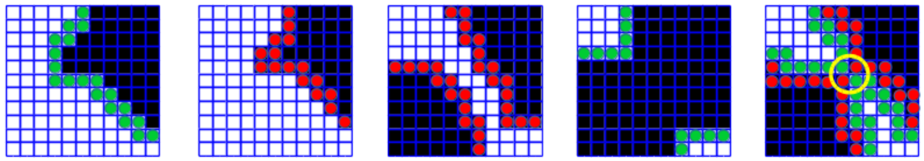
1 Introduction

Markov image models have found widespread application in segmentation [1], [2], denoising [3], compression [4], [5], [6], [7], [8], [9], and deep learning [10]. Such models are often parametrized as Gibbs distributions where the potential functions encode structural constraints between nearby image pixels. For example, in the uniform Ising model with no external field, the simplest non-trivial bilevel Markov field [11], the probability of an image is monotone decreasing in the number of *odd bonds* or pairs of adjacent sites with different values. The boundary of a block of sites within a uniform Ising model can be viewed as consisting of some number of alternating white and black *runs*, where a *run* is a monotone subset of sites with an odd bond at each end of the run. *Maximum a posteriori* (MAP) estimation of a block conditioned on its boundary then amounts to the combinatorial problem of finding the set of configurations within the block that together with the boundary have a minimum number of odd bonds. This paper derives such sets of minimum odd bond block configurations for boundaries with 1 and 2 runs on the boundary.

While the problem of interpolating from a boundary is a natural one to consider, for example as a type of inpainting [12], the problem addressed in the present paper was originally motivated by a sampling and interpolation problem in bilevel images modeled with a uniform Ising model with no external field. In [7], a Manhattan grid cutset consisting of evenly spaced rows and columns of the image is losslessly compressed, and the remaining blocks of unencoded sites are MAP interpolated from the values on their respective boundaries. This interpolation goal for a binary block also arose as a step in interpolating grayscale images from Manhattan grid sampling [13].

For many, if not most, image, and most common block sizes (4 to 32), boundaries with 0, 1, or 2 runs of white and black predominate. In the context of interpolating a block from its boundary, due to the Markov property, it is sufficient to consider a graph consisting solely of the block and its boundary, as the block is conditionally independent of sites beyond its boundary given a configuration on its boundary. The goal of MAP interpolation, i.e., minimizing odd bonds, can be restated as finding a minimum *b-w cut* (W, B), where b and w are the respective sets of black and white pixels on the block boundary, B and W are the sets of black and white pixels in the reconstruction, and the weight of a cut (W, B) is the number of odd bonds between W and B. Picard et al [14] and Vazirani et al [15] have derived iterative approaches to finding all min-cut solutions built upon the well-known Ford-Fulkerson algorithm [16]. While the iterative solutions of [14] and [15] can be applied to more general boundaries, and indeed more general graph topologies, they do not supply any insight into the geometric structure of the resulting minimum odd bond reconstructions. In contrast, the solutions presented in this paper for 1- and 2-run boundaries are closed form. That is, the solution of this paper is not iterative. Rather, it simply computes a handful of numbers, then based on the minimum of these numbers, characterize all min-cuts. Moreover, by exploiting the structure in the particular problem, one is able to explicitly characterize the set of all min-cuts in terms of their semantic structure, essentially by describing so-called *simple paths* delineating the sets W and B in MAP reconstructions. In other words, with respect to the motivating image problem, the solution of this paper is both faster (i.e., closed-form versus iterative) and more informative.

As a key step, Lemma 1 shows that a MAP reconstruction (for *any* boundary) cannot contain an *island*, a monotone subset of sites enclosed by a monotone *loop* of the other color. Any *island-free* reconstruction for a 1-run boundary can be generated with a *reconstruction path* connecting the endpoints of b (or w) in such a way that B (W) becomes the set of sites enclosed by, and including, the path and b (w). This is illustrated in Figure 1 (a) and (b). The monotone set W (B) containing the other boundary run is naturally defined as the complement of that defined by the reconstruction path for b (w). Theorem 1 characterizes the set of reconstruction paths that generate minimum odd bond reconstructions, as well as the numbers of odd bonds resulting from such. These paths are referred



(a) (b) (c) (d) (e)

Fig. 1. (a) A reconstruction generated by a white reconstruction path, shown in green, for a 1-run boundary; (b) the same reconstruction generated by a black reconstruction path, shown in red; (c) for a 2-run boundary, a reconstruction generated by two non-touching black paths whose black set B consists of two non-touching sets and whose white set W is connected; (d) a reconstruction generated by two non-touching white paths whose white set W consists of two non-touching sets and whose black set B is connected; and (e) a reconstruction generated by a pair of black paths and also by a pair of white paths in which both B and W are connected. The two pairs of paths meet at a widget, which is circled.

to as *1-run optimal* reconstruction paths, which also play a key role in generating minimum odd bond reconstructions for 2-run boundaries. The proof shows that such 1-run optimal paths are the shortest possible reconstruction paths, which are referred to as *simple*. It establishes the optimality of simple paths by considering the numbers of odd bonds between adjacent rows (or columns) resulting from reconstruction paths as described above.

For a 2-run boundary, a natural thing to consider is generating a reconstruction by connecting the endpoints of each run of one color with a simple path, giving a monotone set as the union of the two monotone sets determined by the individual simple paths, with the remainder of the block assigned the opposite color. This is illustrated in Figure 1 (c)-(e). The two pairs of reconstruction paths may be non-touching, as illustrated in Figure 1 (c) and (d), or they may touch as in (e). A place where a pair of reconstruction paths touch is a 2×2 checkerboard configuration referred to as a *widget*. Lemma 3 shows that if a minimum odd bond reconstruction contains a widget, the widget must be on the boundary of the block. That is, it must be located at one of the four boundary odd bonds. As a result, it is more convenient to instead view an island-free reconstruction with a widget on the boundary as being generated by three *connecting pairs* of paths, one between each of the three non-boundary odd bonds of the widget, and one of the other three boundary odd bonds. To ensure that a reconstruction (with a widget on the boundary) generated by three connecting pairs of paths is indeed island-free, the concept of a *feasible trio* of connecting pairs of paths, a triple of connecting pairs of paths that do not *cross* each other, is introduced.

If an island-free reconstruction from a 2-run boundary is generated by non-touching reconstruction paths, then the number of odd bonds in the reconstruction is the sum of the numbers of odd bonds resulting from the two individual reconstruction paths. Therefore, if there exist non-touching simple paths, then the resulting reconstructions have the minimum number of odd bonds out of all reconstructions generated by non-touching reconstruction paths. However, there are not always non-touching simple paths, and if there are a pair of simple reconstruction paths that touch, then the resulting reconstruction would have fewer odd bonds than a reconstruction generated by non-touching simple reconstruction paths.

On the other hand, the paper shows that the best island-free reconstructions with a widget on the boundary are generated by a simple feasible trio. As a result, one can find all minimum odd bond reconstructions by comparing sums of the numbers of odd bonds from: (1) two non-touching simple black paths, (2) two non-touching simple white paths, and (3) for each of the four possible locations for a widget on the boundary, three non-touching connecting pairs of paths. However, there are checkable conditions that sometimes allow us to rule out the optimality of reconstructions of one or more of the three types.

In [7] complete lists of all possible MAP solutions were stated without proof for 1 run boundaries, and a partial list of MAP solutions was stated without proof for 2 run boundaries.

The outline of the paper is as follows. The following section introduces notation and terminology on graphs and cuts. Section 3 sets the stage and proves the key result that all minimum b-w cuts are *island-free*. Section 4 introduces simple reconstruction paths. Section 5 derives all possible minimum odd bond reconstructions for 1-run boundaries. Section 6 island-free reconstructions for 2-run boundaries. Section 7 finds all minimum odd bond reconstructions for 2-run boundaries. Finally, Section 8 concludes with a discussion. Due to space limitations some proofs are sketched.

2 Graphs and Cuts

Let $G = (V, E)$ be an $N \times N$ grid graph G, where V is the set of *sites* or *pixels* and E is the set of *edges* consisting of horizontally, vertically, and diagonally adjacent pairs sites, referred to as *neighbors*. Let U denote the *interior* of V, which is the subset of sites of V that have exactly 8 neighbors, and let the set $\partial U = V \setminus U$ be called the *boundary* of U. Given a partition (w, b) of ∂U, a *b-w cut of G* is a partition (W, B) of the sites V such that $b \subset B$, $w \subset W$. One can think of b and w as being the sets of white and black pixels, respectively, on the boundary of a block V in a binary image, and B and W as the black and white pixels in the entire block V after interpolation of the interior U. As such, (W, B) is called a *configuration* on V, and B and W are referred to as *monotone* sets. Moreover, a partition (w, b) is referred to as a configuration on the boundary ∂U, and a corresponding configuration (W, B) is referred to as a *reconstruction from* (w, b). An edge connecting B and W is referred to as an *odd bond*, and a b-w cut (W, B) is said to be a *minimum b-w cut* if it contains a minimum number of odd bonds over all b-w cuts. A minimum b-w cut (W, B) can also be interpreted as a reconstruction with largest probability conditioned on the boundary when configurations on V are modeled with a uniform Ising model with positive correlation parameter and no external field [11].

The problem considered in this paper is that of finding the set of all minimum b-w cuts when there are either two or four edges connecting b and w. When there are two edges between b and w, we say there is one *run* of white and one run of black on the boundary, and refer to (w, b) as a *1-run boundary* for G. Similarly, when there are four edges connecting b and w, we say there is a *2-run boundary* for G, consisting of two runs of white and two runs of black, indicated as $(w, b) = (w_1, b_1, w_2, b_2)$.

3 Island-Free Reconstructions

Given a graph $G = (V, E)$, a *path* is a sequence of sites such that any two successive sites are connected by an edge. Given a configuration on G, a path all of whose sites are the same color is referred to as a *monotone path*. A *4-path* is a path p whose successive sites are joined by a vertical or horizontal edge in E. A subset of sites $A \subset V$ is *connected* if for all distinct sites $i, j \in A$, there exists a path in A from i to j. A subset $A \subset V$ all of whose sites are the same color is referred to as a *monotone set*. A subset $A \subset V$ is said to be 4-connected if any two of its sites i and j are connected by a 4-path. If A is connected but not 4-connected, it is referred to as 8-*connected*. Two sets are said to *touch* if they are disjoint and there is at least one edge consisting of a site from each set. They are said to 4-*touch* if they touch and at least one of the edges connecting them is a vertical or horizontal edge. Two sets are said to 8-*touch* if they touch but do not 4-touch. A *loop* is a path whose first and last sites coincide. A site i is in the *interior* $I(l)$ of a loop l if every 4-path from i to the boundary of the block intersects l. An *island* in a configuration (W, B) is a monotone subset of sites that forms the interior of a monotone loop of the other color. The sites in an island are not connected to the boundary through any monotone 4-path. Alternatively, if each site in the block can be connected to the boundary through a monotone 4-path, then the reconstruction is said to be *island-free*; likewise, an island-free *configuration*. The following lemma shows that all minimum cuts are island-free.

Lemma 1 (No Islands). *Let (w, b) be a partition of the boundary ∂U. There are no islands in a minimum b-w cut. Equivalently, in a minimum b-w cut, for each site in V, there is a monotone 4-path connecting it to the boundary ∂U.*

Figure 2 (a) and (b) illustrate reconstructions with and without islands. Note that (b) is not the only configuration that can be obtained from (a) by filling islands. Lemma 1 implies that in looking for minimum odd bond reconstructions, it suffices to restrict attention to island-free reconstructions. It also implies that in any island-free b-w cut (W, B) of G, both B and W can be partitioned into monotone 4-connected sets, each containing at least one boundary run. For a 1-run boundary and an island-free b-w cut, both B and W are connected and the cut can be specified by just one monotone 4-connected set.

For a 2-run boundary, an island-free b-w cut is most usefully thought of as either a partition of B into 4-connected sets $\{B_1, B_2\}$ where $b_1 \subset B_1$ and

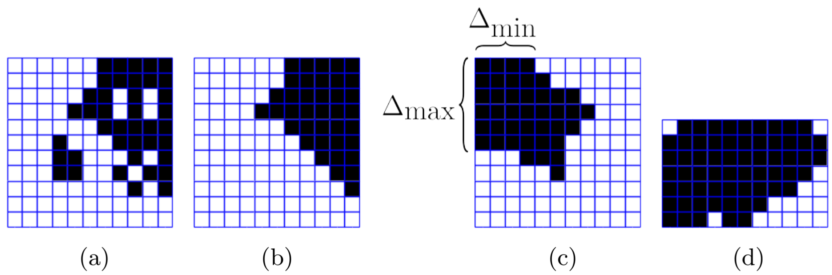


Fig. 2. (a) A configuration with islands, and (b) configuration with fewer odd bonds than (a) obtained by eliminating islands. (c) Configuration illustrating $\Delta_{\max}$ and $\Delta_{\min}$. (d) Configuration useful for illustrating the assertions of Lemma 2 about the numbers of diagonal odd bonds between successive rows.

$b_2 \subset B_2$; or a partition of W into $\{W_1, W_2\}$ where $w_1 \subset W_1$ and $w_2 \subset W_2$. For a given island-free configuration, if B_1 and B_2 4-touch, as in Figure 1 (d), the configuration is referred to as *black-connected*, and in this case W_1 and W_2 neither 4- nor 8-touch. Likewise, if W_1 and W_2 4-touch, as in Figure 1 (c), the configuration is said to be *white-connected*, and in this case B_1 and B_2 neither 4- nor 8-touch. On the other hand, there are configurations in which neither B_1 and B_2 nor W_1 and W_2 are 4-connected, but where both B_1 and B_2, and W_1 and W_2, are 8-connected, as illustrated in Figure 1 (e). Such reconstructions are referred to as *bi-connected*.

A path q is said to be a *reconstruction path* for boundary run r if q is a 4-path connecting the endpoints of r, and q does not pass through any boundary pixels not in r. With this in mind, for a run r and reconstruction path q for r, let $C(r, q)$ denote the set of sites enclosed by and including run r and path q. If $(w, b) = (r, \partial U \setminus r)$ is a 1-run boundary for G, let $\overline{C}(r, q)$ denote the *reconstruction generated by* r and q, meaning the reconstruction with $C(r, q)$ having the color of r and $V \setminus C(r, q)$ having the opposite color. Figure 1 (a) and (b) illustrate this. Note that $\overline{C}(r, q)$ is island-free. Conversely, if (W, B) is an island-free reconstruction for a 1-run boundary (w, b), then there exist black and white reconstruction paths q^b and q^w such that $B = C(b, q^b)$ and $W = C(w, q^w)$.

Let $O(r, q)$ denote the number of odd bonds in the reconstruction $\overline{C}(r, q)$. Furthermore, let $O_1^*(r)$ denote the minimum of $O(r, q)$, over all reconstruction paths q for r, which is also the number of odd bonds in a minimum odd bond reconstruction for $(r, \partial U \setminus r)$. A reconstruction path q that attains $O_1^*(r)$, and the corresponding reconstruction $\overline{C}(r, q)$ will be called *1-run optimal*. Note that $O_1^*(r)$ is defined for a run, not for a 1-run boundary (because it will be used in the analysis of 2-run boundaries).

4 Simple Reconstruction Paths

The following sections will show that the set of minimum *b*-*w* cuts for 1-run and 2-run boundaries are generated by the following easy-to-characterize type of reconstruction path.

Definition 1 (Simple Paths). *A path is* simple *if it is a shortest 4-path between its endpoints and it contains a shortest 8-path between its endpoints,* i.e.*, there is a subsequence of sites in the 4-path that forms an 8-connected path connecting its endpoints that is shortest among all 8-connected paths connecting its endpoints. This 8-path will also be the only (sub) 8-path contained within the simple 4-path, and will be referred to as the* imputed 8-path*.*

Note that a path is simple if and only if it does not have both horizontal and vertical edges, nor both kinds of diagonal edges. A simple path is contained within the smallest rectangle containing the endpoints of the path. Its length is the sum of the height and width of this rectangle, as defined by

$$\Delta_{\mathrm{H}} = |k_2 - k_1|, \text{ and } \Delta_{\mathrm{V}} = |l_2 - l_1|, \qquad (1)$$

respectively, where (l_1, k_1) are the coordinates of one of the endpoints and (l_2, k_2) are the coordinates of the other endpoint. Let $\Delta_{\max}$ and $\Delta_{\min}$ denote the maximum and minimum of these two numbers, which are referred to, respectively, as the *major* and *minor differences*. This is illustrated in Figure 2 (c). One can view this path, which connects the corners of the aforementioned rectangle, as consisting of directed edges by choosing one corner as the start of the path and the other as the end of the path. Viewed in this way, one can make the following observation regarding simple paths. Simple paths are those such that the imputed 8-path does not have both horizontal and vertical edges, do not have both SW-NE and NW-SE diagonal edges, and all edges of a given type have the same direction. In other words, the imputed 8-path will have $\max\{\Delta_{\mathrm{H}}, \Delta_{\mathrm{V}}\}$ edges, where $\min\{\Delta_{\mathrm{H}}, \Delta_{\mathrm{V}}\}$ of the edges are diagonal.

The number c of corners of ∂B contained in a boundary run r, and the quantities $\Delta_{\max}$ and $\Delta_{\min}$ determined by the endpoints of r, are used in the following section to characterize optimal reconstruction paths for 1-run boundaries. For boundary run r, let P^r denote *set of simple reconstruction paths* for r. It is straightforward to see that between two sites there are $\binom{\Delta_{\max}}{\Delta_{\min}}$ simple paths. However, for some runs containing two or three corners of ∂B, some of the simple paths connecting the endpoints of r will not be in P^r, as they intersect boundary pixels not in r. For a run r containing four corners, there is only one simple path connecting its endpoints, which excluding the endpoints themselves, coincides exactly with the set of boundary pixels not in r. Thus for a four corner run r, the set P^r is empty. For a boundary run r that does not contain four corners, the simple reconstruction path that, together with r, encloses the fewest sites is called the *inner path*, while the simple reconstruction path that together with r encloses the most sites is the *outer path*.

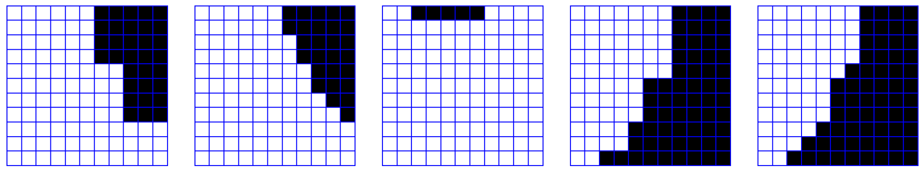


Fig. 3. Some minimum odd bond reconstructions.

5 Map Reconstructions for 1-Run Boundaries

Consider a 1-run boundary (w, b). The following theorems characterizes the set of minimum odd bond reconstructions and gives the numbers of odd bonds in the resulting reconstructions. As shown in the next theorem, simple reconstruction paths optimize $O(r, q)$, with the same qualification that if one of the runs contains 4 corners, the 1-run optimal reconstruction path for that color will not be simple.

Theorem 1 (1-run MAP Reconstructions). *Consider a 1-run boundary configuration (w, b). Let Δ_{max} and Δ_{min} denote the major and minor differences of b, and let c denote the number of corners of the boundary ∂U contained in b. The following characterizes the set of minimum odd bond reconstructions, where $|P^b|$ indicates the number of simple reconstruction paths for run b.*

(a) If $c \in \{0, 1, 2, 3\}$, then (W, B) is a minimum odd bond configurations if and only if $B = C(b, q)$ for some $q \in P^b$.
(b) If $c = 4$, $P^b = \emptyset$, and (W, B) is a minimum odd bond configuration if and only if $B = V \setminus w$.
(c) The number of odd bonds in a minimum odd bond reconstruction is

$$O_1^*(b) = \begin{cases} 3\Delta_{max} + \Delta_{min} - 2c + 5, & 0 \le c \le 3 \\ 3\Delta_{max} - 1, & c = 4 \end{cases}. \qquad (2)$$

(d) For $c \le 3$, the number of minimum odd bond reconstructions is

$$|P^b| = \begin{cases} \binom{\Delta_{max}}{\Delta_{min}} & \text{if } 0 \le c \le 2 \\ \binom{\Delta_{max}-1}{\Delta_{min}-1} & \text{if } c = 3 \end{cases}. \qquad (3)$$

Figure 3 (b), (d), and (e) are minimum odd bond reconstructions, while (a) and (c) are not. Note that Theorem 1 is stated in terms of the black run, but applies equally to the white run, and the proofs for the $c = 3$ and $c = 4$ cases use the fact that in these cases, the white runs contain 1 and 0 corners, respectively. A key step in the proof is the following lemma, which is easy to verify. It can be useful to refer to Figure 2 (d) when reading its assertions.

Lemma 2. *In a reconstruction (W, B), the number of diagonal odd bonds between two successive rows of pixels is*

(a) 0, if the two rows are monotone of the same color,
(b) 1, if one row is monotone and all pixels of the other have the same color except one at its end,
(c) 2, if one row is monotone and the other has the same color except for the pixels on each end of the row,
(d) 2, if one row has black pixels to the left, white pixels to the right and the second row is identical to the first, or has the same form except that the transition from white to black occurs one pixel to the left or to the right of the transition in the first row,
(e) at least 3 in all other cases

Proof. no corners: If b contains no corners, then it is entirely contained within one side of the block, which without loss of generality can be assumed to be the right side. Then $\Delta_{\max}$ is the length of this run minus 1, and $\Delta_{\min} = 0$. It is easy to see that any reconstruction has at least $\Delta_{\max} + 1$ horizontal odd bonds, one for each site of the run, and at least $\Delta_{\min} + 2 = 2$ vertical odd bonds. Since the reconstruction that fills the block entirely with white has this many horizontal and vertical odd bonds, this gives $\Delta_{\max} + \Delta_{\min} + 3$ as the minimum number of odd bonds possible for a reconstruction on B with boundary $x_{\partial B}$ and demonstrates (c) for the $c = 0$ case.

It is easy to see that the all white interior reconstruction is the reconstruction generated by the reconstruction path q consisting of the run b itself. The run b is the only simple reconstruction path, establishing (d). It also easy to see that any other reconstruction path will have at least two additional vertical bonds. Hence, the only optimal reconstructions are those generated by paths in P^b, which establishes (a) for the $c = 0$ case.

four corners: If b contains four corners, then w contains zero corners. Therefore, the set of optimal reconstructions in this case are precisely those for w determined by the previous arguments. As such, one need only verify that they agree with the statements of the theorem when described in terms of b and black reconstruction paths. In particular, the one and only optimal reconstruction for w makes the interior entirely black, which is the one and only reconstruction described in (b). There is only one simple path connecting the endpoints of b, and since this path completely covers w, it is not a reconstruction path. Therefore, $P^b = \emptyset$, as also stated in (b).

Let $\Delta'_{\min}$ and $\Delta'_{\max}$ denote the minor and major differences for the white run w. It is clear that $\Delta_{\min} = \Delta'_{\min} = 0$ and that $\Delta_{\max} = \Delta'_{\max} + 2$. Applying the expression in (c) for the minimum number of odd bonds for c=0, one sees that the minimum number of odd bonds attainable with a boundary run containing four corners is

$$\begin{aligned}\Delta'_{\max} + \Delta'_{\min} + 3 &= \Delta_{\max} - 2 + \Delta_{\min} + 3 \\ &= \Delta_{\max} + \Delta_{\min} + 1 = \Delta_{\max} + 1,\end{aligned}$$

which matches (c) for the $c = 4$ corner case, and completes the proof for this case.

one corner: Without loss of generality, assume the black run b contains the upper right corner of the block and no other corners. Also without loss of generality, assume that $\Delta_{\text{max}} = \Delta_{\text{V}}$ and that the left endpoint of b is on the top side and the right endpoint of b is on the right side of the block. Then, it is easy to see that there are at least $\Delta_{\text{max}}+1$ horizontal odd bonds, and at least $\Delta_{\text{min}}+1$ vertical odd bonds in any reconstruction formed from reconstruction paths. This can be seen in Figure 2 (c). From Lemma 2, there are a minimum of $2\Delta_{\text{max}}$ diagonal odd bonds between the rows intervening the rows containing the left and right endpoints of b. Furthermore, there is at least one additional diagonal odd bond between the rows containing the run endpoint b_2 on the right side of the block and its neighbor to the SW. It is clear that the inner path for b results in a block reconstruction that attains this minimum. It is also straightforward to see that any simple reconstruction path attains this minimum, showing not only (c) but also the 'if' part of (a). Moreover, all simple paths connecting the endpoints of b do not intersect w. Therefore, $|P^b| = \binom{\Delta_{\text{max}}}{\Delta_{\text{min}}}$ as stated in part (d) of the theorem.

It is straightforward to show that any non simple reconstruction path $q \notin P^b$ incurs more odd bonds than the minimum given in (c). From the above arguments, $C(b,q)$ has at least $3\Delta_{\text{max}} + \Delta_{\text{min}} - 2c + 5$ odd bonds since this is the minimum number of odd bonds in a reconstruction. It is straightforward to see that if q is not simple, then there will be a row in $C(b,q)$ that satisfies part (e) of Lemma 2, and hence has at least one additional odd bond. Thus, if q is not simple, then $C(b,q)$ is not minimum odd bond, completing the proof of part (a).

three corners: If the black run contains three corners, then the white run contains one corner. Since the set of minimum odd bond reconstructions in this case are precisely those for w, and w has one corner, these minimum odd bond reconstructions are just those determined by the previous arguments. Therefore, one need only verify that they agree with the statements of the theorem when described in terms of b and black reconstruction paths. In particular, note that there is a one-to-one correspondence between simple reconstruction paths for b and simple reconstruction paths for w. If a reconstruction path for b, then, is not simple, the corresponding reconstruction path for w is not simple and by the arguments above, the resulting reconstruction is not minimum odd bond for the boundary. Conversely, if a reconstruction path is simple for b, the corresponding reconstruction path for w is simple and again by earlier arguments, the resulting reconstruction is minimum odd bond for ∂U. This shows (a) for the three corner case.

To show (c) and (d), let Δ'_{max}, Δ'_{min} and c' refer to the white run and note that $\Delta_{\text{max}} = \Delta'_{\text{max}} + 1$ and $\Delta_{\text{min}} = \Delta'_{\text{min}} + 1$ and $c = c' + 2$. Using these relations and (c) for the one corner case, compute the minimum number of odd bonds as

$$\begin{aligned} 3\Delta'_{\text{max}} + \Delta'_{\text{min}} + 5 - 2c' &= 3(\Delta_{\text{max}} - 1) + (\Delta_{\text{min}} - 1) + 5 - 2(c - 2) \\ &= 3\Delta_{\text{max}} + \Delta_{\text{min}} + 5 - 2c \end{aligned}$$

and since this minimum is attained by simple reconstruction paths for w, it is likewise attained by simple reconstruction paths for b.

Likewise, using the relationship between Δ' and Δ and the correspondence between simple reconstruction paths for b and simple reconstruction paths for w, the number of simple reconstruction paths for b is

$$|P^b| = |P^w| = \binom{\Delta'_{\max}}{\Delta'_{\min}} = \binom{\Delta_{\max}}{\Delta_{\min}}$$

as stated in (c).

two corners: Now suppose the black run b contains two corners, which without loss of generality we take to be the top two corners. Still without loss of generality, assume that the endpoint on the left side of the block is at least as close to the top side as the endpoint on the right side.

By similar arguments as above, there are a minimum of $\Delta_{\max} + \Delta_{\min} + 1$ horizontal and vertical odd bonds in any reconstruction generated by a path. From Lemma 2, one can see that there are at least $2\Delta_{\max}$ diagonal odd bonds between the columns intervening b_1 and b_2. It is also easy to see that any simple reconstruction path for b results in a reconstruction that has precisely this many odd bonds. Hence, the minimum number of odd bonds is $3\Delta_{\max}+\Delta_{\min}+1$ which agrees with (c) and establishes the 'if' part of (a). Again, there are $|P^b| = \binom{\Delta_{\max}}{\Delta_{\min}}$ simple reconstruction paths, showing (d).

To argue that any non simple reconstruction path $q \notin P^b$ will incur more odd bonds than the minimum given in (c), use the same argument as that used in the one-corner case above. From the above arguments, $\overline{C}(b, q)$ has at least $3\Delta_{\max} + \Delta_{\min} - 2c + 5$ odd bonds, since this is the minimum number of odd bonds in a reconstruction. It is straightforward to see that if q is not simple, then there will be a row in $\overline{C}(b, q)$ that satisfies part (e) of Lemma 2, and hence has at least one additional odd bond. This shows that if $\overline{C}(b, q)$ is generated by a non-simple q, then it is not minimum odd bond, proving part (a) and completing the proof of the theorem in the two corner case. □

Note that with the exception of boundary runs that contain 4 corners, a reconstruction path is 1-run optimal if and only if it is simple. Further, if a run of 1-run boundary contains four corners, and consequently, has no simple reconstruction paths, then the run of the other color contains no corners and the unique 1-run optimal reconstruction path for it is simple. Thus, any minimum odd bond reconstruction for a 1-run boundary can be generated by a simple reconstruction path.

6 Island-Free Reconstructions for 2-Run Boundaries

To begin the discussion of finding minimum odd bond reconstructions for a 2-run boundary (w_1, b_1, w_2, b_2), let O_2^* denote this minimum number of odd bonds within V over reconstructions from the boundary. First note that if one run of a 2-run boundary contains 4 corners, then the unique minimum odd bond reconstruction is obtained by filling the interior with the same color as the 4 corner run. Henceforth, assume that no run contains 4 corners, and therefore 1-run optimal and simple are synonymous when referring to reconstruction paths.

Let $O_{2,b}$ and $O_{2,w}$ denote the minimum number of odd bonds in reconstructions generated by non-touching pairs of black, respectively white, reconstruction paths for the 2-run boundary, where dependence on the boundary has been supressed. Note that $O_{2,w}$ and $O_{2,w}$ are easily computable using the formulas for $O_1^*(b)$ and $O_1^*(w)$ in Thm. 1. However, there are minimum odd bond island-free reconstructions that cannot be generated from non-touching reconstruction paths. As such, in order to determine O_2^* it is not sufficient to compute $O_{2,b}$ and $O_{2,w}$. Moreover, there are boundaries from which no reconstruction can be generated by non-touching black reconstruction paths, for example. In such cases $O_{2,b}$ is defined to be infinity, and likewise for $O_{2,w}$. We start by considering how to generate reconstructions from non-touching reconstruction paths, and then introduce additional concepts needed to find all minimum odd bond reconstructions for 2-run boundaries.

As mentioned in the introduction and illustrated in Figure 1, a simple way to generate a reconstruction for a 2-run boundary is to choose reconstruction paths for each run of one color, and *merge* the reconstructions generated by each path. For example, if q^{b_1} and q^{b_2} are reconstruction paths for black runs b_1 and b_2, respectively, then the reconstruction *generated by* q^{b_1} *and* q^{b_2} has black region $B = C(b_1, q^{b_1}) \cup C(b_2, q^{b_2})$. This reconstruction is said to be a merging of the 1-run reconstructions for b_1 and b_2 generated by q^{b_1} and q^{b_2}, respectively. Likewise, the reconstruction generated by reconstruction paths q^{w_1} and q^{w_2} for white boundary runs w_1 and w_2, respectively, has white region $W = C(w_1, q_1^w) \cup C(w_2, q_2^w)$. Notice that if k_1 and k_2 are runs of color k, and if q^{k_1} and q^{k_2} do not overlap or touch, then the reconstruction generated by them is island-free and the number of odd bonds in the merged reconstruction is $O(k_1, q^{k_1}) + O(k_2, q^{k_2})$. If q^{k_1} and q^{k_2} are simple, then the reconstruction generated by them attains the minimum over all reconstructions generated by non-touching reconstruction paths of color k, in which case the number of odd bonds is $O_1^*(k_1) + O_1^*(k_2)$, and this number is $O_{2,k}$.

The presence of diagonal edges means that 8-touching of reconstruction paths reduces the number of odd bonds, and consequently reconstructions generating by 8-touching paths can sometimes be better than any reconstruction generated by non-touching paths. For example, if there is a single pixel black run in a corner, and the two white runs are each a single pixel, then both the inner white paths and the inner black paths touch, and the unique minimum odd bond reconstruction is generated by a pair of reconstruction paths that meet at a 2×2 subblock, the upper right and lower left corners of which are black (or white) and the upper left and lower right corners of which are white (or black). This subblock is referred to as a *widget*, which is illustrated in Figure 1 (e) and Figure 4 (a)-(g).

Therefore, in addition to considering the best reconstructions generated by non-touching reconstruction paths, one needs to consider the best island-free reconstructions with a widget. It turns out that whenever a minimum odd bond reconstruction contains a widget, the widget must necessarily be on the boundary, as stated in the following lemma.

Lemma 3 (No interior widgets). *A widget in an island-free (*e.g.*, minimum odd bond) reconstruction must lie on the boundary,* i.e.*, at least two of its pixels are in ∂U.*

This paper considers the problem of finding the set of MAP reconstructions of an $N \times N$ block condititioned on a boundary configuration consisting of 1 or 2 alternating runs of black and white within a uniform Ising model with no external field. When the boundary contains a single run, the set of minimum odd bond reconstructions are described by *simple paths* connecting the endpoints of either the black or white run. When the boundary consists of 2 runs, the set of minimum odd bond reconstructions are formed in one or more of the following ways: by simple paths connecting the endpoints of the two black runs; by simple paths connecting the two white runs; or by three simple paths connecting one of the boundary odd bonds to each of the other three. A closed form solution for determining all minimum odd bond reconstructions for a 2-run boundary is provided.

Since there are four odd bonds on the boundary, there are at most four possible locations for a minimum odd bond reconstruction to have a widget. When considering an island-free reconstruction (W, B) with a widget at the ith odd bond, without loss of generality orient the block so that boundary odd bond i is on top of the block and such that the white pixel of the odd bond is on the left, as illustrated in Figure 4 (a)-(c). This can be achieved through a rotation and/or reflection of the block. The white boundary run containing the white boundary widget pixel is denoted w_1, and the black run containing the black boundary widget pixel is denoted b_1. Now index the four boundary odd bonds in counterclockwise order as OB_0, OB_1, OB_2, OB_3 with OB_0 being boundary odd bond i. Similarly, index the four odd bonds in the boundary widget in counterclockwise order as ob_0, ob_1, ob_2, ob_3, with ob_0 again being boundary odd bond i. The six odd bonds OB_1, OB_2, OB_3, ob_1, ob_2, ob_3 are referred to as *sub* odd bonds with respect to boundary odd bond i.

As illustrated in Figure 4 (d)-(f), an island-free reconstruction with a widget at boundary odd bond i can then be described by three *connecting pairs* of paths $q_1 = (q_1^b, q_1^w)$, $q_2 = (q_2^b, q_2^w)$, $q_3 = (q_3^b, q_3^w)$, where $q_j = (q_j^b, q_j^w)$ connects ob_j to OB_j. With this in mind, a white path q^w and a black path q^b are said to be *complementary* to each other if q^w connects the white sites of two odd bonds OB_1 and OB_2, q_b connects the black sites of these odd bonds, q^w and q^b do not overlap, and all interior (i.e., non-endpoint) sites of q^w are 8- or 4-connected to an interior site of q^b, and likewise, all interior sites of q^b are 8- or 4-connected to an interior site of q^w. Refer to $q = (q^b, q^w)$ as a *connecting pair of paths* (for odd bonds OB_1 and OB_2) and say that q^w and q^b are *members* of connecting pair q.

Such a reconstruction is partitioned into four monotone sets. Two of these are the regions $C_{b,1} = C(b_1, q_3^b)$ and $C_{w,1} = C(w_1, q_1^w)$ enclosed, respectively,

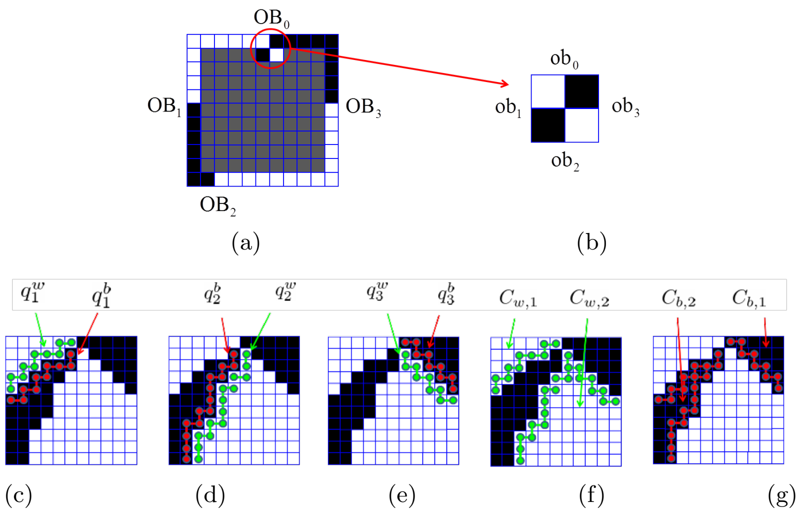


Fig. 4. (a) Boundary odd bonds $\mathrm{OB}_0, \ldots, \mathrm{OB}_3$ relative to the boundary on the top of the block; (b) widget odd bonds $\mathrm{ob}_0, \ldots, \mathrm{ob}_3$; (c) Complementary reconstruction paths q_1^b and q_1^w connecting odd bounds OB_1 and ob_1; (d) complementary reconstruction paths q_2^b and q_2^w connecting odd bonds OB_2 and ob_2; (e) complementary reconstruction paths q_3^b and q_3^w connecting odd bonds OB_3 and ob_3; (f) monotone white region $C(w_1, q_1^w)$ generated by run w_1 and reconstruction path q_1^w, and monotone white region $C(w_2, q_2^w, q_3^w)$ generated by run w_2 and white reconstruction paths q_2^w and q_3^w; and (g) monotone black region $C(b_2, q_1^b, q_2^b)$ generated by the black run b_2 and black reconstruction paths q_1^b and q_2^b, and monotone black region $C(b_1, q_3^b)$ generated by run b_1 and black reconstruction path q_3^b.

by b_1 and q_3^b, and w_1 and q_1^w. The other two monotone regions are defined as follows: let $C_{b,2} = C(b_2, q_1^b, q_2^b)$ denote the set of pixels enclosed by run b_2 and paths q_1^b and q_2^b, and let $C_{w,2} = C(w_2, q_2^w, q_3^w)$ denote the set of pixels enclosed by run w_2 and paths q_2^w and q_3^w. Because there are no islands, the only widget is that at boundary odd bond i. So the odd bonds in the reconstruction are those of q_1, those of q_2, those of q_3, plus an additional odd bond between $w_{1,2}$ and $b_{1,1}$, which are, respectively, the white and black interior widget pixels. Thus, for an island-free reconstruction, the number of odd bonds is $O(q_1)+O(q_2)+O(q_3)+1$. Conversely, any trio of paths q_1, q_2, q_3 generate a reconstruction in two ways. Specifically, $C_{b,1} \cup C_{b,2}$ generates a reconstruction from the black paths, and $C_{w,1} \cup C_{w,2}$ generates a reconstruction from the white paths. Such regions are illustrated in Figure 4 (g) and (h).

In general, the reconstructions $C_{b,1} \cup C_{b,2}$ and $C_{w,1} \cup C_{w,2}$ will be distinct. By placing the following restriction on q_1, q_2, and q_3, these reconstructions will be identical and give us a means of generating a unique reconstruction from a trio of connecting pairs of paths. A trio of connecting pairs of paths q_1, q_2, q_3 is said to be a *feasible trio* if (a) each q_j connects ob_j to OB_j, (b) q_1^w and q_2^w touch only at the widget, and (c) q_2^b and q_3^b touch only at the widget. Note that a trio of connecting pairs of paths (q_1, q_2, q_3) describing an island-free bi-connected reconstruction is necessarily a feasible trio. As such, a feasible trio exists for i if it is possible to form a bi-connected, island-free reconstruction with a widget at boundary odd bond i. Conversely, from any feasible trio, the same bi-connected, island-free reconstruction with a widget at boundary odd bond i can be generated by either its black paths or its white paths. A pair of odd bonds ob and OB that can be connected by a connecting pair of simple paths is said to be *properly oriented*. If all components of a feasible trio are simple connecting paths, this is referred to as a *simple feasible trio*.

Define $O_{\text{bi},i}$ to be the minimum number of odd bonds in a bi-connected, island-free reconstruction with a widget at boundary odd bond i, if there exists one, and ∞ otherwise.

7 MAP Reconstructions for 2-Run Boundaries

We now discuss finding the set of all minimum odd bond reconstructions for a boundary with 2 runs. Recall that an island-free reconstruction from a 2-run boundary can be generated in one or more of the following six ways: by two non-touching black reconstructions paths connecting the endpoints of the two black runs; by two non-touching white reconstruction paths connecting the endpoints of the white runs; and by three pairs of connecting paths connecting one of the four boundary odd bounds to each of the other three boundary odd bonds. Therefore,

$$O_2^* = \min\{O_{2,b}, O_{2,w}, O_{\text{bi},0}, O_{\text{bi},1}, O_{\text{bi},2}, O_{\text{bi},3}\}. \tag{4}$$

One could, in principle, determine the set of minimum odd bond reconstructions by identifying which of the above terms are minimal. Then the set of minimum odd bond reconstructions are all those corresponding to the terms that are minimal.

It turns out, however, that one is not able to compute all terms of the above for all possible block boundary configurations. However, one is able to compute *proxies* for each of these terms. Specifically, if O denotes one of the six terms above, then a *proxy* for it is a function $\tilde{O}$ with the property that for every 2-run boundary $\underline{r}$,

$$O = O_2^* \implies \tilde{O} = O_2^* \tag{5}$$

and

$$O > O_2^* \implies \tilde{O} > O_2^*. \tag{6}$$

If one can establish proxies for each of the six terms in (4), then it will follow from (4)-(6) that

$$\tilde{O}_2^* \triangleq \min\{\tilde{O}_{2,b}, \tilde{O}_{2,w}, \tilde{O}_{\text{bi},1}, \tilde{O}_{\text{bi},2}, \tilde{O}_{\text{bi},3}, \tilde{O}_{\text{bi},4}\} = O_2^*. \tag{7}$$

Moreover, if the proxy $\tilde{O}$ for some particular type of reconstruction equals $\tilde{O}_2^*$, then $\tilde{O} = O_2^*$, and the set of all reconstructions of this type that are optimal is precisely the set of those with $\tilde{O}$ odd bonds. Conversely, if $\tilde{O} > \tilde{O}_2^*$ for some type of reconstruction, then $O > O_2^*$, and consequently, no reconstruction of this type can be optimal. In summary, the set of all optimal reconstructions for a given boundary $\underline{r}$ is the union over those of the six types whose proxy $\tilde{O}$ equals $\tilde{O}_2^*$ of the reconstructions of that type that attain their respective proxy.

It remains now to find the six proxies. As will be seen, each will be based on generating reconstructions from 1-run optimal reconstruction connecting paths, so that the values of the proxies are straightforward to compute using Thm 1. As will be discussed in more detail subsequently, there are situations when one can immediately rule out the existence of one of the six types of reconstructions from the values on the boundary, in which case one can set the value of the corresponding proxy to be infinite.

First, the proxy for $O_{2,b}$ is based on generating a reconstruction by connecting the endpoints of each black run with a 1-run optimal reconstruction path, and is defined as

$$\tilde{O}_{2,b} \triangleq \begin{cases} \infty & \text{if single-pixel white run in a corner} \\ O_1^*(b_1) + O_1^*(b_2) & \text{else} \end{cases} \tag{8}$$

The quantity $\tilde{O}_{2,b}$ is the number of odd bonds in a reconstruction generated by two non-touching simple black reconstruction paths, if such a pair of paths exist. If there exist such, then $\tilde{O}_{2,b}$ is obviously the least number of odd bonds among reconstructions generated by non-touching black paths, *i.e.*, it equals $O_{2,b}$. Note that boundaries with a single-pixel white run in a corner are the only boundaries for which $O_{2,b}$ is infinite, but are not the only ones for which $O_{2,b} \neq O_1^*(b_1)+O_1^*(b_2)$. For example, if there is a single-pixel black run in a corner and each of the white runs is a single pixel, then while there is a reconstruction generated by non-touching black reconstruction paths, there is no reconstruction generated by non-touching simple black reconstruction paths. The proxy $\tilde{O}_{2,w}$ for $O_{2,w}$ is defined similarly.

Note that if there are two single-pixel runs in corners, they must be the same color. Therefore, at least one of $\tilde{O}_{2,w}$ and $\tilde{O}_{2,b}$ will be finite.

Lemma 4. *$\tilde{O}_{2,b}$ and $\tilde{O}_{2,w}$ are proxies for $O_{2,b}$ and $O_{2,w}$, respectively.*

Proof. By symmetry it suffices to show that $\tilde{O}_{2,b}$ is a proxy for $O_{2,b}$, which we do by demonstrating (5) and (6).

When $O_{2,b} = O_2^*$, every pair of simple black paths do not touch, because if a pair did touch, then merging the 1-run reconstructions generated by each path would create a reconstruction of a different type that had fewer odd bonds than $O_{2,b}$, in which case $O_{2,b} > O_2^*$. Any such non-touching simple black paths generate a reconstruction with $\tilde{O}_{2,b}$ odd bonds. Therefore $\tilde{O}_{2,b} = O_{2,b} = O_2^*$, which demonstrates (5).

Now assume $O_{2,b} > O_2^*$. If $O_{2,b} = \infty$, there do not exist non-touching black reconstruction paths, which means that there is a single-pixel white run in a corner. Thus, $\tilde{O}_{2,b} = \infty$ and hence $\tilde{O}_{2,b} > O_2^*$. Now assume that $\infty > O_{2,b} > O_2^*$.

If $O_{2,b} > \tilde{O}_{2,b}$, then all pairs of simple black paths touch, and again there is a reconstruction of a different type better than that counted by $\tilde{O}_{2,b}$, so $\tilde{O}_{2,b} > O_2^*$. If $O_{2,b} = \tilde{O}_{2,b}$, then $\tilde{O}_{2,b} > O_2^*$. This completes the proof of (6) and the lemma.

For future reference, note that if a 2-run boundary $\underline{r}$ has $\tilde{O}_{2,b} = \tilde{O}_2^*$ (in which case every reconstruction generated by simple black reconstructions paths is optimal), then no pair of simple black paths can touch or overlap, for if they did, a reconstruction with fewer than $\tilde{O}_{2,b}$ odd bonds would be formed. The same sort of thing applies when $\tilde{O}_{2,w} = \tilde{O}_2^*$.

To find proxies for the $O_{\mathrm{bi},i}$'s, one first needs to develop some terminology for analyzing and generating bi-connected, island-free reconstructions using connecting paths.

In defining a proxy for $O_{\mathrm{bi},i}$, first note that there are two cases for which there does not exist an island-free bi-connected reconstruction with a widget at odd bond i, and consequently, that $O_{\mathrm{bi},i} = \infty$. The first is when a widget cannot exist at i, which happens when and only when one of the pixels of odd bond i is a corner pixel and is contained in a run of length greater than 1. The second is when a widget is possible at i but there is no feasible trio, which occurs when and only when one of the other boundary odd bonds contains a single-pixel run in a corner. In these cases there does not exist a feasible trio of connecting paths for a widget at odd bond i.

Recall from Section 4 that a pair of odd bonds is properly oriented if and only if it can be connected with a simple connecting pair of paths. If for boundary odd bond i one of the pairs of sub odd bonds is not properly oriented, then that pair of sub odd bonds cannot be connected with simple connecting paths, and therefore, there does not exist a simple feasible trio for i. Now define the following quantity, which will soon be demonstrated to be a proxy for $O_{\mathrm{bi},i}$:

$$\tilde{O}_{\mathrm{bi},i} \triangleq \begin{cases} O_1^*(\mathrm{ob}_1, \mathrm{OB}_1) + O_1^*(\mathrm{ob}_2, \mathrm{OB}_2) + O_1^*(\mathrm{ob}_3, \mathrm{OB}_3) + 1, & \text{s.f.t.} \\ \infty, & \text{not s.f.t.} \end{cases}$$

where the condition s.f.t. indicates that a simple feasible trio is possible at boundary odd bond i, meaning that all sub odd bonds for i are properly oriented, odd bond i does not contain a site that is a corner pixel that is contained in a run of length greater than one, and none of the other boundary odd bonds contain a single-pixel in a corner. Note that even if condition s.f.t. holds, there still might not be a simple feasible trio. However, the quantity $\tilde{O}_{\mathrm{bi},i}$ is still computable, and as we will now argue, is a proxy for $O_{\mathrm{bi},i}$. We begin with the following lemma, which implies that if a feasible trio is not simple, then the reconstruction generated by it is not optimal.

Lemma 5 (Bi-Connected MAP Implies Simple Feasible Trio). *Suppose* (w_1, b_1, w_2, b_2) *is a 2-run boundary for which* (W, B) *is a bi-connected minimum odd bond reconstruction with a widget at the* i*th boundary odd bond. Then the feasible trio* q_1, q_2, q_3 *that generates* (W, B) *is simple (*i.e.*, each* q_j *is 1-run optimal), and consequently, all sub odd bonds* (ob_j, OB_j) *are properly oriented.*

Proof. We will demonstrate the contrapositive, namely, that any bi-connected, island-free reconstruction (W, B) for a 2-run boundary (w_1, b_2, w_2, b_2) generated by a non-simple feasible trio is not optimal. Accordingly, let (W, B) be such a reconstruction generated by a feasible trio q_1, q_2, q_3 with at least one non-simple member, and let $C_{b,1}$, $C_{b,2}$, $C_{w,1}$, $C_{w,2}$ denote its 4-connected monotone regions.

For any $C \subset V$, let $O(C)$ denote the number of odd bonds in the reconstruction with $B = C$ and $W = V \setminus C$, or equally, the reconstruction with $W = C$ and $B = V \setminus C$. Then, one can easily see that the number of odd bonds in (W, B) can be expressed in any of the following ways

$$\begin{aligned} O(C_{w,1} \cup C_{w,2}) &= O(C_{w,1}) + O(C_{w,2}) - 2 = O(C_{b,1}) + O(C_{b,2}) - 2 \\ &= O(C_{b,1} \cup C_{b,2}). \end{aligned}$$

If q_1 is not simple, one can replace it with a simple path q'_1, creating $C'_{w,1} = C(w_1, q'^w_1)$ and generating reconstruction $C'_{w,1} \cup C_{w,2}$ with $O(C'_{w,1} \cup C_{w,2})$ odd bonds. Since q'_1 is 1-run optimal and q^w is not, $O(C'_{w,1}) < O(C_{w,1})$. Moreover, the sets $C'_{w,1}$ and $C_{w,2}$ may overlap or touch, and consequently,

$$O(C'_{w,1} \cup C_{w,2}) \leq O(C'_{w,1}) + O(C_{w,2}) - 2.$$

It then follows that

$$\begin{aligned} O(C'_{w,1} \cup C_{w,2}) \leq O(C'_{w,1}) + O(C_{w,2}) - 2k \quad &< \quad O(C_{w,1}) + O(C_{w,2}) - 2 \\ = O(C_{w,1} \cup C_{w,2}), & \end{aligned} \tag{9}$$

which implies that (W, B) is not a minimum odd bond configuration. The same argument shows that (W, B) is not minimum odd bond if q_3 is not simple.

Now assume that both q_1 and q_3 are simple, but q_2 is not. First note that one can replace q_1 with $\bar{q}_1$ such that $\bar{q}^w_1$ is the inner path for the white run w_1; and likewise replace q_3 with $\bar{q}_3$ such that $\bar{q}^b_3$ is the inner path for the black run b_1. The resulting trio is still feasible, and the new $\bar{q}_1$ and $\bar{q}_3$ are still simple with the same numbers of odd bonds as q_1 and q_3, respectively, *i.e.*, $O(\bar{q}_1) = O(q_1)$ and $O(\bar{q}_3) = O(q_3)$.

Next, it can be easily seen that q_2 can be replaced with a simple $\bar{q}_2$ such that either $\bar{q}^w_2$ does not cross $\bar{q}^w_3$, or $\bar{q}^b_2$ does not cross $\bar{q}^b_1$. If $\bar{q}^w_2$ does not cross $\bar{q}^w_3$, consider the reconstruction generated by $\bar{C}_{w,1} \cup C(w_2, \bar{q}^w_3, \bar{q}^w_2)$. On the other hand, if $\bar{q}^b_2$ does not cross $\bar{q}^b_1$, consider the reconstruction generated by $\bar{C}_{b,1} \cup C(b_2, \bar{q}^b_1, \bar{q}^b_2)$. In either case, the resulting three paths form a simple feasible trio, and an argument like that in (9) shows that the reconstruction has fewer odd bonds than the one with which we started, implying that the latter is not optimal, and completing the proof of the two-corner case, and the theorem itself.

Lemma 6. *$\tilde{O}_{\mathrm{bi},i}$ is a proxy for $O_{\mathrm{bi},i}$.*

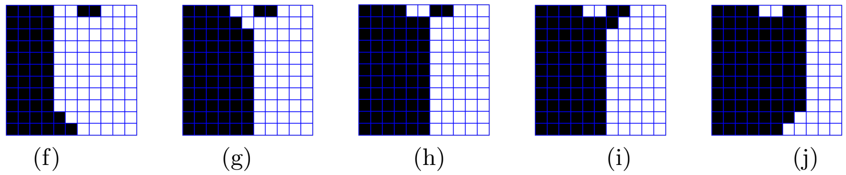


Fig. 5. Minimum odd bond reconstructions for a given 2-run boundary.

Proof. We need to show that $O_{\text{bi},i} = O_2^*$ implies $\tilde{O}_{\text{bi},i} = O_2^*$ and $O_{\text{bi},i} > O_2^*$ implies $\tilde{O}_{\text{bi},i} > O_2^*$.

If $O_{\text{bi},i} = O_2^*$, then all simple trios for i are feasible. For if there was a simple trio that was not feasible, then there would be a reconstruction of a different type with fewer odd bonds than a bi-connected, island-free reconstruction with a widget at i, which would contradict $O_{\text{bi},i} = O_2^*$. Since all simple trios are feasible, $\tilde{O}_{\text{bi},i} = O_{\text{bi},i} = O_2^*$.

Now assume $O_{\text{bi},i} > O_2^*$. First, suppose that all pairs of sub odd bonds are properly oriented. If all simple trios are feasible, then $\tilde{O}_{\text{bi},i} = O_{\text{bi},i} > O_2^*$. If there is a simple trio that is not feasible, then $O_{\text{bi},i} > \tilde{O}_{\text{bi},i}$ and there exists a reconstruction of a type other than an island-free bi-connected reconstruction with a widget at i with fewer odd bonds. Therefore $\tilde{O}_{\text{bi},i} > O_2^*$. Next suppose that one of the pairs of sub odd bonds for i is not properly oriented. By its definition, $\tilde{O}_{\text{bi},i} = \infty > O_2^*$.

Having established proxies for each term in (4), it follows that $O_2^* = \tilde{O}_2^*$. The following theorem summarizes the set of all minimum odd bond reconstructions for 2-run boundaries.

Theorem 2 (2-Run MAP Reconstructions). *Consider a 2-run boundary* $\underline{r} = (b_1, w_1, b_2, w_2)$ *with no boundary run containing four corners.*

(A) $O_2^* = \tilde{O}_2^* \triangleq \min\left\{\tilde{O}_{2,b}, \tilde{O}_{2,w}, \tilde{O}_{\text{bi},1}, \tilde{O}_{\text{bi},2}, \tilde{O}_{\text{bi},3}, \tilde{O}_{\text{bi},4}\right\}$.

(B) If $\tilde{O}_{2,w} = \tilde{O}_2^*$*, then all reconstructions generated by a pair of simple white reconstruction paths are optimal, and no pair of simple white reconstruction paths touch.*

(C) If $\tilde{O}_{2,b} = \tilde{O}_2^*$*, then all reconstructions generated by a pair of simple black reconstruction paths are optimal, and no pair of simple black reconstruction paths touch*

(D) If $\tilde{O}_{\text{bi},i} = \tilde{O}_2^*$*, then all bi-connected, island-free reconstructions with a widget at boundary odd bond* i *generated by a simple trio of reconstruction paths are optimal. Such trios are feasible, and the reconstructions are the only bi-connected, island-free reconstructions with a widget at odd boundary bond* i *that are optimal.*

(E) If an $\tilde{O}$ *terms in (A) does not equal* $\tilde{O}_2^*$*, then no reconstruction of the corresponding type will be optimal.*

(F) There are no other optimal reconstructions.

8 Concluding Remarks

This paper has derived MAP estimates for blocks conditioned on a boundary with 1 or 2 runs. Unlike traditional applications of Ford-Fulkerson, our solutions are closed-form and are semantically informative with respect to the motivating image reconstruction problem. Looking forward, the solutions in this paper for 1- and 2-run boundaries will be useful for boundaries with more than 2 runs, as island-free reconstructions are determined by those runs that contain a corner. Moreover, there are potentially fruitful connections to be explored between our motivating sampling and reconstruction problem and recent developments in the connections between Markov image models and deep learning [10].

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