1. In an MLB division series, teams A and B face off in a series of games where the first team to win three games wins the series. If team A wins the series, how many possible scenarios are there for the series to play out? (For example, one possible scenario is for team A to win the first game, team B to win the second, and team A to win the third and fourth games.)

   Proposed by Jiming Chen and Kevin Chen

   Answer: $\boxed{10}$

   Solution: There is 1 way for the series to end in 3 games; there are 3 ways for the series to end in 4 games; and there are $\binom{4}{2} = 6$ ways for the series to end in 5 games. So the answer is $1 + 3 + 6 = \boxed{10}$ ways.

2. For each integer m between 1 and 2024! (inclusive) and each integer n between 1 and 2024 (inclusive), Bradley writes down the remainder of m when divided by n. Find the mean value of all $2024! \cdot 2024$ remainders that Bradley wrote down.

   Proposed by Bradley Guo

   Answer: $\boxed{\frac{2023}{4}}$

   Solution: Since 2024! is divisible by each integer between 1 and 2024, inclusive, if n is fixed, then the expected value of the remainder is

$$\frac{0 + 1 + \cdots + (n-1)}{n} = \frac{n-1}{2}.$$

   So the answer is

$$\frac{1}{2024} \sum_{n=1}^{2024} \frac{n-1}{2} = \frac{1}{2024} \frac{2024 \cdot 2023}{2 \cdot 2} = \boxed{\frac{2023}{4}.}$$

3. A farmer from Ithaca has a 30 acre farm and 22 days to plant bundles of crops where he plants one bundle at a time. The farmer can either plant a bundle of wheat which takes 3 acres and 1 day to plant, or he can plant a bundle of corn which takes 2 acres and 2 days to plant. Once a piece of land is used to plant a bundle of crop, it cannot be used again to plant another bundle of crop in the future. What is the maximum number of bundles of crops that the farmer can plant?

   Proposed by Rowan Hess

   Answer: $\boxed{13}$

   Solution: Say the farmer plants x bundles of wheat and y bundles of corn. We wish to maximize $x + y$. From the conditions of the problem,

$$3x + 2y \leq 30$$
$$x + 2y \leq 22.$$

   Adding these inequality, we see that $x + y \leq 13$. We can reach equality (and satisfy the two above inequalities) if we set $x = 4$ and $y = 9$, so the answer is $\boxed{13.}$

4. Let $\triangle ABC$ be a triangle with $AB = 2023$, $BC = 2024$, and $CA = 2025$. A point P is chosen uniformly at random on the interior of $\triangle ABC$. Compute the probability that the closest side of $\triangle ABC$ to P is AB.

   Proposed by Kevin Chen and Bradley Guo

   Answer: $\boxed{\frac{2023}{6072}}$

Solution: Let I be the incenter of $\triangle ABC$. Recall that the line AI is the angle bisector of $\angle BAC$, and so the line AI is the locus of all points on the place that are equidistant to the lines AB and AC. Likewise, the line BI is the locus of all points that are equidistant to the lines AB and CB. Hence, the region of points whose closest side is AB is precisely $\triangle ABI$.

Recall the area of $\triangle ABI = \frac{1}{2}r \cdot AB$ where r is the inradius. We have similar formulas for the areas of $\triangle BCI$ and $\triangle CAI$. Hence, the answer is

$$\begin{aligned}\frac{\text{area of } \triangle ABI}{\text{area of } \triangle ABC} &= \frac{\text{area of } \triangle ABI}{(\text{area of } \triangle ABI) + (\text{area of } \triangle BCI) + (\text{area of } \triangle CAI)} \\ &= \frac{\frac{1}{2}r \cdot AB}{\frac{1}{2}r(AB + BC + CA)} \\ &= \frac{AB}{AB + BC + CA} \\ &= \boxed{\frac{2023}{6072}}.\end{aligned}$$

5. Find the sum of all odd integers $k \geq 1$ such that the base-2 representation of $\frac{1}{k}$ is periodic with period exactly 16.

Proposed by Kevin Chen

Answer: $\boxed{111024}$

Solution: An integer k satisfies the given condition if and only if k divides $2^{16} - 1$ but does not divide $2^8 - 1$. Notice that

$$2^{16} - 1 = (2^8 - 1)(2^8 + 1) = 255 \cdot 257.$$

Because 257 is prime, we find all such k take the form $k = 257 \cdot d$ where $d \mid 255$. As the prime factorization of 255 is $3 \cdot 5 \cdot 17$, the sum of all such k is

$$257(1 + 3)(1 + 5)(1 + 17) = \boxed{111024.}$$

6. Alex has a tetrahedron $ABCD$ with side lengths $AB = AC = AD = 1$ and $BC = CD = DB = \sqrt{2}$. After each second, he moves each of the four vertices A, B, C, D *instantaneously* so that each vertex is moved to the centroid of the triangle formed by the other three vertices. Suppose after n seconds, vertex A is at point X_n, where we let X_0 denote the point that A originally starts at. Compute the length of X_0X_{2024}.

Proposed by Kevin Chen

Answer: $\boxed{\frac{\sqrt{3}}{4}\left(1 - \frac{1}{3^{2024}}\right)}$

Solution: Let

$$a_0 = (0, 0, 0), \qquad b_0 = (1, 0, 0), \qquad c_0 = (0, 1, 0), \qquad d_0 = (0, 0, 1).$$

Then

$$a_{n+1} = \frac{b_n + c_n + d_n}{3} = \frac{4g - a_n}{3}.$$

where $g = (1/4, 1/4, 1/4)$ is the centroid of $ABCD$. This means

$$a_n = \frac{4}{3}g\left(1 - \frac{1}{3} + \frac{1}{9} - \cdots + \frac{1}{(-3)^{n-1}}\right) + \frac{1}{(-3)^{n-1}}a_0 = g\left(1 - \frac{1}{(-3)^n}\right).$$

The answer is thus

$$\|a_{2024}\| = \boxed{\frac{\sqrt{3}}{4}\left(1 - \frac{1}{3^{2024}}\right)}.$$

7. How many non-empty subsets S are there of $\{0, 1, 2, 3, 4, 5, 6\}$ such that for all $a, b \in S$ (where a and b are not necessarily distinct), the remainder when ab is divided by 7 is not in S?

   Proposed by Rowan Hess

   Answer: $\boxed{13}$

   Solution: Clearly, 0 cannot be in S. By looking at the exponent of a generator modulo 7, the problem is equivalent to finding subsets T of $\{0, 1, 2, 3, 4, 5\}$ where the remainder of the sum of any two elements in T when divided by 6 is not in T. Separate the problem into cases by the number of odds in this set.

   - 0: T is either the singleton containing 2 or 4 $\to$ 2.
   - 1: If the odd is 3, then T can either contain 2 or 4 or neither. If the odd is 1, then the only even in T can be 4. 5 works the same as 1, but with 2 instead of 4. $\to$ 7.
   - 2: T cannot contain any even numbers $\to$ 3.
   - 3: T cannot contain any even numbers $\to$ 1

   The answer is thus $2 + 7 + 3 + 1 = \boxed{13.}$

8. Mikh independently samples 80 points $x_1, x_2, \ldots, x_{80}$ uniformly at random from the interval $[0, 1]$. For each pair of distinct integers $1 \le i, j \le 80$ such that the interval $[x_i, x_j]$ contains the point $\frac{1}{2}$, Mikh creates a new piece of string with length $x_j - x_i$. What is the expected combined length of all strings that Mikh creates?

   Proposed by Michael Ngo and Minh Pham

   Answer: $\boxed{790}$

   Solution: Let X_{ij} be the length of string contributed by either $[x_i, x_j]$ or $[x_j, x_i]$. The probability that neither $[x_i, x_j]$ or $[x_j, x_i]$ contains $\frac{1}{2}$ is $\frac{1}{2}$. Otherwise, the expected length of string will be $0.75 - 0.25 = 0.5$. Hence,

$$\mathbb{E}[X_{ij}] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0.5 = \frac{1}{4}.$$

   Hence, the answer is

$$\binom{80}{2} \cdot \frac{1}{4} = \boxed{790.}$$

9. Find the unique positive integer N that minimizes the ratio between N and the cube of the number of factors of N.

   Proposed by Bradley Guo

   Answer: $\boxed{2520}$

   Solution: For each prime factor p, multiplying N which has a factors of p by another factor of p is equivalent to multiplying the ratio by

$$\frac{p}{(a+2)^3/(a+1)^3}$$

   For each prime p, we should find the minimum possible a such that $p > \frac{(a+2)^3}{(a+1)^3}$.

- $p = 2 \implies a = 3$.
- $p = 3 \implies a = 2$.
- $p = 5 \implies a = 1$.
- $p = 7 \implies a = 1$.
- If $p > 7$, then $a = 0$.

Hence, $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = \boxed{2520.}$

10. Let $n = 2024$. Define real numbers $0 < w_1, w_2, \ldots, w_n < 1$ such that $w_1 + w_2 + \cdots + w_n = 1$ and

$$\frac{w_1 + w_2 + \cdots + w_k}{w_1} = 2^{k-1}$$

for all $1 \le k \le n$. Given n points $X_1, X_2, \ldots, X_n \in \mathbb{R}^2$, define the *epicenter* of the n-tuple of points $(X_1, X_2, \ldots, X_n)$ to be the point

$$w_1X_1 + w_2X_2 + \cdots + w_nX_n \in \mathbb{R}^2.$$

Alex starts with $n+1$ points $A_0, A_1, \ldots, A_n \in \mathbb{R}^2$ such that $A_0A_1 \cdots A_n$ is a regular $(n+1)$-gon inscribed in the unit circle centered at the origin of $\mathbb{R}^2$. Starting with the 0th second, Alex begins moving points after each second where at the k-th second, he moves the point A_k to the epicenter of the n-tuple of points $(A_{k+1}, A_{k+2}, \ldots, A_{k+n})$, where indices are taken modulo $n + 1$. Let d_k denote the distance between the origin of $\mathbb{R}^2$ and the point A_0 at the k-th second. Given that d_k converges to d as $k \to \infty$, compute d^{-1}.

Proposed by Kevin Chen

Answer: $\boxed{(2^{2024} - 1)\sqrt{\dfrac{5 - 4\cos(\frac{2\pi}{2025})}{1 + 2^{4048} - 2^{2025}\cos(\frac{2\pi}{2025})}}}$

Solution: We work in $\mathbb{C}$ instead of $\mathbb{R}^2$. Given *any* points $B_1, \ldots, B_n \in \mathbb{C}$, define the sequence $\{B_k\}_{k=1}^{\infty}$ by

$$B_k = w_1B_{k-n} + w_2B_{k-(n-1)} + \cdots + w_nB_{k-1}$$

for each $k \ge n + 1$. Then let $f(B_1, \ldots, B_n)$ denote the limit $\lim_{k\to\infty} B_k$. (i.e. f is a function of the form $\mathbb{C}^n \to \mathbb{C}$.) We then desire to compute

$$\frac{1}{|f(A_1, \ldots, A_n)|}.$$

Let $b, B'_1, \ldots, B'_n \in \mathbb{C}$. Also, let $B = (B_1, \ldots, B_n) \in \mathbb{C}^n$ and $B' = (B'_1, \ldots, B'_n) \in \mathbb{C}^n$. Then observe that

$$f(bB) = \lim_{k\to\infty} bB_k = b \lim_{k\to\infty} B_k = bf(B_1, \ldots, B_n) \quad (1)$$

$$f(B + B') = \lim_{k\to\infty} (B_k + B'_k) = \lim_{k\to\infty} B_k + \lim_{k\to\infty} B'_k = f(B) + f(B') \quad (2)$$

If we let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^n$ where the 1 appears in the i-th position, then

$$\begin{aligned} f(B_1, \ldots, B_n) &= f(B_1\mathbf{e}_1 + \cdots + B_n\mathbf{e}_n) \\ &= f(B_1\mathbf{e}_1) + \cdots + f(B_n\mathbf{e}_n) && \text{from Equation (2)} \\ &= B_1f(\mathbf{e}_1) + \cdots + B_nf(\mathbf{e}_n) && \text{from Equation (1).} \end{aligned}$$

If we let $\beta_i = f(\mathbf{e}_i)$, then this can be rewritten as

$$f(B_1, \ldots, B_n) = \beta_1B_1 + \cdots + \beta_nB_n \quad (3)$$

We will find β_i explicitly. Notice that the sequences $\{B_k\}_{k=1}^{\infty}$ and $\{B_{k+1}\}_{k=1}^{\infty}$ both converge to the same value. This means

$$f(B_1, \dots, B_n) = f(B_2, B_3, \dots, B_n, w_1 B_1 + \cdots + w_n B_n)$$

since the limit remains the same by simply shifting the sequence one to the left. We can use Equation (3) to rewrite the abvoe as

$$\begin{aligned}\beta_1 B_1 + \cdots + \beta_n B_n &= \beta_1 B_2 + \beta_2 B_3 + \cdots + \beta_{n-1} B_n + \beta_n (w_1 B_1 + \cdots + w_n B_n). \\ &= \beta_n w_1 B_1 + (\beta_1 + \beta_n w_2) B_2 + (\beta_2 + \beta_n w_3) B_3 + \cdots + (\beta_{n-1} + \beta_n w_n) B_n\end{aligned}$$

Equating coefficients of the B_i's yields

$$\begin{aligned}\beta_1 &= \beta_n w_1 \\ \beta_2 &= \beta_1 + \beta_n w_2 = \beta_n (w_1 + w_2) \\ \beta_3 &= \beta_2 + \beta_n w_3 = \beta_n (w_1 + w_2 + w_3) \\ &\vdots \\ \beta_n &= \beta_{n-1} + \beta_n w_n = \beta_n (w_1 + \cdots + w_n).\end{aligned}$$

Hence, we have the following ratios:

$$\begin{aligned}\beta_1 : \beta_2 : \cdots : \beta_n &= w_1 : w_1 + w_2 : \cdots : w_1 + \cdots + w_n \\ &= 1 : 2 : 2^2 : \cdots : 2^{n-1}\end{aligned}$$

where the last equality uses the assumption on w_i given in the problem statement. Moreover, if we let $B_k = 1$ for $1 \le k \le n$, then the sequence $\{B_k\}_{k=1}^{\infty}$ is just the constant sequence $1, 1, \dots$ since the the epicenter of $1, 1, \dots, 1$ is 1. In particular, the sequence $\{B_k\}_{k=1}^{\infty}$ converges to 1, and so from Equation (3),

$$1 = f(1, 1, \dots, 1) = \beta_1 + \cdots + \beta_n.$$

Combining the previous two equations yields

$$f(B_1, \dots, B_n) = \frac{B_1 + 2B_2 + 2^2 B_3 + \cdots + 2^{n-1} B_n}{1 + 2 + \cdots + 2^{n-1}}.$$

Let A denote the point that A_0 converges to, and let $\omega = e^{2\pi i/(n+1)}$. Then

$$A = f(A_1, \dots, A_n) = \frac{\omega + 2\omega^2 + \cdots + 2^{n-1}\omega^n}{1 + 2 + \cdots + 2^{n-1}} = \frac{1}{2^n - 1} \cdot \frac{\omega - 2^n \omega^{n+1}}{1 - 2\omega} = \frac{1}{2^n - 1} \cdot \frac{\omega - 2^n}{1 - 2\omega}.$$

The answer is then

$$|A|^{-1} = (2^n - 1)\frac{|1 - 2\omega|}{|\omega - 2^n|} = \boxed{(2^n - 1)\sqrt{\frac{5 - 4\cos(\frac{2\pi}{n+1})}{1 + 2^{2n} - 2^{n+1}\cos(\frac{2\pi}{n+1})}}}.$$