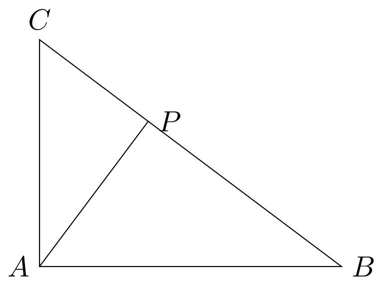
1. Let $\triangle ABC$ be a non-degenerate triangle. Suppose there exists a point P on side BC such that AP splits $\triangle ABC$ into two non-degenerate triangles that are both similar to $\triangle ABC$. Prove that $\angle BAC = 90^\circ$.

   Proposed by Kevin Chen

   Solution: Since $\angle APC + \angle APB = 180^\circ$, if $\angle APC$ was equal to some angle of $\triangle ABP$ that is not $\angle APB$, then the remaining angle would have to be 0°. Thus, it must be that $\angle APC = \angle APB = 90^\circ$. Thus, $\triangle ABC$ has a right angle. As $\angle APC = \angle APB = 90^\circ$, both $\angle ACP$ and $\angle ABP$ are acute, it must be $\angle BAC = 90^\circ$, as desired.



2. Prove that if x, y, z are integers such that

$$5x^2 + 2y^2 - z^2 = 2xy + 2yz,$$

   then $x = y = z = 0$.

   Proposed by Kevin Chen

   Solution: Let (*) denote the equation. Suppose (x, y, z) is solution with not all x, y, z being 0. Since (*) on is homogeneous, we can divide by common factors to reach an integer solution that is relatively prime with each other. WLOG assume that (x, y, z) is an integer solution such that $\gcd(x, y, z) = 1$. If we take (*) modulo 2, then

$$x^2 \equiv z^2 \pmod 2 \implies x \equiv z \pmod 2.$$

   If $x \equiv z \equiv 0 \pmod 2$, then taking (*) modulo 4 yields

$$2y^2 \equiv 0 \pmod 4 \implies y \equiv 0 \pmod 2.$$

   But this contradicts $\gcd(x, y, z) = 1$. Hence, $x \equiv z \equiv 1 \pmod 2$. If we take (*) modulo 4 again, then

$$2y^2 \equiv 2xy + 2yz \pmod 4 \implies y^2 \equiv xy + yz \equiv 0 \pmod 2 \implies y \equiv 0 \pmod 2.$$

   Let $y = 2a$ for $a \in \mathbb{Z}$. If we take (*) modulo 8, then

$$4 \equiv 4a(x + z) \pmod 8 \implies 1 \equiv a(x + z) \equiv 0 \pmod 2.$$

   But this is clearly a contradiction. Hence, there are no nontrivial solutions to (*).

3. Let $n \geq 3$ be an integer. The integers from 1 to n, inclusive, are written around a circle in some order. We say an unordered pair of integers on the circle is *Cornellian* if they don't occupy neighboring positions and at least one of the two arcs they enclose contains exclusively integers that are smaller than both of the pair. For example, suppose $n = 6$ and the integers are placed around the circle in the following order: 1, 4, 3, 2, 5, 6. Then the pair $\{4, 5\}$ is *Cornellian* because the arc between 4 and 5 containing 2 and 3 only contains integers that are less than both 4 and 5.

For each integer $n \geq 3$, find all integers $k \geq 0$ such that there exists a configuration of the integers 1 to n, inclusive, on the circle with exactly k *Cornellian* pairs.

Proposed by Lucas Sandleris

Solution: We will see that the only such k is $n-3$. We proceed by induction. For $n=3$ this is trivial, as any two numbers will occupy neighboring positions on the circle, so there can be no Cornellian pairs (and $n-3 = 3-3 = 0$). Now, suppose that, for some $n \geq 3$, any configuration has $n-3$ Cornellian pairs. Consider a configuration of the numbers from 1 through $n+1$ on the circle now. Let i, j be the neighbors to number 1. As $n+1 \geq 4$, i and j cannot occupy neighboring positions. And, since one of the arcs they form only contains number $1 \leq i, j$, $\{i, j\}$ is a Cornellian pair. Now, consider the configuration resulting from removing number 1 from our original configuration, and then subtracting 1 to each one of the remaining numbers. This clearly yields a configuration of the numbers 1 through n. Now, as 1 is less than all other numbers in the original configuration, adding/removing 1 doesn't change whether an arc contains exclusively elements less than the numbers in both its ends. Also, number 1 cannot be part of any Cornellian pair by the same reason. Therefore, the only change in the set of Cornellian pairs after removing number 1 is that i, j now occupy neighboring positions, so $\{i, j\}$ is not a Cornellian pair anymore. Then, subtracting 1 to every number in the circle makes no change in the number of Cornellian pairs, as it preserves ordering. Therefore, the new configuration with the numbers 1 through n has exactly one Cornellian pair less than the original configuration. But, by the inductive hypothesis, it also has exactly $n-3$ Cornellian pairs. Therefore, the original configuration of the numbers 1 through $n+1$ had exactly $n-3+1 = (n+1)-3$ Cornellian pairs, concluding the proof.

4. Let $\{x_n\}_{n\geq 0}$ be a sequence given by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+2} = x_n + \sqrt{21x_{n+1}^2 + 4}$$

for all integers $n \geq 0$. Show that all terms of the sequence are integers.

Proposed by Lucas Sandleris

Solution: The key observation is that $x_{n+2} - x_n = \sqrt{21x_{n+1}^2 + 4}$ looks like the square root of the discriminant of a quadratic, so we might guess that x_n and x_{n+2} are the roots of a quadratic with coefficients involving x_{n+1}. In fact, we will show that

$$x_n^2 - 5x_n x_{n+1} + x_{n+1}^2 = 1 \tag{1}$$

for each integer $n \geq 0$. We use induction. It is trivial to check that it works for $n = 0$. Now assume the result holds for a fixed integer $n \geq 0$. Then

$$\begin{aligned} x_{n+2}^2 - 5x_{n+2}x_{n+1} + x_{n+1}^2 &= \left(x_n + \sqrt{21x_{n+1}^2 + 4}\right)^2 - 5x_{n+1}\left(x_n + \sqrt{21x_{n+1}^2 + 4}\right) + x_{n+1}^2 \\ &= (x_n^2 - 5x_n x_{n+1} + x_{n+1}^2) + (21x_{n+1}^2 + 4) + (2x_n - 5x_{n+1})\sqrt{21x_{n+1}^2 + 4} \\ &= 1 + (21x_{n+1}^2 + 4) + (2x_n - 5x_{n+1})\sqrt{21x_{n+1}^2 + 4} \end{aligned}$$

Notice that this is equal to 1 if and only if

$$0 = (21x_{n+1}^2 + 4) + (2x_n - 5x_{n+1})\sqrt{21x_{n+1}^2 + 4} \iff x_n = \frac{5x_{n+1} - \sqrt{21x_{n+1}^2 + 4}}{2}.$$

But this is true since x_n is a root of the quadratic $t^2 - 5x_{n+1}t + x_{n+1}^2$ from the induction hypothesis and $x_n < x_{n+1}$. This proves Equation (1).

We now use induction to show x_n is an integer for each integer $n \geq 0$. This is clearly true for $n = 0$ and $n = 1$. Now assume that, for some $n \geq 1$, both x_n and x_{n+1} are integers. By Equation (1), the values x_n and x_{n+2} are roots of the quadratic $t^2 - 5x_{n+1}t + x_{n+1}^2$. In particular, the sum of the roots is

$$x_n + x_{n+2} = 5x_{n+1} \implies x_{n+2} = 5x_{n+1} - x_n.$$

As x_n and x_{n+1} are integers, so is x_{n+2}. This completes the induction and thus the proof.