1. Define the sequence $\{a_n\}_{n\geq 0}$ such that $a_0 = a_1 = 2024^{2024!}$ and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for all integers $n \geq 2$. Compute a_{2024}.

   Proposed by Kevin Chen

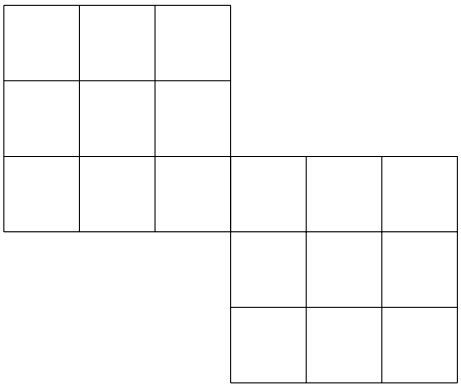
   Answer: $\boxed{1}$

   Solution: We can compute the first values of a_n in terms of a_0 and a_1 in the following table:

n	a_n
0	a_0
1	a_1
2	a_1/a_0
3	$1/a_0$
4	$1/a_1$
5	a_0/a_1
6	a_0
7	a_1

   Therefore, the sequence has period 6. As $2024 \equiv 2 \pmod 6$, we conclude $a_{2024} = a_2 = a_1/a_0 = \boxed{1.}$

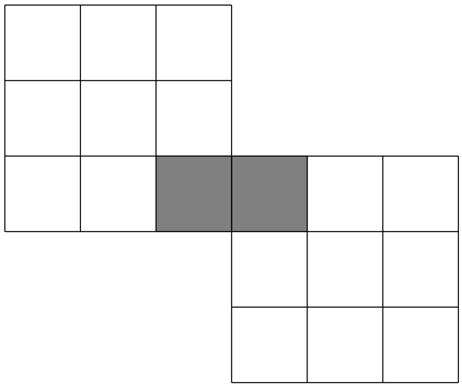
2. Two 3×3 squares are placed next to each other so that they share an edge of length 1 as shown in the figure below. How many ways can we cover all 18 squares with 2×1 tiles?



   Proposed by Rowan Hess

   Answer: $\boxed{16}$

   Solution: We need to put a tile in the grey region since otherwise, we would have to put tiles in a 9×9 grid, which is impossible since each tile occupies an even amount of grids. After putting the that tile down, it is not hard to see that there are 4 ways to cover the central square of the upper left 3×3 square, and the same holds for the bottom left 3×3 square. Each of the ways uniquely determines the complete covering of each of the 3×3 squares.
   The answer is then $4 \cdot 4 = \boxed{16.}$



3. A fair 10-sided die with sides labeled $1, 2, \ldots, 10$ is rolled three times. What is the probability that the median of these three rolls is 3?

   Proposed by Rowan Hess

   Answer: $\boxed{\frac{14}{125}}$

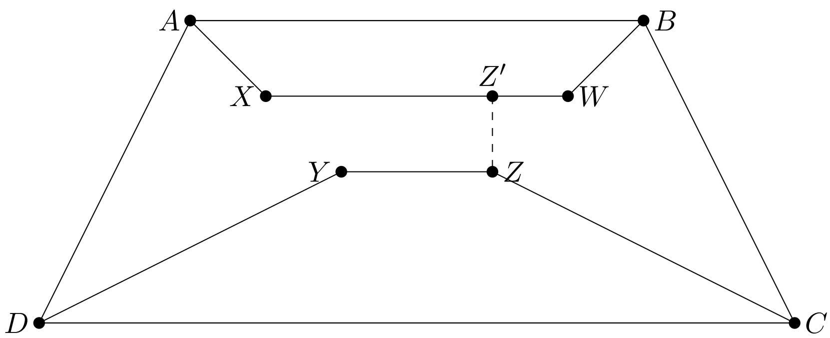
   Solution: Let $n = 10$. By counting the number of 3s that appear in the rolls, we see that the three rolls must be one of the following:

$$x, 3, y; \qquad 3, 3, y; \qquad x, 3, 3; \qquad 3, 3, 3.$$

   where $1 \leq x \leq 2$ and $4 \leq y \leq n$. This occurs with probability

$$\frac{6 \cdot 2 \cdot (n-3) + 3 \cdot (n-3) + 3 \cdot 2 + 1}{n^3} = \frac{15(n-3)+7}{n^3} = \boxed{\frac{14}{125}.}$$

4. Given an isosceles trapezoid $ABCD$ with $AB \parallel CD$, let W, X, Y, Z be points inside $ABCD$ such that $WXAB$ and $YZCD$ are isosceles trapezoids that do not overlap each other and with $WX \parallel AB$ and $YZ \parallel CD$. Suppose that $AB + WX = CD + YZ = 20$ and $ZZ' = 4$ where Z' is the point on line WX such that $ZZ' \perp WX$. Given that the height of trapezoid $ABCD$ is 29, compute the combined area of trapezoids $WXAB$ and $YZCD$.



   Proposed by Kevin Chen

   Answer: $\boxed{250}$

   Solution: Let h_1 and h_2 denote the heights of trapezoids $WXAB$ and $YZCD$. Notice that $h_1 + h_2 = 29 - 5 = 24$. Their combined area is then

$$\frac{1}{2}h_1(WX + AB) + \frac{1}{2}h_2(YZ + CD) = \frac{1}{2}h_1 \cdot 20 + \frac{1}{2}h_2 \cdot 20 = 10(h_1 + h_2) = 10 \cdot 25 = \boxed{250.}$$

5. Alex has 42 pairwise distinct positive integers. He takes each integer and computes its remainder when divided by 6. Of these 42 remainders, exactly 7 of them evaluate to 0, exactly 7 of them evaluate to 1, exactly 7 of them evaluate to 2, and so on. Of the original 42 integers that Alex started with, what is the maximum number of prime numbers could he have had?

   Proposed by Kevin Chen

   Answer: $\boxed{16}$

   Solution: All integers that are 0, 2, or 4 modulo 6 are divisible by 2. All integers that are 3 modulo 6 are divisible by 3. Hence, the only prime numbers that are 0, 2, 3, or 4 modulo 6 are 2 and 3. It is easy to find 7 primes that are 1 modulo 6 and and 7 primes that are 5 modulo 6:

$$\begin{aligned} &\text{primes} \equiv 1 \text{ modulo } 6: \quad 7, 13, 19, 31, 37, 43, 61 \\ &\text{primes} \equiv 5 \text{ modulo } 6: \quad 5, 11, 17, 23, 29, 41, 47 \end{aligned}$$

   (Alternatively, one could use Dirichlet's theorem on primes in an arithmetic progression for a non-constructive proof that such an example exists.) The answer is then $7+7+1+1 = \boxed{16.}$

6. Bradley can perform one of two operations on an integer: he can either square it or add 1 to it. If Bradley starts with the integer 1, what is the minimum number of operations that Bradley needs to perform to reach exactly 1000.

   Proposed by Bradley Guo

   Answer: $\boxed{50}$

   Solution: By induction, for each integer $n \geq 1$, one can see that the optimal strategy to get to n is to recursively perform the optimal strategy to get $\lfloor\sqrt{n}\rfloor$, square it, and then keep adding 1 until you get to n. The optimal strategy to get to 1000 is then given by the sequence

$$1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 25 \longrightarrow 26 \longrightarrow 27 \longrightarrow \cdots \longrightarrow 31$$

$$\longrightarrow 961 \longrightarrow 962 \longrightarrow 963 \longrightarrow \cdots \longrightarrow 1000.$$

   Thus, the answer is $\boxed{50.}$

7. Let $S(n)$ denote the sum of all digits of n in base 10. Find the number of integers $1 \leq n \leq 2024$ such that 11 divides $n - S(n)$.

   Proposed by Rowan Hess

   Answer: $\boxed{99}$

   Solution: Write $n = \overline{abcd}$ in base 10. Then

$$n - S(n) = (1000a + 100b + 10c + d) - (a + b + c + d) = 999a + 99b + 9c \equiv 9a + 9c \pmod{11}.$$

   Hence, 11 divides $n - S(n)$ if and only if 11 divides $a + c$. Since $n \leq 2024$, this is true if and only if $a = c = 0$. Therefore, the number of such integers n is the number of possibilities for (b, d) that make $1 \leq \overline{0b0d} \leq 2024$, which is $10 \cdot 10 - 1 = \boxed{99.}$

8. Let $\triangle ABC$ be a right triangle such that $\angle ABC = 90^\circ$ and the altitude from B onto AC has length $\sqrt{6}$. If AB^2 and BC^2 are both integers, find the maximum possible area of $\triangle ABC$.

   Proposed by Bradley Guo

Answer: $\boxed{\frac{7\sqrt{6}}{2}}$

Solution: Let $x = AB^2$ and $y = BC^2$. Then

$$xy = 6(x+y) \iff (x-6)(y-6) = 36.$$

There are only finitely many choices for x and y since they are integers. Recall we wish to maximize the area which is given by $\frac{1}{2}\sqrt{xy}$. If we assume $x \le y$, then we have the following cases:

$$\begin{aligned}
x-6 = 1, y-6 = 36 &\implies xy = 7 \cdot 42 = 294 \\
x-6 = 2, y-6 = 18 &\implies xy = 8 \cdot 24 = 192 \\
x-6 = 3, y-6 = 12 &\implies xy = 9 \cdot 18 = 162 \\
x-6 = 4, y-6 = 9 &\implies xy = 10 \cdot 15 = 150 \\
x-6 = 6, y-6 = 6 &\implies xy = 12 \cdot 12 = 144.
\end{aligned}$$

Hence, the answer is $\boxed{\frac{7\sqrt{6}}{2}.}$

9. Find the number of permutations $a_1, a_2, \ldots, a_{20}$ of the integers $1, 2, \ldots, 20$ such that for all integers $1 \le i, j \le 20$, if i divides j, then a_i divides a_j.

Proposed by Kevin Chen

Answer: $\boxed{24}$

Solution: Suppose $a_1, \ldots, a_{20}$ satisfies the desired property. We proceed with a divide-and-conquer method. Notice that no integer in $[11, 20]$ divides any integer in $[1, 20]$ except itself. This means $a_{11}, a_{12}, \ldots, a_{20}$ is a permutation of $11, \ldots, 20$. It then follows that $a_1, \ldots, a_{10}$ is a permutaiton of $1, \ldots, 10$. By doing the same process of halfing everything, we see that $a_1, \ldots, a_5$ is a permutation of $1, \ldots, 5$, and $a_6, \ldots, a_{10}$ is a permutation of $6, \ldots, 10$.

We will show $a_i = i$ for all $1 \le i \le 10$. Notice that a_1 divides a_i for all i. This can only happen if $a_1 = 1$. Because $a_1, \ldots, a_5$ is a permutation of $1, \ldots, 5$, the condition $a_2 \mid a_4$ can only be true if we set $a_1 = 1, a_2 = 2, a_4 = 4$ and $\{a_3, a_5\} = \{3, 5\}$.

We now use the fact that $a_6, \ldots, a_{10}$ is a permutation of $6, \ldots, 10$.

- 7 is prime $\implies a_7 = 7$. (If 7 is equal to a_i for some composite i, then a_i would be divisible by $a_j \ne 1$ where $1 < j < i$, which is impossible.)
- $4 = a_4 \mid a_8 \implies a_8 = 8$.
- We have $a_2, a_3 \mid a_6$ and $a_2, a_5 \mid a_{10}$. Recall $a_2 = 2$ and $\{a_3, a_5\} = \{3, 5\}$. Hence, $\{a_6, a_{10}\} = \{6, 10\}$. This leaves $a_9 = 9$, which must be divisible by a_3. Thus, $a_3 = 3$, $a_5 = 5$, $a_6 = 6$, and $a_{10} = 10$.

This completes showing $a_i = i$ for all $1 \le i \le 10$. We will now use casework again to show $a_i = i$ for all $11 \le i \le 20$ such that i is not prime.

- Because $11, 13, 17, 19$ are prime, $a_{11}, a_{13}, a_{17}, a_{19}$ is a permutation of $11, 13, 17, 19$. (Like before, if one of these primes is equal to a_i for some composite i, then a_i would be divisible by $a_j \ne 1$ where $1 < j < i$, which is impossible.)
- If $11 \le i \le 20$ is even with $i = 2j$ for some $1 \le j \le 10$, then a_i is divisible by $a_2 = 2$ and $a_j = j$, so $a_i = 2 \cdot j = i$.
- If $11 \le i \le 20$ is odd and not prime, then $i = 15$. Because a_{15} is divisible by $a_3 = 3$ and $a_5 = 5$ which are both relatively prime, $a_{15} = 3 \cdot 5 = 15$.

From permuting the four primes $11, 13, 17, 19$ at the positions $a_{11}, a_{13}, a_{17}, a_{19}$, we see the answer is $4! = \boxed{24.}$

10. Let $ABCD$ be a rhombus with side length 4. Suppose the circumcircle of $\triangle ABD$ intersects line segment CD at P. Given that $CP = 1$, find the area of $ABCD$.

    Proposed by Kevin Chen

    Answer: $\boxed{6\sqrt{7}}$

    Solution: Let x and y denote the lengths of the diagonals AC and BD, respectively. Also, let R denote the circumradius $\triangle ABD$. The area of $\triangle ABC$ can be calculated in two ways:

    $$\frac{4 \cdot 4 \cdot y}{4R} = \frac{1}{4}xy \implies Rx = 16.$$

    By power of the point at C,

    $$(x - 2R)x = 4 \implies x^2 = 4 + 2Rx = 36 \implies x = 6.$$

    It follows that $y = 2\sqrt{4^2 - 3^2} = 2\sqrt{7}$. Hence, the area of the rhombus is $\frac{1}{2}xy = \boxed{6\sqrt{7}.}$

11. Let $S = \{1, 2, \ldots, 10\}$. Suppose $f\colon S \to S$ is a function chosen uniformly at random among all possible functions from S to S. Find the probability that $f(f(f(f(1)))) = 1$.

    Proposed by Bradley Guo

    Answer: $\boxed{\frac{601}{2500}}$

    Solution: Fix $x \in S$. For each integer $k \geq 1$, let $P(k)$ denote the probability that the smallest integer n such that $f^n(x) = x$ is $n = k$. Then

    $$\begin{aligned} P(1) &= \frac{1}{10} \\ P(2) &= \frac{9}{10} \cdot \frac{1}{10} \\ P(4) &= \frac{9}{10} \cdot \frac{8}{10} \cdot \frac{7}{10} \cdot \frac{1}{10} \end{aligned}$$

    By linearity of expectation, the desired answer is then

    $$P(1) + P(2) + P(4) = \boxed{\frac{601}{2500}}.$$

12. Let $n = 2024$ and $\omega = e^{2\pi i/n}$. For each integer $1 \leq k \leq n$, let S_k be the set of the first n positive integers with the integer k removed. (For example, $S_3 = \{1, 2, 4, 5, 6, \ldots, n\}$.) Also, for each integer $1 \leq k \leq n$, define

    $$a_k = \prod_{j \in S_k} (2 + \omega^j)$$

    where the product is taken over all values $j \in S_k$. Compute $a_1 + a_2 + \cdots + a_n$.

    Proposed by Kevin Chen

    Answer: $\boxed{2024 \cdot 2^{2023}}$

Solution: Let $P(x) = x^n - 1$. We can rewrite the sum as

$$a_1 + a_2 + \cdots + a_n = (-1)^{n-1}(-2-\omega)(-2-\omega^2)\cdots(-2-\omega^n)\sum_{j=1}^{n} \frac{1}{-2-\omega^j}$$
$$= (-1)^{n-1}P(-2)\sum_{j=0}^{n-1} \frac{1}{-2-\omega^j}.$$

Let $r_j = 2 + \omega^j$. We wish to calculate $\sum_{j=0}^{n-1} 1/r_j$. Notice that $r_0, \ldots, r_{n-1}$ are roots of the polynomial $(x-2)^n - 1$. This means

$$0 = (r_j - 2)^n - 1 = [(-2)^n - 1] + \sum_{k=1}^{n} \binom{n}{k} r_j^k (-2)^{n-k} = P(-2) + \sum_{k=1}^{n} \binom{n}{k} r_j^k (-2)^{n-k}$$

and so,

$$\frac{1}{r_j} = -\frac{1}{P(-2)} \sum_{k=1}^{n} \binom{n}{k} r_j^{k-1} (-2)^{n-k}$$

Recall $\sum_{j=0}^{n-1} \omega^{j\ell} = 0$ for all integers ℓ not divisible by n. In particular, for all integers $1 \le \ell \le n-1$, we have

$$\sum_{j=0}^{n-1} r_j^\ell = \sum_{j=0}^{n-1} (\omega^j + 2)^\ell = n2^\ell + \sum_{j=1}^{n-1}\sum_{r=1}^{\ell} \binom{\ell}{r} \omega^{jr} 2^{\ell - r} = n2^\ell + \sum_{r=1}^{\ell}\sum_{j=1}^{n-1} \binom{\ell}{r} \omega^{jr} 2^{\ell - r} = n2^\ell$$

Using this, we see that summing over all j in the previous equation yields

$$\sum_{j=0}^{n-1} \frac{1}{r_j} = -\frac{1}{P(-2)} \sum_{k=1}^{n} \binom{n}{k} (n2^{k-1})(-2)^{n-k} = -\frac{n(-2)^{n-1}}{P(-2)} \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k}$$
$$= -\frac{n(-2)^{n-1}}{P(-2)}.$$

Plugging this into the original sum shows

$$a_1 + a_2 + \cdots + a_n = n2^{n-1} = \boxed{2024 \cdot 2^{2023}.}$$

Another approach to finding $\sum_{j=0}^{n-1} 1/r_j$ is by writing the summation as a sum of n infinite series as such:

$$\sum_{j=0}^{n-1} \frac{1}{2+\omega^j} = \frac{1}{2}\sum_{j=0}^{n-1} \frac{1}{1+\omega^j/2} = \frac{1}{2}\sum_{j=0}^{n-1}\sum_{r=0}^{\infty} \frac{\omega^j}{(-2)^j}$$
$$= \frac{1}{2}\sum_{r=0}^{\infty} \frac{1}{(-2)^j} \sum_{j=0}^{n-1} \omega^{rj}$$
$$= \frac{1}{2}\sum_{r'=0}^{\infty} \frac{n}{(-2)^{r'n}}$$
$$= \frac{1}{2} \cdot \frac{n}{1 - 1/(-2)^n}$$
$$= \frac{1}{2} \frac{n \cdot (-2)^n}{P(-2)}$$
$$= -\frac{n(-2)^{n-1}}{P(-2)},$$

which is the same result as before.

13. Find the number of injective functions $f\colon \{1, 2, \ldots, 2024\} \to \{1, 2, \ldots, 2024\}$ such that

$$f(x+y) \equiv f(x) + f(y) \pmod{2024}$$

for all integers $1 \le x, y \le 2024$.

(For non-empty sets X and Y, we say a function $f\colon X \to Y$ is *injective* if $f(x) \neq f(x')$ for all distinct $x, x' \in X$.)

Proposed by Kevin Chen

Answer: $\boxed{\phi(n) = 880}$

Solution: Let $n = 2024$. We say f is additive if it satisfies the congruence in the problem. Suppose f is additive. We claim f is injective if and only if $\gcd(f(1), n) = 1$. Notice that the value of $f(1)$ fully determines f since the additive condition implies $f(k) \equiv kf(1) \pmod{n}$ for all integers k. Moreover, if $\gcd(f(1), n) = r > 1$, then

$$f(n/r) = \frac{n}{r} f(1) = n \equiv 0 \pmod{n}.$$

Hence, f is not bijective. Conversely, if f is not bijective, then there exists some nonzero $k \in \mathbb{Z}/n\mathbb{Z}$ such that $kf(1) \equiv f(k) \equiv 0 \pmod{n}$. But this implies $\gcd(f(1), n) > 1$. Hence, f is bijective if and only if $\gcd(f(1), n) = 1$. The answer is then given by $\phi(n) = 880$, as desired.

14. Jiming flips 50 coins and records the resulting sequence of coin flips. Let a be the number of times he flips two heads in a row and b be the number of times he flips two tails in a row. For example, the sequence TTTHHHTT would yield $a = 2$ and $b = 3$. What is the expected value of the product ab?

Proposed by Bradley Guo

Answer: $\boxed{141}$

Solution: Let c_{ij} be 1 if the i^{th} pair of coin flips are both heads and the j^{th} pair of coin flips are both tails. Then the expected sum of c_{ij} over all i and j is the expected value of ab. The expected value of c_{ij} is $\frac{1}{16}$ if $|i-j| \ge 2$ and 0 otherwise. Since there are $n-1$ pairs of coin flips, the number of flips that are at least 2 apart can be calculated using complementary counting as $(n-1)^2 - (n-1) - 2(n-2) = (n-2)(n-3)$. Thus, when $n = 50$, the expected value of ab is $\frac{48 \cdot 47}{16} = \boxed{141.}$

Alternate Solution: We can also approach this problem with recursion. WLOG let the last coin flip after n-1 flips be heads. Let a_i be the number of consecutive heads flips after i flips and b_i be the number of consecutive tails flips after i flips. If the n^{th} coin flip is tails, then a and b remain the same, but if the n^{th} coin flip is heads, then a increases by 1. Thus,

$$\mathbb{E}[a_n b_n] = \frac{1}{2}\mathbb{E}[a_{n-1}b_{n-1}] + \frac{1}{2}\mathbb{E}[a_{n-1}(b_{n-1}+1)] = \mathbb{E}[a_{n-1}b_{n-1}] + \frac{1}{2}\mathbb{E}[a_{n-1}]$$

But since we know that the $(n-1)^{th}$ coin flip is heads, $a_{n-1} = a_{n-2}$. There are $n-3$ pairs of coin flips, each having a $\frac{1}{4}$ probability of being a pair of heads, so the expected number of head pairs after $n-2$ random coin flips is $\frac{n-3}{4}$. The closed form of the recursion $f(n) = f(n-1) + \frac{n-3}{8}$ can be easily shown to be $\frac{(n-2)(n-3)}{16}$, giving $\boxed{141}$ at $n = 50$.

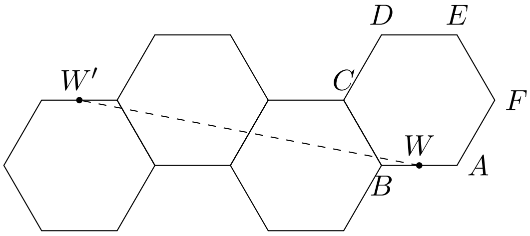
15. Let $ABCDEF$ be a regular hexagon with side length 1, and let W be the midpoint of side AB. Suppose X, Y, and Z are points on sides BC, DE, and FA, respectively, such that W, X, and Z are not collinear. Find the minimum possible value of the perimeter of quadrilateral $WXYZ$.

Proposed by Kevin Chen

Answer: $\boxed{\frac{\sqrt{39}}{2}}$

Solution: Reflect the hexagon along the sides as shown below. This unravels the quadrilateral into a path from W to W' that is composed of four line segments. Finding the optimal quadrilateral is then equivalent to finding the shortest path from W to W', which is a straight line. Therefore, the answer is

$$WW' = \sqrt{3^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \boxed{\frac{\sqrt{39}}{2}.}$$



16. Let $n = 2024$. Given a point $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and an integer $r > 0$, Alex can r-*amplify* the point X to get a new point $X' \in \mathbb{R}^n$ given by

$$X' = (x_1, rx_1 + x_2, r^2x_1 + rx_2 + x_3, \ldots, r^{n-1}x_1 + r^{n-2}x_2 + \cdots + x_n)$$

where the k-th coordinate of X' is $\sum_{i=1}^{k} r^{k-i}x_i$.

Suppose Alex starts with the point $A_0 = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n$ where

$$a_0 = 1, \qquad a_{2k-1} = 0, \qquad a_{2k} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (-1)^k i_1^2 i_2^2 \cdots i_k^2$$

for each integer $k \ge 1$. (The summation is taken over all integers $1 \le i_1 < i_2 \cdots < i_k \le n$.) Alex first 1-amplifies A_0 to get a new point A_1. He then 2-amplifies A_1 to get a new point A_2. He continues this process until he n-amplifies A_{n-1} to get a new point A_n. Compute the sum of the n coordinates of A_n.

Proposed by Kevin Chen

Answer: $\boxed{2025! - 2024! \text{ or } 2024 \cdot 2024!}$

Solution: We use generating functions. The coordinates of A_1 can be expressed by the first $n-1$ coefficients of the polynomial

$$f_1(x) = (1-x^2)(1-2^2x^2)\cdots(1-n^2x^2).$$

Notice that A_k is then given by the first $n-1$ coefficients of

$$f_k(x) = \frac{f(x)}{(1-x)(1-2x)\cdots(1-kx)}.$$

In particular,

$$\begin{aligned} f_n(x) &= \frac{[(1-x)(1-2x)\cdots(1-nx)] \cdot [(1+x)(1+2x)\cdots(1+nx)]}{(1-x)(1-2x)\cdots(1-nx)} \\ &= (1+x)(1+2x)\cdots(1+nx). \end{aligned}$$

The answer is then

$$f_n(1) - n! = (n+1)! - n! = \boxed{2025! - 2024!} = \boxed{2024 \cdot 2024!.}$$