Cornell University®

Big Red Math Competition
**Individual Round**

Name: ____________________

Team ID: ____________________

10/28/2023 10:45–12:45am

**INSTRUCTIONS — PLEASE READ THIS NOW**

- **Write your Name and Team ID.**
- **Simplification.** Please simplify all answers as much as possible. For example, $\frac{\cos\pi}{\frac{1}{2}}$ can be simplified to $-2$.
- **You have 2 hour to complete this exam.**
- This is a closed book exam. You are **NOT** allowed to use a calculator, computer, notes, or any other resources.

**Please sign below to indicate that you have read and agree to these instructions.**

Signature, Print Name: ____________________

**Answer Sheet:**

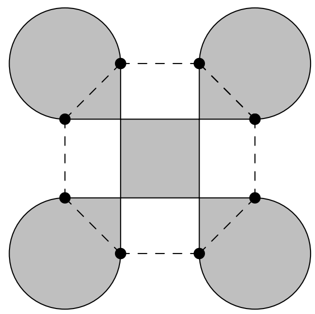
| | | | |
|---|---|---|---|
| 1. ______ | 5. ______ | 9. ______ | 13. ______ |
| 2. ______ | 6. ______ | 10. ______ | 14. ______ |
| 3. ______ | 7. ______ | 11. ______ | 15. ______ |
| 4. ______ | 8. ______ | 12. ______ | 16. ______ |

**Please don't mark anything below on the page, it's for official use only.**

| | | |
|---|---|---|
| Grader 1 Name: ______ | Grader 2 Name: ______ | Grader 3 Name: ______ |
| Tally: ______ | Tally: ______ | Tally: ______ |
| Score: ______ | Score: ______ | Score: ______ |

1. Find the largest 4 digit integer that is divisible by 2 and 5, but not 3.

2. The diagram below shows the eight vertices of a regular octagon of side length 2. These vertices are connected to form a path consisting of four crossing line segments and four arcs of degree measure 270°. Compute the area of the shaded region.



3. Consider the numbers formed by writing full copies of 2023 next to each other, like so:

$$2023202320232023\ldots$$

   How many copies of 2023 are next to each other in the smallest multiple of 11 that can be written in this way?

4. A positive integer $n$ with base-10 representation $n = a_1a_2 \ldots a_k$ is called *powerful* if the digits $a_i$ are nonzero for all $1 \leq i \leq k$ and

$$n = a_1^{a_1} + a_2^{a_2} + \cdots + a_k^{a_k}.$$

   What is the unique four-digit positive integer that is powerful?

5. Six (6) chess players, whose names are Alice, Bob, Crystal, Daniel, Esmeralda, and Felix, are sitting in a circle to discuss future content pieces for a show. However, due to fights they've had, Bob can't sit beside Alice or Crystal, and Esmeralda can't sit beside Felix. Determine the amount of arrangements the chess players can sit in. Two arrangements are the same if they only differ by a rotation.

6. Given that the infinite sum $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$ is equal to $\frac{\pi^4}{90}$, compute the value of

$$\frac{\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots}{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots}$$

7. Triangle $ABC$ is equilateral. There are 3 distinct points, $X$, $Y$, $Z$ inside $\triangle ABC$ that each satisfy the property that the distances from the point to the three sides of the triangle are in ratio $1:1:2$ in some order. Find the ratio of the area of $\triangle ABC$ to that of $\triangle XYZ$.

8. For a fixed prime $p$, a finite non-empty set $S = \{s_1, \ldots, s_k\}$ of integers is *$p$-admissible* if there exists an integer $n$ for which the product

$$(s_1 + n)(s_2 + n) \cdots (s_k + n)$$

is not divisible by $p$. For example, $\{4, 6, 8\}$ is 2-admissible since $(4+1)(6+1)(8+1) = 315$ is not divisible by 2. Find the size of the largest subset of $\{1, 2, \ldots, 360\}$ that is two-, three-, and five-admissible.

9. Kwu keeps score while repeatedly rolling a fair 6-sided die. On his first roll he records the number on the top of the die. For each roll, if the number was prime, the following roll is tripled and added to the score, and if the number was composite, the following roll is doubled and added to the score. Once Kwu rolls a 1, he stops rolling. For example, if the first roll is 1, he gets a score of 1, and if he rolls the sequence $(3, 4, 1)$, he gets a score of $3 + 3 \cdot 4 + 2 \cdot 1 = 17$. What is his expected score?

10. Let $\{a_1, a_2, a_3, ...\}$ be a geometric sequence with $a_1 = 4$ and $a_{2023} = \frac{1}{4}$. Let $f(x) = \frac{1}{7(1+x^2)}$. Find

$$f(a_1) + f(a_2) + ... + f(a_{2023}).$$

11. Let $\mathcal{S}$ be the set of quadratics $x^2 + ax + b$, with $a$ and $b$ real, that are factors of $x^{14} - 1$. Let $f(x)$ be the sum of the quadratics in $\mathcal{S}$. Find $f(11)$.

12. Find the largest integer $0 < n < 100$ such that $n^2 + 2n$ divides $4(n-1)! + n + 4$.

13. Let $\omega$ be a unit circle with center $O$ and radius $OQ$. Suppose $P$ is a point on the radius $OQ$ distinct from $Q$ such that there exists a **unique** chord of $\omega$ through $P$ whose midpoint when rotated $120^\circ$ counterclockwise about $Q$ lies on $\omega$. Find $OP$.

14. A sequence of real numbers $\{a_i\}$ satisfies

$$n \cdot a_1 + (n-1) \cdot a_2 + (n-2) \cdot a_3 + \cdots + 2 \cdot a_{n-1} + 1 \cdot a_n = 2023^n$$

for each integer $n \geq 1$. Find the value of $a_{2023}$.

15. In $\triangle ABC$, let $\angle ABC = 90$ and let $I$ be its incenter. Let line $BI$ intersect $AC$ at point $D$, and let line $CI$ intersect $AB$ at point $E$. If $ID = IE = 1$, find $BI$.

16. For a positive integer $n$, let $S_n$ be the set of permutations of the first $n$ positive integers. If $p = (a_1, \ldots, a_n) \in S_n$, then define the bijective function $\sigma_p : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\sigma_p(i) = a_i$ for all integers $1 \le i \le n$.

    For any two permutations $p, q \in S_n$, we say $p$ and $q$ are *friends* if there exists a third permutation $r \in S_n$ such that for all integers $1 \le i \le n$,

    $$\sigma_p(\sigma_r(i)) = \sigma_r(\sigma_q(i)).$$

    Find the number of friends, including itself, that the permutation $(4, 5, 6, 7, 8, 9, 10, 2, 3, 1)$ has in $S_{10}$.