Big Red Math Competition
Individual Round

Name: ____________________

Student ID: ____________________

10/26/2024 10:45am–12:45pm

INSTRUCTIONS — PLEASE READ THIS NOW

- **Write your Name and Student ID.**
- **Simplification.** Please simplify all answers as much as possible. For example, $\frac{\cos\pi}{\frac{1}{2}}$ can be simplified to -2.
- **You have 2 hour to complete this exam.**
- This is a closed book exam. You are **NOT** allowed to use a calculator, computer, notes, or any other resources.

Please sign below to indicate that you have read and agree to these instructions.

Signature, Print Name: ____________________

Answer Sheet:

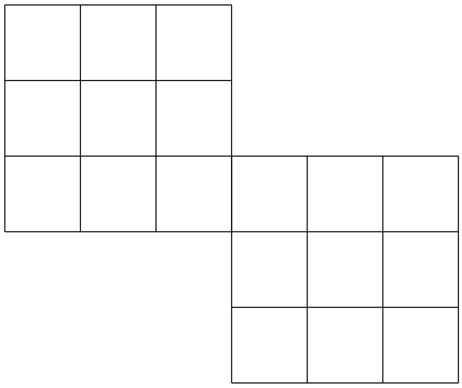
1. ______	5. ______	9. ______	13. ______
2. ______	6. ______	10. ______	14. ______
3. ______	7. ______	11. ______	15. ______
4. ______	8. ______	12. ______	16. ______

Please don't mark anything below on the page, it's for official use only.

Grader 1 Name: ______	Grader 2 Name: ______	Grader 3 Name: ______
Tally: ______	Tally: ______	Tally: ______
Score: ______	Score: ______	Score: ______

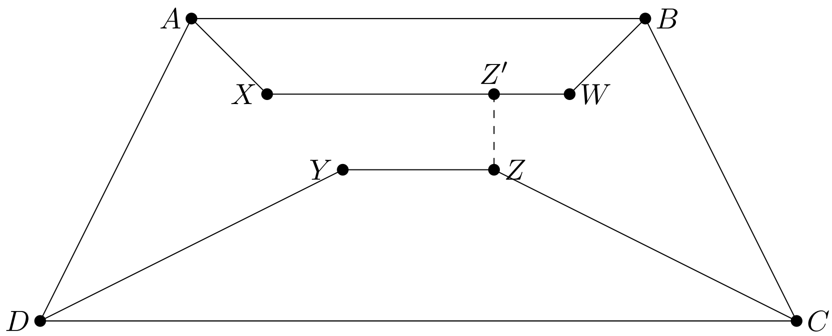
1. Define the sequence $\{a_n\}_{n\geq 0}$ such that $a_0 = a_1 = 2024^{2024!}$ and $a_n = \frac{a_{n-1}}{a_{n-2}}$ for all integers $n \geq 2$. Compute a_{2024}.

2. Two 3×3 squares are placed next to each other so that they share an edge of length 1 as shown in the figure below. How many ways can we cover all 18 squares with 2×1 tiles?



3. A fair 10-sided die with sides labeled $1, 2, \ldots, 10$ is rolled three times. What is the probability that the median of these three rolls is 3?

4. Given an isosceles trapezoid $ABCD$ with $AB \parallel CD$, let W, X, Y, Z be points inside $ABCD$ such that $WXAB$ and $YZCD$ are isosceles trapezoids that do not overlap each other and with $WX \parallel AB$ and $YZ \parallel CD$. Suppose that $AB + WX = CD + YZ = 20$ and $ZZ' = 4$ where Z' is the point on line WX such that $ZZ' \perp WX$. Given that the height of trapezoid $ABCD$ is 29, compute the combined area of trapezoids $WXAB$ and $YZCD$.



5. Alex has 42 pairwise distinct positive integers. He takes each integer and computes its remainder when divided by 6. Of these 42 remainders, exactly 7 of them evaluate to 0, exactly 7 of them evaluate to 1, exactly 7 of them evaluate to 2, and so on. Of the original 42 integers that Alex started with, what is the maximum number of prime numbers could he have had?

6. Bradley can perform one of two operations on an integer: he can either square it or add 1 to it. If Bradley starts with the integer 1, what is the minimum number of operations that Bradley needs to perform to reach exactly 1000.

7. Let $S(n)$ denote the sum of all digits of n in base 10. Find the number of integers $1 \leq n \leq 2024$ such that 11 divides $n - S(n)$.

8. Let $\triangle ABC$ be a right triangle such that $\angle ABC = 90^\circ$ and the altitude from B onto AC has length $\sqrt{6}$. If AB^2 and BC^2 are both integers, find the maximum possible area of $\triangle ABC$.

9. Find the number of permutations $a_1, a_2, \ldots, a_{20}$ of the integers $1, 2, \ldots, 20$ such that for all integers $1 \leq i, j \leq 20$, if i divides j, then a_i divides a_j.

10. Let $ABCD$ be a rhombus with side length 4. Suppose the circumcircle of $\triangle ABD$ intersects line segment CD at P. Given that $CP = 1$, find the area of $ABCD$.

11. Let $S = \{1, 2, \ldots, 10\}$. Suppose $f\colon S \to S$ is a function chosen uniformly at random among all possible functions from S to S. Find the probability that $f(f(f(f(1)))) = 1$.

12. Let $n = 2024$ and $\omega = e^{2\pi i/n}$. For each integer $1 \le k \le n$, let S_k be the set of the first n positive integers with the integer k removed. (For example, $S_3 = \{1, 2, 4, 5, 6, \ldots, n\}$.) Also, for each integer $1 \le k \le n$, define

$$a_k = \prod_{j \in S_k} (2 + \omega^j)$$

where the product is taken over all values $j \in S_k$. Compute $a_1 + a_2 + \cdots + a_n$.

13. Find the number of injective functions $f\colon \{1, 2, \ldots, 2024\} \to \{1, 2, \ldots, 2024\}$ such that

$$f(x + y) \equiv f(x) + f(y) \pmod{2024}$$

for all integers $1 \le x, y \le 2024$.

(For non-empty sets X and Y, we say a function $f\colon X \to Y$ is *injective* if $f(x) \neq f(x')$ for all distinct $x, x' \in X$.)

14. Jiming flips 50 coins and records the resulting sequence of coin flips. Let a be the number of times he flips two heads in a row and b be the number of times he flips two tails in a row. For example, the sequence TTTHHHTT would yield $a = 2$ and $b = 3$. What is the expected value of the product ab?

15. Let $ABCDEF$ be a regular hexagon with side length 1, and let W be the midpoint of side AB. Suppose X, Y, and Z are points on sides BC, DE, and FA, respectively, such that W, X, and Z are not collinear. Find the minimum possible value of the perimeter of quadrilateral $WXYZ$.

16. Let $n = 2024$. Given a point $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and an integer $r > 0$, Alex can *r-amplify* the point X to get a new point $X' \in \mathbb{R}^n$ given by

$$X' = (x_1, rx_1 + x_2, r^2x_1 + rx_2 + x_3, \ldots, r^{n-1}x_1 + r^{n-2}x_2 + \cdots + x_n)$$

where the k-th coordinate of X' is $\sum_{i=1}^{k} r^{k-i} x_i$.

Suppose Alex starts with the point $A_0 = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n$ where

$$a_0 = 1, \qquad a_{2k-1} = 0, \qquad a_{2k} = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (-1)^k i_1^2 i_2^2 \cdots i_k^2$$

for each integer $k \ge 1$. (The summation is taken over all integers $1 \le i_1 < i_2 \cdots < i_k \le n$.) Alex first 1-amplifies A_0 to get a new point A_1. He then 2-amplifies A_1 to get a new point A_2. He continues this process until he n-amplifies A_{n-1} to get a new point A_n. Compute the sum of the n coordinates of A_n.