Cornell University®

Big Red Math Competition

Proof Round

Name: ______________________

Student ID: ______________________

10/26/2024 9:30am–10:30am

INSTRUCTIONS — PLEASE READ THIS NOW

- **Show your work.** To receive full credit, your answers must be neatly written and logically organized.
- **Simplification.** Please simplify all answers as much as possible. For example, $\frac{\cos\pi}{\frac{1}{2}}$ can be simplified to -2.
- **You have 1 hour to complete this exam.**
- This is a closed book exam. You are **NOT** allowed to use a calculator, computer, notes, or any other resources.

Please sign below to indicate that you have read and agree to these instructions.

Signature of Student: ______________________

Please don't mark anything below on the page, it's for official use only.

Grader 1: ______	Grader 2: ______	Grader 3: ______
1. ____ / 10 ____	1. ____ / 10 ____	1. ____ / 10 ____
2. ____ / 20 ____	2. ____ / 20 ____	2. ____ / 20 ____
3. ____ / 20 ____	3. ____ / 20 ____	3. ____ / 20 ____
4. ____ / 30 ____	4. ____ / 30 ____	4. ____ / 30 ____
Total: ____ / 80 ____	Total: ____ / 80 ____	Total: ____ / 80 ____

1. Let $\triangle ABC$ be a non-degenerate triangle. Suppose there exists a point P on side BC such that AP splits $\triangle ABC$ into two non-degenerate triangles that are both similar to $\triangle ABC$. Prove that $\angle BAC = 90°$.

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2. Prove that if x, y, z are integers such that

$$5x^2 + 2y^2 - z^2 = 2xy + 2yz,$$

then $x = y = z = 0$.

(This page is intentionally left blank to give you more space for your proof.)

3. Let $n \geq 3$ be an integer. The integers from 1 to n, inclusive, are written around a circle in some order. We say an unordered pair of integers on the circle is *Cornellian* if they don't occupy neighboring positions and at least one of the two arcs they enclose contains exclusively integers that are smaller than both of the pair. For example, suppose $n = 6$ and the integers are placed around the circle in the following order: 1, 4, 3, 2, 5, 6. Then the pair $\{4, 5\}$ is *Cornellian* because the arc between 4 and 5 containing 2 and 3 only contains integers that are less than both 4 and 5.

   For each integer $n \geq 3$, find all integers $k \geq 0$ such that there exists a configuration of the integers 1 to n, inclusive, on the circle with exactly k *Cornellian* pairs.

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4. Let $\{x_n\}_{n\geq 0}$ be a sequence given by $x_0 = 0$, $x_1 = 1$, and

$$x_{n+2} = x_n + \sqrt{21x_{n+1}^2 + 4}$$

for all integers $n \geq 0$. Show that all terms of the sequence are integers.

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