Question 1

(25 points) A consumer has utility function $u(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$, where x_1, x_2, x_3, and x_4 are non-negative amounts of four goods that the consumer chooses. The consumer's budget constraint is

$$p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 \leq m,$$

where $p_1, ..., p_4$ are strictly positive prices and m is income, also assumed to be strictly positive. The consumer takes prices and income as given and maximizes utility.

A Write down the formal problem that the consumer solves. (5 points)

B Does a solution to the problem exist? Explain. (5 points.)

C Solve the problem for all possible values of prices and income, i.e. leaving $p_1, ..., p_4$ and m as variables, in contrast with what happens in Part D below. (10 points.)

D Suppose the consumer has $m = 12$, and that in the consumer's current location, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, and $p_4 = 4$. The consumer can drive out-of-town at cost D and "make groceries" at cheaper prices: $p_1 = 1$, $p_2 = 1$, $p_3 = 2$, and $p_4 = 2$. However, the consumer has only $m - D$ to spend out-of-town. How much is the consumer willing to pay at most in transportation costs D? Explain. (5 points.)

$$\max_{x_1 \dots x_4} U = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$$

s.t. $p_1 x_1 + \dots + p_4 x_4 \leq m$

$x_1 \dots x_4 \geq 0$

$p_1 \dots p_4 > 0$

$$\mathcal{L} = \sqrt{x_1 x_2} + \sqrt{x_3 x_4} - \lambda \left[p_1 x_1 + \dots + p_4 x_4 - m \right]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}} - \lambda p_1 = 0 \Rightarrow \frac{1}{2}\sqrt{\frac{x_2}{x_1}} = \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{2}\sqrt{\frac{x_1}{x_2}} - \lambda p_2 = 0 \Rightarrow \frac{1}{2}\sqrt{\frac{x_1}{x_2}} = \lambda p_2$$

$$\frac{p_1}{p_2} = \sqrt{\frac{x_2}{x_1}} \Big/ \sqrt{\frac{x_1}{x_2}} \Rightarrow \frac{p_1}{p_2} = \frac{x_2}{x_1} \Rightarrow x_2 = \frac{p_1 x_1}{p_2}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 0 \Rightarrow \frac{1}{2}\sqrt{\frac{x_4}{x_3}} = \lambda p_3 \qquad \frac{p_3}{p_4} = \frac{x_4}{x_3}$$

$$\frac{\partial \mathcal{L}}{\partial x_4} = 0 \Rightarrow \frac{1}{2}\sqrt{\frac{x_3}{x_4}} = \lambda p_4 \qquad \Rightarrow x_4 = \frac{p_3 x_3}{p_4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow p_1 x_1 + \dots + p_4 x_4 = m$$

$$\Rightarrow p_1 x_1 + \cancel{p_2}\left(\frac{p_1 x_1}{\cancel{p_2}}\right) + p_3 x_3 + \cancel{p_4}\left[\frac{p_3 x_3}{\cancel{p_4}}\right] = m$$

$$\Rightarrow 2p_1 x_1 + 2p_3 x_3 = m \Rightarrow \underbrace{p_1 x_1 + p_3 x_3}_{} = \frac{m}{2}$$

$$\Rightarrow 2p_2 x_2 + 2p_3 x_3 = m \Rightarrow p_2 x_2 + p_3 x_3 = m/2$$

$$\Rightarrow 2p_4 x_4 + \dots = m \Rightarrow p_3 x_3 + p_4 x_4 = m/2$$

$$\Rightarrow 2p_1 x_1 + 2p_4 x_4 = m \Rightarrow \dots p_1 x_1 = m/2$$

$\Rightarrow 2p_2x_2 + 2p_4x_4 = m \Rightarrow p_2x_2 + p_4x_4$

$p_1x_1 = m/4 \Rightarrow x_1 = \frac{m}{4p_1}, \quad x_4 = \frac{m}{4p_4}$

$\Rightarrow x_2 = \frac{m}{4p_2}, \quad x_3 = \frac{m}{4p_3}$

A $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 4, \quad m = 12$

B $p_1 = 1, p_2 = 1, p_3 = 2, p_4 = 2, \quad m = 12 - D$

A $\to x_1 = \frac{m}{4p_1} = 3, \quad x_2 = \frac{3}{2}, \quad x_3 = 1, \quad x_4 = \frac{3}{4}$

$u = \sqrt{x_1x_2} + \sqrt{x_3x_4} = \sqrt{3 \cdot 3/2} + \sqrt{1 \cdot \frac{3}{4}} = \frac{3}{\sqrt{2}} + \frac{\sqrt{3}}{2}$

B $\to x_1 = \frac{m}{4p_1} = \frac{12-D}{4}, \quad x_2 = \frac{12-D}{4}, \quad x_3 = \frac{12-D}{8}, \quad x_4 = \frac{12-D}{8}$

$u = \sqrt{\frac{12-D}{4} \cdot \frac{12-D}{4}} + \sqrt{\frac{12-D}{8} \cdot \frac{12-D}{8}} = \frac{12-D}{4} + \frac{12-D}{8}$

$u = \frac{36-3D}{8} = \frac{3}{\sqrt{2}} + \frac{\sqrt{3}}{2} \Rightarrow D = \_\_\_\_$

Question 2

(25 points) A firm is looking to produce a total quantity equal to Y in the cheapest possible way. The firm has two plants. The cost of producing y_1 units in plant 1 is $2\sqrt{y_1}$. The cost of producing y_1 units in plant 2 is $4\sqrt{y_2}$. The firm wants to determine the best way to divide total production Y into y_1 and y_2 and how much that costs.

A Write down the formal problem that the firm solves. (5 points)

B Does a solution to the problem exist? Explain. (5 points.)

C Solve the problem for all possible values of Y. (10 points.)

D How much does the cost of the firm increase with a marginal increase in Y, taking into account the optimal division into y_1 and y_2? Explain. (5 points.)

$$\min_{y_1, y_2} C = 2\sqrt{y_1} + 4\sqrt{y_2}$$

s.t.

$$y_1 + y_2 \leq Y$$

$$y_1, y_2 \geq 0$$

$$\mathcal{L} = 2\sqrt{y_1} + 4\sqrt{y_2} - \lambda\left[y_1 + y_2 - Y\right]$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = 0 \Rightarrow \frac{1}{\sqrt{y_1}} = \lambda \qquad y_1 = \frac{y_2}{4}$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = 0 \Rightarrow \frac{2}{\sqrt{y_2}} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow y_1 + y_2 = Y$$

$$\frac{y_2}{4} + y_2 = Y \Rightarrow y_2 = \frac{4Y}{5}, \quad \boxed{y_1 = \frac{Y}{5}}$$

$$\lambda = \frac{1}{\sqrt{y_1}} = \frac{1}{\sqrt{Y/5}}$$

OR

$$\lambda = \frac{2}{\sqrt{y_2}} = \frac{2}{\sqrt{\frac{4Y}{5}}} = \frac{1}{\sqrt{Y/5}}$$