Question 1

(25 points) A consumer has utility function $u(x_1, x_2, x_3) = \sqrt{x_1 x_2} + x_3$, where x_1, x_2, and x_3 are non-negative amounts of three goods that the consumer chooses. The consumer's budget constraint is

$$x_1 + x_2 + px_3 \leq m,$$

where p is a strictly positive price and m is income, also assumed to be strictly positive. The consumer takes prices and income as given and maximizes utility.

A Write down the formal problem that the consumer solves. (5 points)

B Does a solution to the problem exist? Explain. (5 points.)

C Solve the problem for all possible values of prices and income. (10 points.)

D Suppose the consumer has $m = 12$, and that in the consumer's current location, $p = 2.25$ The consumer can drive out-of-town and "make groceries" at cheaper prices: $p = 1.5$. How much income is the consumer willing to give up at most in transportation costs? Explain. (5 points.)

Q1a

$$\max_{x_1, x_2, x_3} U = \sqrt{x_1 x_2} + x_3$$

s.t. $x_1 + x_2 + px_3 \leq m$

$x_1, x_2, x_3 \geq 0$

$p > 0$

b)

Because $x_1, x_2, x_3 \geq 0$ then $U \geq 0$ and is continuous.

The feasible region is closed because $x_1 + x_2 + px_3 \leq m$ and in the first octant because $x_1, x_2, x_3 \geq 0$.

c) Let $V = \sqrt{x_1 x_2}$. Then $U = V + x_3$. This is linear utility.

$$x_3 = \begin{cases} m/p & \text{if } V < x_3 \\ 0 & \text{if } V > x_3 \end{cases}$$

Looking at V, $\frac{\partial V}{\partial x_1} = \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2}$ and $\frac{\partial V}{\partial x_2} = \frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2}$.

Looking at v, $\frac{\partial v}{\partial x_1} = \frac{1}{2}\left(\frac{\quad}{x_1}\right)$ and $\frac{\quad}{\partial x_2}$

mrs has to be same $\Rightarrow$ $\frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{1}{2}\left(\frac{x_1}{x_2}\right)^{1/2} \Rightarrow x_1 = x_2$.

Thus we have:

$$\max\ U = x_1 + x_3 \quad \text{s.t.} \quad 2x_1 + p x_3 \le m$$

$$\Rightarrow \quad x_1 = \begin{cases} m/2 & \text{if } p > 2 \\ 0 & \text{if } p < 2. \end{cases}$$

$$\Rightarrow \quad x_2 = \begin{cases} m/2 & \text{if } p > 2 \\ 0 & \text{if } p < 2 \end{cases}$$

$$\Rightarrow \quad x_3 = \begin{cases} m/p & \text{if } p < 2 \\ 0 & \text{if } p > 2. \end{cases}$$

d.) $m = 12$, $p = 2.25$

$\Rightarrow x_1 = 12/2 = 6 = x_2 \Rightarrow x_3 = 0$

$U = \sqrt{6(6)} + 0 = 6$

When $m = 12 - d$, $p = 1.5$,

$x_3 = \frac{12-d}{1.5}$, $x_1 = x_2 = 0$

$\Rightarrow U = \sqrt{0 \cdot 0} + \frac{12-d}{1.5} = 6$

$\Rightarrow d = 12 - 6(1.5) = 3$.

He can give up to \$3.

Question 2

(25 points) A firm is looking to produce a total quantity equal to Y in the cheapest possible way. The firm has two plants. The cost of producing y_1 units in plant 1 is $2\sqrt{y_1}$. The cost of producing y_1 units in plant 2 is $4\sqrt{y_2}$. The firm wants to determine the best way to divide total production Y into y_1 and y_2 and how much that costs.

A Write down the formal problem that the firm solves. (5 points)

B Does a solution to the problem exist? Explain. (5 points.)

C Solve the problem for all possible values of Y. (10 points.)

D How much does the cost of the firm increase with a marginal increase in Y, taking into account the optimal division into y_1 and y_2? Explain. (5 points.)

Q2a.

$$\min \quad Cost = 2\sqrt{y_1} + 4\sqrt{y_2}$$

$$\text{s.t.} \quad y_1 + y_2 \leq Y$$

$$y_1, y_2 \geq 0.$$

$$\Rightarrow \max \quad -Cost = -2\sqrt{y_1} - 4\sqrt{y_2}$$

$$\text{s.t.} \quad y_1 + y_2 \leq y$$

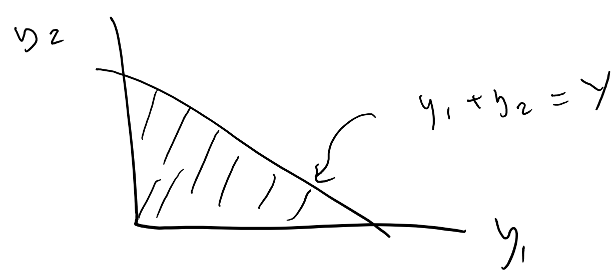
$$y_1, y_2 \geq 0$$

Q2b.

Let $W = -Cost$. Because $y_1 \geq 0$ & $y_2 \geq 0$ then

$W \leq 0$ and is continuous.

The feasible region is closed & bounded as seen in the diagram below:



Q2c

$$\frac{\partial \mathcal{L}}{\partial y_1} = 0 \Rightarrow \frac{-2}{\sqrt{y_1}}\left(\frac{1}{2}\right) = \lambda \Rightarrow \frac{-1}{\sqrt{y_1}} = \lambda \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = 0 \Rightarrow \frac{-4}{\cancel{\sqrt{y_1}}\ \sqrt{y_2}}\left(\frac{1}{2}\right) = \lambda \Rightarrow \frac{-2}{\sqrt{y_2}} = \lambda, \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow y_1 + y_2 = Y \qquad (3)$$

From (1) & (2) $\Rightarrow \dfrac{1}{\sqrt{y_1}} = \dfrac{2}{\sqrt{y_2}} \Rightarrow y_2 = 4y_1$

$$\Rightarrow y_1 + [4y_1] = Y \Rightarrow y_1 = Y/5$$

$$\Rightarrow y_2 = \frac{4Y}{5}.$$

Q2d

From (1) $\lambda = \dfrac{-1}{\sqrt{y_1}} = \dfrac{-1}{\sqrt{Y/5}} = -\sqrt{\dfrac{5}{Y}}$

$\Rightarrow$ W falls by $\sqrt{\frac{5}{Y}}$ when Y rises by 1.

$\Rightarrow$ Cost rises by $\sqrt{\frac{5}{Y}}$ when Y rises by 1.

Note that increase in Cost depends on Y. The increase in cost (MC) falls when Y rises.