Framework for Dynamic Portfolio Management in the Presence of Parameter Uncertainty

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Contents

1 Introduction

Brockwell [1] discusses a Fractional Kelly investment framework to describe the evolution of a *leveraged portfolio* over time. In such a portfolio, we maintain a fixed proportion of total capital in each of a number m of investment instruments, and accrue/pay interest at a risk-free rate on the remaining cash/debt. In this context, Kelly's formula gives the same prescription for leverage as the solution of a Markowitz mean-variance optimization problem, and it is possible to characterize the distribution of portfolio returns over any period of time.

However, the results rely on knowledge of the drift and diffusion parameters of the risky instruments in the portfolio. In practical applications, one must use estimates of these quantities, since the true values are unknown. These parameter estimates are corrupted by noise in the realization of past data. In addition, they commonly exhibit random variation over time, which makes it difficult even to quantify what we hope to achieve with an estimator, let alone ensure good long-term portfolio behavior.

In this paper we generalize the framework of [1] to account for both of these types of uncertainty. Specifically, we state a general result that relates long-term performance to the joint limiting distribution of various quantities. In most cases, knowledge of the leverage selection process itself, along with some minimal assumptions, is sufficient to determine these limiting distributions, and consequently, to evaluate long-term performance.

We provide an example that illustrates performance analysis for a fractional-Kelly portfolio when the covariance matrix (diffusion parameters) is taken to be known, but means (drift parameters, scale-equivalent to Sharpe ratios) are unknown and must be causally estimated. The respective unknown quantities in the fractional-Kelly leverage formula are replaced by their estimates. We then see how portfolio growth can be decomposed into three parts: a base term familiar from contemporary portfolio theory, a second term representing potential improvement due to accurate tracking of a time-varying Sharpe ratio, and a third term representing losses incurred by the tracking process itself. The first of these three terms is known and well-understood. To the author's knowledge, this paper is the first to quantify the impact of active portfolio management and uncertainty that is encompassed in the additional terms.

2 Time-Varying Diffusion Models for Prices

2.1 Notation

In [1], we introduced a standard multivariate geometric Brownian motion describing the evolution of prices of m investment instruments in our portfolio. We now generalize that formulation to allow for time-varying parameters. Assume the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, and on the space let

$$\{\mathbf{P}_t = (P_{t,1}, P_{t,2}, \ldots, P_{t,m})^T,\ t \in \mathbb{R}\} \tag{1}$$

denote a price process satisfying the system of stochastic differential equations

$$d\mathbf{P}_t = \mathrm{diag}(\mu_t)\mathbf{P}_t dt + \mathrm{diag}(\sigma_{\mathbf{t}})\mathrm{diag}(\mathbf{P}_t)d\mathbf{U}_t, \tag{2}$$

where $\{\mathbf{U}_t \in \mathbb{R}^m\}$ is a multivariate Brownian motion with time-varying correlation matrix R_t, so that

$$\mathbf{E}\left[d\mathbf{U}_t\right] = 0, \quad \mathrm{Var}\left(d\mathbf{U_t}\right) = R_t dt, \tag{3}$$

and diag$(\cdot)$ represents a square matrix with diagonal elements given by the vector argument, and zeros in all off-diagonal positions. It will be convenient to define the covariance matrix

$$\Sigma_t = \mathrm{diag}(\sigma_t) R_t \mathrm{diag}(\sigma_t). \tag{4}$$

As argued in the previous work, when μ_t and Σ_t are constant over time, (2) provides a fairly good description of the evolution of prices of financial instruments over time. Specifically, they evolve according to a multivariate geometric Brownian motion. Here we have simply generalized that model to allow for time-heterogeneity. The variance and covariance terms $\sigma_t,\ R_t$ and Σ_t are referred to as the *diffusion parameters*, while the components of the mean vector μ_t are referred to as the *drift parameters*.

The drift and diffusion parameters may be known or unknown, but much of this paper will address the scenario where Σ_t is known and μ_t is unknown, and study the impact on leverage selection and portfolio performance. Leverage selection was the focus of [1], where we saw how best to amplify returns of individual instruments to obtain good aggregate portfolio performance. In this paper, we also allow for leverage to vary with time,

$$\mathbf{k}_t = (k_{t,1}, \ldots, k_{t,m})^T. \tag{5}$$

As in [1], we assume that at any given point in time t, for each $j = 1, 2, \ldots, m$, the portfolio maintains proportions k_j of total capital A_t in each of the respective instruments, and allocates the remainder to cash holdings (if positive) or debt (if negative). As before we define the *total leverage*

$$\kappa_t = \sum_{j=1}^{m} k_{j,t}. \tag{6}$$

We also assume that cash earns risk-free interest rate r, while debt pays that same rate, and for notational convenience, we define the vector

$$\mathbf{r} = (r, r, \ldots, r)^T \in \mathbb{R}^m. \tag{7}$$

In the time-varying framework of this paper, We need to be careful with regard to causality, so we will generally require

Assumption 2.1 *Assume that $\{\mu_t\}$, $\{\Sigma_t\}$ and $\{\mathbf{k}_t\}$ are almost surely continuous in t, and adapted to the filtration $\{\mathcal{F}_t.\}$*

Intuitively, this is a statement that the future does not get tangled up with the past. For example, choice of $\mathbf{k}_t = \mu_{t+\delta},\ \delta > 0$ would violate this assumption.

By a straightforward adaptation of the argument in [1] based in Itô's formula, when Assumption 2.1 holds, we find that

$$d\log(\mathbf{P}_t) = (\mu_t - \sigma_t^2/2)dt + d\mathbf{Q}_t. \tag{8}$$

where $\{\mathbf{Q}_t\}$ is a multivariate Brownian motion with

$$E[d\mathbf{Q}_t] = 0, \quad \mathrm{Var}\,(d\mathbf{Q}_t) = \Sigma_t dt. \tag{9}$$

Furthermore, the total *capital* $\{A_t\}$ generated by the portfolio is a geometric Brownian motion satisfying

$$d\log(A_t) = r + (\mathbf{k}_t \cdot (\mu_t - \mathbf{r}) - \mathbf{k}_t^T \Sigma_t \mathbf{k}_t/2)dt + (\mathbf{k}_t^T \Sigma_t \mathbf{k}_t)^{1/2} dW_t, \tag{10}$$

where $\{W_t\}$ is a standard Brownian motion.

2.2 Ergodicity

The primary task of the investor is to choose the leverage vector $\mathbf{k}_t$. When μ_t and Σ_t are known and constant, it is straightforward to balance growth rate against volatility to make this choice. In this paper we will investigate the far more complex problem of choosing leverage in the face of uncertainty.

The first step is to develop a framework for analysis. To do so, we will need to bring in the machinery of ergodic theory for stochastic processes. Some early results in this field are due to [4], but more recent developments that apply directly to solutions of stochastic differential equations can be found in the literature.

2.3 Relating Uncertainty to Performance

Uncertainty in portfolio parameters is a known problem, and has been addressed in various ways. One notable approach is the use of Bayesian priors, which can be effective, but introduces an element of subjectivity to the problem. Here we adopt a different approach. Our goal is to measure the impact of this uncertainty on performance directly without resorting to Bayesian methods, and to provide explicit statements on long-term performance and its connection with uncertainty.

The following result is the key to analysis of portfolio behavior. It shows that long-term performance is governed by the limiting joint distribution of three time-varying random processes.

Theorem 2.2 (Ergodic Investment Theorem) *Suppose that the price process $\{\mathbf{P}_t\}$ satisfies (2) and we apply a time-varying leverage vector $\mathbf{k}_t$ to its components. If Assumption 2.1 holds, and the process*

$$\{(\mu_t, \mathbf{k}_t, \Sigma_t),\ t \geq 0\} \tag{11}$$

is ergodic with limiting distribution π, then the long-term expected log-return per unit time of capital $\{A_t\}$ is

$$L = \lim_{t\to\infty} \mathbf{E}\left[\log A_t - \log A_0\right]/t = \int (\mathbf{k}\cdot\mu - \mathbf{k}^T\Sigma\mathbf{k}/2)d\pi(\mathbf{k}, \mu, \Sigma). \tag{12}$$

Furthermore, the long-term log-return variance per unit time is

$$V = \lim_{t\to\infty} Var\left(\log A_t - \log A_0\right)/t = \int \mathbf{k}^T\Sigma\mathbf{k}d\pi(\mathbf{k}, \mu, \Sigma). \tag{13}$$

Proof: Under the conditions of the theorem,

$$\log A_t - \log A_0 \quad = \quad \int_0^t d\log A_t \tag{14}$$

$$= \quad \int_0^t (\mathbf{k}_t \cdot \mu_t - \mathbf{k}_t^T\Sigma_t\mathbf{k}/2)dt + \int_0^t (\mathbf{k}_t^T\Sigma_t\mathbf{k})^{1/2}dW_t. \tag{15}$$

Taking expectations on both sides and dividing by t, we have

$$\mathbf{E}\left[\log A_t - \log A_0\right]/t \quad = \quad \frac{1}{t}\mathbf{E}\left[\int_0^t (\mathbf{k}_u \cdot \mu_u - \mathbf{k}_u^T\Sigma_t\mathbf{k}_u/2)du\right] \tag{16}$$

$$\tag{17}$$

Taking limits as $t \to \infty$, the result then follows from the ergodicity of $(\mathbf{k}_t, \mu_t)$ □

In financial terms, Theorem 2.2 directly quantifies the effect of uncertainty on portfolio performance. It covers uncertainty caused by various real-time parameter estimation schemes that may feed into the choice of $\mathbf{k}_t$, as well as uncertainty due to the random drift of μ_t over time, for example, due to regime changes. The result states that knowledge of the limiting joint distribution of $\mathbf{k}_t, \mu_t$, and Σ_t, is sufficient to determine long-term performance.

Following from the arguments in [1], we will be particularly interested in the case where $\mathbf{k}_t$ is chosen using a fractional Kelly approach. However, it is worth noting that Theorem 2.2 applies to far more general schemes for leverage selection.

3 Uncertainty in Leverage Selection

In the previous section we established a result that explains how long-term portfolio performance depends on the limiting distribution of the quantities $\mathbf{k}_t$, μ_t and Σ_t. These quantities may vary randomly over time. To apply the result, we need to

1. make assumptions about the (random) evolution of these processes,
2. specify our (causal) procedure for leverage-selection at every point in time, and
3. determine the joint limiting distribution of $\mathbf{k}_t$, μ_t, and Σ_t.

In this section we demonstrate one approach to carry out such analysis. We will take $\Sigma_t = \Sigma$ to be a known constant, since this simplifies the analysis. Of course Theorem 2.2 still applies in the more general setting, and future work could address the impact on performance of time-varying and/or unknown Σ_t.

3.1 Time-Varying Drift Parameter Model

To determine long-term performance of our portfolio, we need to describe the random evolution of μ_t over time. However, to construct a model that can be easily interpreted, we do this indirectly, first applying a change of basis

$$\mathbf{S}_t = \Sigma^{-1/2}\mu_t, \tag{18}$$

where $\Sigma^{-1/2}$ is the upper triangular part of the Cholesky decomposition of the inverse of Σ, so that

$$\Sigma^{-1} = (\Sigma^{-1/2})^T(\Sigma^{-1/2}). \tag{19}$$

The process $\{\mathbf{S}_t\}$ can be thought of as a vector of time-varying Sharpe ratios of a "whitened" or "de-correlated" version of our portfolio.

We next present a formal model for these time-varying Sharpe ratios. Let $\{\mathbf{S}_t\}$ be the stationary and causal solution of

$$d\mathbf{S}_t = -h(\mathbf{S}_t - \mathbf{S})dt + C^{1/2}d\epsilon_t, \tag{20}$$

where $h > 0$, $\mathbf{S} \in \mathbb{R}^m$ is a vector, $C^{1/2}$ is the lower triangular component in the Cholesky decomposition of a positive definite matrix $C = C^{1/2}(C^{1/2})^T$, and $\{\epsilon_t\}$ is a standard m-dimensional Brownian motion with

$$\mathbf{E}\left[\epsilon_t\right] = 0, \quad \mathrm{Var}\left(\epsilon_t\right) = tI_{m\times m}. \tag{21}$$

Equation (20) defines a standard Ornstein-Uhlenbeck process, sometimes also referred to as a continuous-time first-order autoregressive process (see, e.g. [2]). It has the following properties.

1. $\mathbf{E}[S_t] = \mathbf{S}$, and $\text{Var}(S_t) = (2h)^{-1}C$.
2. The vector $\mathbf{S}_t$ "reverts" to $\mathbf{S}$, that is, when components of $\mathbf{S}_t$ are larger/smaller than the corresponding components of $\mathbf{S}$, there is a negative/positive instantaneous drift in the respective part of $d\mathbf{S}_t$.

For convenience we will also define the vector

$$\mu = (\Sigma^{-1/2})^{-1}\mathbf{S}, \tag{22}$$

and we define the portfolio Sharpe ratio by

$$S = (\mu^T \Sigma^{-1} \mu)^{-1/2}, \tag{23}$$

even though the vectors $\mathbf{S}$ and μ are typically unknown. It is easily verified that

$$\mathbf{S}^T\mathbf{S} = \mu^T(\Sigma^{-1/2})^T\Sigma^{-1/2}\mu = \mu^T\Sigma^{-1}\mu = S^2. \tag{24}$$

3.2 Drift Parameter Estimation

We can measure prices $\mathbf{P}_t$ over time, but we do not directly observe μ_t or its long-term mean μ, or the corresponding (basis-changed vectors) $\mathbf{S}_t$ and $\mathbf{S}$, at any point in time. Even so, we can come up with useful estimators for these quantities, by constructing an Ornstein-Uhlenbeck process. Let us define the *rate* of our estimator by a constant

$$g > 0. \tag{25}$$

Then define the m-dimensional process

$$\begin{aligned} \mathbf{M}_0 &= 0, & (26) \\ d\mathbf{M}_t &= -g[\mathbf{M}_t dt - \Sigma^{-1/2}(d\log \mathbf{P}_t - \sigma_t^2 dt/2)]. & (27) \end{aligned}$$

We already know that

$$d\log \mathbf{P}_t = (\mu_t - \sigma_t^2/2)dt + d\mathbf{Q}_t, \quad d\mathbf{Q}_t \sim \mathrm{N}(0, \Sigma dt), \tag{28}$$

so it follows directly that

$$d\mathbf{M}_t = -g(\mathbf{M}_t - \mathbf{S}_t)dt + g\Sigma^{-1/2}d\mathbf{Q}_t. \tag{29}$$

In the form (27), $d\mathbf{M}_t$ is expressed as a function of *only* the current value of the estimator $\mathbf{M}_t$ itself, the (vector) price process $\{\mathbf{P}_t\}$, and the known vector σ. Hence the estimator $\mathbf{M}_t$ is computable at time t using only current and past observable quantities. [1]

In the equivalent form (29), we see that the estimator evolves as an Ornstein-Uhlenbeck process over time, hence we can easily establish a range of useful properties. In fact, we will construct a joint process starting with equations (20) and (29) as follows. Combining the two stochastic differential equations, we have

$$d\begin{bmatrix} \mathbf{S}_t \\ \mathbf{M}_t \end{bmatrix} = \begin{bmatrix} -hI_{m\times m} & 0 \\ gI_{m\times m} & -gI_{m\times m} \end{bmatrix} \begin{bmatrix} \mathbf{S}_t - \mathbf{S} \\ \mathbf{M}_t - \mathbf{S} \end{bmatrix} + d\mathbf{Z}_t, \tag{30}$$

where $\{Z_t\}$ is a $2m$-dimensional Brownian motion with

$$\mathbf{E}\left[d\mathbf{Z}_t\right] = 0, \quad \mathrm{Var}(d\mathbf{Z}_t) = \begin{bmatrix} C & 0 \\ 0 & g^2 I_{m\times m} \end{bmatrix} dt. \tag{31}$$

Applying the results in Appendix A.1, we can establish the following properties.

1. $\begin{bmatrix} \mathbf{S}_t \\ \mathbf{M}_t \end{bmatrix}$ is multivariate normal, since it is the solution of a Gaussian Ornstein-Uhlenbeck stochastic differential equation.

2. $$\lim_{t\to\infty} \mathbf{E}\begin{bmatrix} \mathbf{S}_t \\ \mathbf{M}_t \end{bmatrix} = \begin{bmatrix} \mathbf{S} \\ \mathbf{S} \end{bmatrix}. \tag{32}$$

   Among other things, this means that $\mathbf{M}_t$ is an asymptotically unbiased estimator of $\mathbf{S}_t$, and $\mathbf{M}_t$ is an asymptotically unbiased estimator of $\mathbf{S}$.

3. $$\lim_{t\to\infty} \mathrm{Var}\left(\begin{bmatrix} \mathbf{S}_t \\ \mathbf{M}_t \end{bmatrix}\right) = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{bmatrix} \tag{33}$$

   where

   $$V_{11} = C/(2h), \quad V_{12} = gV_{11}/(g+h), \quad V_{22} = V_{12} + gI_{m\times m}/2. \tag{34}$$

3.3 Plug-In Fractional Kelly Portfolio Performance

We now need to specify our leverage vector $\mathbf{k}_t$. If we knew the underlying value of μ_t, it would make sense to use the fractional Kelly leverage $\alpha\Sigma^{-1}\mu_t$ for some appropriately-chosen

[1] In practice, we would typically use an Euler approximation to evaluate $\{\mathbf{M}_t,\ t = 0, \delta, 2\delta, \ldots\}$.

risk level $\alpha \in [0,1]$. In this case we do not observe μ_t, but since $\mathbf{M}_t$ is an asymptotically unbiased estimator of $\mathbf{S}_t = \Sigma^{-1/2}\mu_t$, it is natural to use the "plug-in" leverage

$$\mathbf{k}_t = \alpha(\Sigma^{-1/2})^T \mathbf{M}_t. \tag{35}$$

In this section we apply Theorem 2.2 to examine portfolio performance using (35).

3.3.1 Limiting Distributions

As a first step, we need to find the means of the limiting distributions of $\mathbf{k}_t \cdot \mu_t$ and $\mathbf{k}_t^T \Sigma \mathbf{k}_t$.

It follows from (32), (33) and (35) that

$$\begin{aligned}
\lim_{t\to\infty} \mathbf{E}\left[\mathbf{k}_t^T \cdot \mu_t\right] &= \lim_{t\to\infty} \alpha \mathbf{E}\left[\mathbf{M}_t^T (\Sigma^{-1/2})(\Sigma^{-1/2})^{-1}\mathbf{S}_t\right] \\
&= \lim_{t\to\infty} \alpha \mathbf{E}\left[\mathbf{M}_t^T \mathbf{S}_t\right] = \lim_{t\to\infty} \alpha \mathbf{E}\left[\mathrm{tr}\left(\mathbf{M}_t^T \mathbf{S}_t\right)\right] \\
&= \alpha \lim_{t\to\infty} \mathbf{E}\left[\mathrm{tr}\left(\mathbf{S}_t \mathbf{M}_t^T\right)\right] = \alpha \lim_{t\to\infty} \mathrm{tr}\left(\mathbf{E}\left[\mathbf{S}_t \mathbf{M}_t^T\right]\right) \\
&= \alpha \mathrm{tr}\left((V_{12} + \mathbf{S}\mathbf{S}^T)\right) = \alpha(\mathrm{tr}\,(V_{12}) + \mathrm{tr}\left(\mathbf{S}\mathbf{S}^T\right)) \\
&= \alpha\left(\gamma g/(g+h) + S^2\right),
\end{aligned} \tag{36}$$

where we define

$$\gamma = (2h)^{-1}\mathrm{tr}\,(C)\,. \tag{37}$$

We also have

$$\begin{aligned}
\lim_{t\to\infty} \mathbf{E}\left[\mathbf{k}_t \Sigma \mathbf{k}_t\right] &= \lim_{t\to\infty} \alpha^2 \mathbf{E}\left[\mathbf{M}_t^T \Sigma^{-1/2}\Sigma(\Sigma^{-1/2})^T \mathbf{M}_t\right] \\
&= \alpha^2 \lim_{t\to\infty} \mathbf{E}\left[\mathrm{tr}(\mathbf{M}_t^T \mathbf{M}_t)\right] \\
&= \alpha^2 \lim_{t\to\infty} \mathrm{tr}\left(\mathbf{E}\left[(\mathbf{M}_t \mathbf{M}_t^T\right]\right) \\
&= \alpha^2 \mathrm{tr}\left(V_{22} + \mathbf{S}\mathbf{S}^T\right) && (38) \\
&= \alpha^2 \mathrm{tr}\left(V_{12} + gI_{m\times m}/2 + \mathbf{S}\mathbf{S}^T\right) && (39) \\
&= \alpha^2(\gamma g/(g+h) + mg/2 + S^2). && (40)
\end{aligned}$$

Now we can make the following statement, which is a direct application of Theorem 2.2.

Corollary 3.1 *Suppose that the m-dimensional price process $\{\mathbf{P}_t\}$ follows (2), and that the time-varying unobserved process $\{\mu_t\}$, after change of basis to the "Sharpe process" $\mathbf{S}_t = \Sigma^{-1/2}\mu_{\mathbf{t}}$, satisfies (20). Let the "Sharpe estimator" $\mathbf{M}_t$ be defined defined by (26,27). If the leverage vector is*

$$\mathbf{k}_t = \alpha(\Sigma^{-1/2})^T \mathbf{M}_t, \tag{41}$$

then the portfolio has long-term expected log-return per unit time

$$L = L_0 + L_1 + L_2, \tag{42}$$

where

$$\begin{aligned} L_0 &= (\alpha - \alpha^2/2)S^2, & (43)\\ L_1 &= (\alpha - \alpha^2/2)\gamma g(g+h)^{-1}, & (44)\\ L_2 &= -\alpha^2 mg/4, & (45) \end{aligned}$$

with $\gamma = (2h)^{-1} tr(C)$, *and* $S = (\mu^T \Sigma^{-1} \mu)^{1/2}$ *denoting the (unknown) portfolio Sharpe ratio. Furthermore, the portfolio's long-term log-return variance per unit time is*

$$V = \alpha^2 [S^2 + \gamma g/(g+h) + mg/2]. \tag{46}$$

Proof: This is a straightforward application of Theorem 2.2, making use of equations (36) and (40). □

Corollary 3.1 has significant implications for optimal portfolio management. The components of growth rate in (42) have natural interpretations.

1. L_0 is the base performance that we could obtain if μ_t and Σ_t were both constant and known. This term is essentially at the core of the bulk of modern portfolio theory.
2. L_1 represents a potential improvement in performance obtained by tracking Sharpe ratios over time with an estimator whose rate of variation is specified by g. L_1 is only positive if the following conditions hold:
   (a) $g > 0$, that is, we are tracking Sharpe ratios, and
   (b) $\gamma > 0$, that is, there is some natural variation in Sharpe to be tracked.
3. L_2 represents loss in performance incurred by active modification of leverage over time. It is negative, and its proportionality to portfolio dimension m can be regarded as a "curse of dimensionality" penalty to performance when tracking Sharpe ratios.

4 Discussion

Building on the framework of [1], we have developed theory that allows us to analyze the performance of an investment scheme, accounting for two very important properties of investment portfolios: imprecise measurement of prior Sharpe ratios, and the time-varying nature of these Sharpe ratios.

Using the framework, we have established new results. To the author's knowledge, Theorem 2.2 has not been stated in the literature, although it has critical implications for long-term performance of managed portfolios. Furthermore, Corollary 3.1 illustrates how standard tracking approaches to portfolio weighting both help by adjusting as Sharpe ratios change, and hurt by injecting additional noise into the process.

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A Supporting Results

A.1 Multivariate Ornstein-Uhlenbeck Process Behavior

First we establish a property of (multivariate) Ornstein-Uhlenbeck processes.

Lemma A.1 *Suppose that a multivariate process* $\{\mathbf{X}_t\}$ *satisfies the system of stochastic differential equations*

$$d\mathbf{X}_t = -A(\mathbf{X}_t - \mu)dt + d\mathbf{V}_t, \tag{47}$$

$$\mathbf{X}_0 = \mathbf{x}_0, \tag{48}$$

where $\{\mathbf{V}_t\}$ *is a multivariate Brownian motion with*

$$d\mathbf{V}_t \sim N(\mathbf{0}, Sdt). \tag{49}$$

If the eigenvalues of A *are strictly positive, then*

$$\mathbf{E}[\mathbf{X}_t] = \exp(-At)\mathbf{x}_0 + (I - \exp(-At))\mu \tag{50}$$

and

$$Var(\mathbf{X}_t) \to V, \tag{51}$$

where V *satisfies*

$$VA^T + AV = S. \tag{52}$$

In some cases, we can write down the component equations of (52) and solve the system. Explicit methods for solving equation (52) can also be found in [3].

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