Loan-Level ABS Estimation: Supplemental Material

The following is intended as an online companion supplement to the manuscript, *Estimating the time-to-event distribution for loan-level data within an asset-backed security*. Please attribute any citations to the original manuscript. This companion includes proofs of all major results, a numeric illustration of the likelihood function under right-censoring, a reference of derivative calculations for implementation, and simulation instructions.

A Proofs

Please see Sections 3 and 4 for complete statements.

A.1 Proof of Theorem 3.1

Proof. Without loss of generality, let $\Delta = 0$. It is equivalent to find the stationary points of the loglikelihood, $\log \mathcal{L}(\boldsymbol{\Theta} \mid \mathcal{S}_n)$. To handle the linear restrictions imposed by $\mathcal{C}$, we will proceed with the technique of Lagrange multipliers (e.g., Ravishanker and Dey, 2002, §2.9, pg. 69). Hence, the Lagrangian function is

$$\log \mathcal{L}(\boldsymbol{g}, p, \pi \mid \mathcal{S}_n) = -\log \alpha + \sum_{v=1}^{m} \sum_{u=v}^{\omega} \hat{h}_{uv} \{\log f(u \mid p) + \log g_v\} + \pi \left(1 - \sum_{v=1}^{m} g_v\right).$$

We now show $\hat{\pi} = 0$. Observe first from (2),

$$\frac{\partial \alpha}{\partial g_v} = \sum_{u=v}^{\omega} f(u \mid p), \quad v \in \mathcal{V},$$

and

$$\frac{\partial \alpha}{\partial p} = \sum_{u=1}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \left(\sum_{v=1}^{\min(u,m)} g_v \right) \equiv \sum_{v=1}^{m} g_v \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right).$$

For convenience of notation, define $\ell \coloneqq \log \mathcal{L}(\boldsymbol{g}, p, \pi \mid \mathcal{S}_n)$. Therefore, for $v \in \mathcal{V}$,

$$\begin{aligned}\frac{\partial \ell}{\partial g_v} &= -\frac{1}{\alpha}\frac{\partial \alpha}{\partial g_v} + \frac{\partial}{\partial g_v}\sum_{v=1}^{m}\sum_{u=v}^{\omega}\hat{h}_{uv}\log g_v - \pi \\ &= -\frac{1}{\alpha}\sum_{u=v}^{\omega} f(u \mid p) + \frac{\hat{h}_{\bullet v}}{g_v} - \pi.\end{aligned}$$

Observe,

$$g_v\left(\frac{\partial \ell}{\partial g_v}\right) = 0 \implies -\frac{1}{\alpha} g_v \sum_{u=v}^{\omega} f(u \mid p) + \hat{h}_{\bullet v} - \pi g_v = 0.$$

That is,

$$\sum_{v=1}^{m} g_v\left(\frac{\partial \ell}{\partial g_v}\right) = 0 \implies -\frac{1}{\alpha}\sum_{v=1}^{m} g_v\left(\sum_{u=v}^{\omega} f(u \mid p)\right) + \sum_{v=1}^{m}\hat{h}_{\bullet v} - \pi\sum_{v=1}^{m} g_v = 0.$$

Because $\sum_v \hat{h}_{\bullet v} = 1$, $g_v > 0$ by assumption, and (2), we must have $\hat{\pi} = 0$. Thus, any stationary point of the unconstrained optimization of (3) will also be a stationary point of the constrained optimization of (3) with solutions restricted to the convex subset, $\mathcal{C}$. This proves the final sentence of Theorem 3.1. Proceeding,

$$\left.\frac{\partial \ell}{\partial g_v}\right|_{\hat{\pi}} = -\frac{1}{\alpha}\sum_{u=v}^{\omega} f(u \mid p) + \frac{\hat{h}_{\bullet v}}{g_v} = 0 \iff g_v = \frac{\alpha \hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}. \tag{S.1}$$

Further,

$$\begin{aligned}\frac{\partial \ell}{\partial p} &= -\frac{1}{\alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial}{\partial p}\sum_{v=1}^{m}\sum_{u=v}^{\omega}\hat{h}_{uv}\log f(u \mid p) \\ &= -\frac{1}{\alpha}\sum_{v=1}^{m} g_v\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p} f(u \mid p)\right) + \sum_{v=1}^{m}\sum_{u=v}^{\omega}\frac{\hat{h}_{uv}}{f(u \mid p)}\frac{\partial}{\partial p} f(u \mid p).\end{aligned}$$

Hence, by (S.1),

$$\left.\frac{\partial \ell}{\partial p}\right|_{g_v} = -\frac{1}{\alpha}\left[\sum_{v=1}^{m}\left(\frac{\alpha \hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p} f(u \mid p)\right)\right] + \sum_{v=1}^{m}\sum_{u=v}^{\omega}\frac{\hat{h}_{uv}}{f(u \mid p)}\frac{\partial}{\partial p} f(u \mid p)$$

$$= -\sum_{v=1}^{m}\left(\frac{\hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p}f(u \mid p)\right) + \sum_{v=1}^{m}\sum_{u=v}^{\omega}\frac{\hat{h}_{uv}}{f(u \mid p)}\frac{\partial}{\partial p}f(u \mid p).$$

Thus,

$$\left.\frac{\partial \ell}{\partial p}\right|_{g_v} = 0 \iff \sum_{v=1}^{m}\left(\frac{\hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p}f(u \mid p)\right) = \sum_{v=1}^{m}\sum_{u=v}^{\omega}\frac{\hat{h}_{uv}}{f(u \mid p)}\frac{\partial}{\partial p}f(u \mid p).$$

This proves (7). Finally, recall the constraint $\sum_v g_v = 1$. Hence, returning to (S.1), we must have

$$1 = \sum_{v \in \mathcal{V}} g_v = \sum_{v \in \mathcal{V}} \frac{\alpha \hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)} \implies \alpha = \left[\sum_{k=1}^{m}\frac{\hat{h}_{\bullet k}}{S(k \mid p)}\right]^{-1}.$$

Therefore, for any $\hat{p} \in \hat{\mathcal{P}}$ and all $v \in \mathcal{V}$,

$$\hat{g}_v = \frac{\alpha(\hat{p})\hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid \hat{p})} = \frac{\hat{h}_{\bullet v}}{S(v \mid \hat{p})}\left[\sum_{k=1}^{m}\frac{\hat{h}_{\bullet k}}{S(k \mid \hat{p})}\right]^{-1}.$$

This recovers (5) and completes the proof. □

A.2 Proof of Corollary 3.1.1

Proof. Without loss of generality, assume $\Delta = 0$. The proof closely follows the proof of Theorem 3.1, and so we omit repetitive details. Recall the form of the likelihood in (9) to define the equivalent Lagrangian function

$$\log \mathcal{L}(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_n') = -\log \alpha + \sum_{v=1}^{m}\sum_{u=v}^{\omega}\hat{h}_{uv}\{\log f(u \mid \boldsymbol{p}) + \log g_v\} + \pi\left(1 - \sum_{v=1}^{m} g_v\right).$$

Because

$$\frac{\partial \alpha}{\partial g_v} = \sum_{u=v}^{\omega} f(u \mid \boldsymbol{p}), \quad v \in \mathcal{V},$$

$$\frac{\partial \alpha}{\partial p_j} = \sum_{u=1}^{\omega}\frac{\partial}{\partial p_j}f(u \mid \boldsymbol{p})\left(\sum_{v=1}^{\min(u,m)} g_v\right) \equiv \sum_{v=1}^{m} g_v\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p_j}f(u \mid \boldsymbol{p})\right),$$

for $j = 1, \dots, r$, and

$$\frac{\partial \log \mathcal{L}(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}'_n)}{\partial g_v} = -\frac{1}{\alpha} \sum_{u=v}^{\omega} f(u \mid \boldsymbol{p}) + \frac{\hat{h}_{\bullet v}}{g_v} - \pi,$$

for all $v \in \mathcal{V}$, it follows that $\hat{\pi} = 0$. Further,

$$\left. \frac{\partial \log \mathcal{L}(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}'_n)}{\partial g_v} \right|_{\hat{\pi}} = 0 \iff g_v = \frac{\alpha \hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid \boldsymbol{p})}. \tag{S.2}$$

Thus, from (S.2) and

$$\frac{\partial \log \mathcal{L}(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}'_n)}{\partial p_j} = -\frac{1}{\alpha} \sum_{v=1}^{m} g_v \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p_j} f(u \mid \boldsymbol{p}) \right) + \sum_{v=1}^{m} \sum_{u=v}^{\omega} \frac{\hat{h}_{uv}}{f(u \mid \boldsymbol{p})} \frac{\partial}{\partial p_j} f(u \mid \boldsymbol{p}),$$

it follows that

$$\left. \frac{\partial \log \mathcal{L}(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}'_n)}{\partial p_j} \right|_{g_v} = -\xi_1(j) + \xi_2(j) = 0 \iff \xi_1(j) = \xi_2(j), \forall j = 1, \dots, r.$$

The set of simultaneous solutions, $\hat{\boldsymbol{p}}$, recovers the estimator (11). The proof is complete by replacing $\hat{\boldsymbol{p}}$ in (S.2) and using the constraint $\sum_{\mathcal{V}} g_v = 1$ to recover (10). □

A.3 Proof of Theorem 3.2

Proof. Observe

$$\begin{aligned}
\mathbf{E}[\psi(X_i, Y_i, p)] &= \sum_{v=\Delta+1}^{\Delta+m} \mathbf{E}\left[\left(\frac{\sum_{u=v}^{\omega} W_i}{\sum_{u=v}^{\omega} f(u \mid p)} \right) \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right) - \sum_{u=v}^{\omega} \frac{W_i}{f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p) \right] \\
&= \sum_{v=\Delta+1}^{\Delta+m} \left(\frac{\sum_{u=v}^{\omega} \mathbf{E}[W_i]}{\sum_{u=v}^{\omega} f(u \mid p)} \right) \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right) - \sum_{u=v}^{\omega} \frac{\mathbf{E}[W_i]}{f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p).
\end{aligned}$$

But $\mathbf{E}[W_i(u, v)] = h_*(u, v)$ and so for any $v \in \{\Delta + 1, \dots, \Delta + m\}$,

$$\left(\frac{\sum_{u=v}^{\omega} \mathbf{E}[W_i]}{\sum_{u=v}^{\omega} f(u \mid p)} \right) \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right) - \sum_{u=v}^{\omega} \frac{\mathbf{E}[W_i]}{f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p)$$

$$
\begin{aligned}
&= \left(\frac{\sum_{u=v}^{\omega} h_*(u,v)}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p)\right) - \sum_{u=v}^{\omega} \frac{h_*(u,v)}{f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p) \\
&= \left(\frac{g_v \sum_{u=v}^{\omega} f(u \mid p)}{\alpha \sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p)\right) - \sum_{u=v}^{\omega} \frac{f(u \mid p) g_v}{\alpha f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p) \\
&= \frac{g_v}{\alpha} \sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) - \frac{g_v}{\alpha} \sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \\
&= 0.
\end{aligned}
$$

Hence, $\mathbf{E}[\psi(X_i, Y_i, p)] = 0$. Further,

$$
\Psi_n(p) = \frac{1}{n} \sum_{i=1}^{n} \psi(X_i, Y_i, p),
$$

and so $\Psi_n(p) \xrightarrow{\mathbf{P}} \mathbf{E}[\psi(X_i, Y_i, p)]$ by the Law of Large Numbers (Lehmann and Casella, 1998, Theorem 8.2, pg. 54-55). That $\Psi_n(\hat{p}_n) = 0$ is immediate by the conditions of (7). The remainder follows the standard Taylor series analysis (e.g., van der Vaart, 1998, §5.3, pg. 51-52), with $\partial/\partial p(\psi)$ following by the quotient rule (Rudin, 1976, Theorem 5.3, pg. 104). □

A.4 Proof of Corollary 3.2.1

Proof. The result (12) follows from Theorem 3.2 and Slutsky's Theorem (Lehmann and Casella, 1998, Theorem 8.10, pg. 58). The latter result is a classical result of maximum likelihood theory (e.g., van der Vaart, 1998, §5.5). □

A.5 Proof of Theorem 3.3

Proof. From the definition of the survival function in (6), the left-hand side of (7) becomes

$$
\begin{aligned}
\sum_{v=\Delta+1}^{\Delta+m} \left(\frac{\hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p)\right) &= \sum_{v=\Delta+1}^{\Delta+m} \left(\frac{\hat{h}_{\bullet v}}{\sum_{u=v}^{\omega} f(u \mid p)}\right) \frac{\partial}{\partial p}\left(\sum_{u=v}^{\omega} f(u \mid p)\right) \\
&= \sum_{v=\Delta+1}^{\Delta+m} \left(\frac{\hat{h}_{\bullet v}}{S(v \mid p)}\right) \frac{\partial}{\partial p} S(v \mid p)
\end{aligned}
$$

$$= \sum_{v=\Delta+1}^{\Delta+m} \hat{h}_{\bullet v} \frac{\partial}{\partial p} \ln S(v \mid p)$$
$$= \frac{\partial}{\partial p} \sum_{v=\Delta+1}^{\Delta+m} \hat{h}_{\bullet v} \ln S(v \mid p).$$

Similarly, on the right-hand side of (7),

$$\sum_{v=\Delta+1}^{\Delta+m} \sum_{u=v}^{\omega} \frac{\hat{h}_{uv}}{f(u \mid p)} \frac{\partial}{\partial p} f(u \mid p) = \frac{\partial}{\partial p} \sum_{u=\Delta+1}^{\omega} \hat{h}_{u\bullet} \ln f(u \mid p).$$

Thus, (7) may equivalently be stated as

$$\left\{ p \in \mathcal{P} : \frac{\partial}{\partial p} \sum_{v=\Delta+1}^{\Delta+m} \hat{h}_{\bullet v} \ln S(v \mid p) = \frac{\partial}{\partial p} \sum_{u=\Delta+1}^{\omega} \hat{h}_{u\bullet} \ln f(u \mid p) \right\},$$

or

$$\frac{\partial}{\partial p} \left(\sum_{v=\Delta+1}^{\Delta+m} \hat{h}_{\bullet v} \ln S(v \mid p) - \sum_{u=\Delta+1}^{\omega} \hat{h}_{u\bullet} \ln f(u \mid p) \right) = 0. \tag{S.3}$$

But,

$$\sum_{v=\Delta+1}^{\Delta+m} \hat{h}_{\bullet v} \ln S(v \mid p) - \sum_{u=\Delta+1}^{\omega} \hat{h}_{u\bullet} \ln f(u \mid p) = \sum_{v=\Delta+1}^{\Delta+m} \ln S(v \mid p)^{\hat{h}_{\bullet v}} - \sum_{u=\Delta+1}^{\omega} \ln f(u \mid p)^{\hat{h}_{u\bullet}}$$
$$= \ln \left(\frac{\prod_{v=\Delta+m}^{\Delta+m} S(v \mid p)^{\hat{h}_{\bullet v}}}{\prod_{u=\Delta+1}^{\omega} f(u \mid p)^{\hat{h}_{u\bullet}}} \right).$$

Therefore, (S.3) may equivalently be written as

$$\frac{\partial}{\partial p} \ln \left(\frac{\prod_{v=\Delta+m}^{\Delta+m} S(v \mid p)^{\hat{h}_{\bullet v}}}{\prod_{u=\Delta+1}^{\omega} f(u \mid p)^{\hat{h}_{u\bullet}}} \right) = 0. \tag{S.4}$$

Because $f(u \mid p) > 0$ for all $u \in \mathcal{U}$, $p \in \mathcal{P}$ by assumption (and, by extension, $S(u \mid p) > 0$ for all $u \in \mathcal{U}$, $p \in \mathcal{P}$), (S.4) is true if and only if,

$$\frac{\partial}{\partial p} \frac{\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{h}_{\bullet v}}}{\prod_{u=\Delta+1}^{\omega} f(u \mid p)^{\hat{h}_{u\bullet}}} = 0.$$

This recovers (13) and completes the proof. □

A.6 Proof of Theorem 3.4

Proof. Without loss of generality, let $\Delta = 0$. Given (14), the survival function becomes

$$S_T(u \mid p) = (1-p)^{u-1}, \quad u \in \{1, \ldots, \omega\}.$$

Hence, (13) reduces to

$$\frac{\partial}{\partial p} \frac{\prod_{v=1}^{m} S(v \mid p)^{\hat{h}_{\bullet v}}}{\prod_{u=1}^{\omega} f(u \mid p)^{\hat{h}_{u\bullet}}} = \frac{\partial}{\partial p} \frac{(1-p)^a}{p^b} = \frac{(1-p)^a}{p^b} \left[\frac{a}{1-p} - \frac{b}{p} \right].$$

Because $0 < p < 1$,

$$\frac{(1-p)^a}{p^b} \left[\frac{a}{1-p} - \frac{b}{p} \right] = 0 \iff \frac{a}{1-p} - \frac{b}{p} = 0 \implies \hat{p} = \frac{b}{b-a},$$

which is unique. Trivially, $\hat{p} \in \mathcal{C}$. To find $\hat{\boldsymbol{g}}$, observe

$$S_T(u \mid \hat{p}) = \left(\frac{a}{a-b} \right)^{u-1},$$

for $u \in \mathcal{U}$. Hence, replace $S_T(u \mid \hat{p})$ in (5). That $\hat{\boldsymbol{g}}$ is unique follows from the uniqueness of $\hat{p}$. Further, by Theorem 3.1, $\hat{\boldsymbol{g}} \in \mathcal{C}$.

To see that $\hat{p}$, $\hat{\boldsymbol{g}}$ are together the global maximum of $\mathcal{L}$, it is sufficient to examine the behavior of $\ell(\boldsymbol{g}, p \mid \mathcal{S}_n) \equiv \ell(\boldsymbol{g}, p, \hat{\pi} \mid \mathcal{S}_n)$ for the boundaries of $\mathcal{C}$ (recall the convexity of $\mathcal{C}$). When $p = 0$, $f_T(u \mid p) = 0$ for all $u \in \{1, \ldots, \omega - 1\}$. Thus, for any $u \in \{1, \ldots, \omega - 1\}$, $\log f_T(u \mid p) \downarrow -\infty$ and $\ell(\boldsymbol{g}, p \mid \mathcal{S}_n)$ cannot obtain a maximum. When $p = 1$, $f_T(u \mid p) = 0$ for all $u \in \{1, \ldots, \omega\}$. Thus, $\log f_T(u \mid p) \downarrow -\infty$ for all $u \in \{1, \ldots, \omega\}$, and $\ell(\boldsymbol{g}, p \mid \mathcal{S}_n)$ similarly cannot obtain a maximum. For the boundaries of $\mathcal{C}$ in terms of $\boldsymbol{\mathcal{G}}$, the constraint $\sum_v g_v = 1$ requires at least one $g_v = 0$ for any $g_v = 1$ (or there is a $g_v = 0$ directly). Hence,

$\log g_v \downarrow -\infty$ and a maximum cannot be obtained. Therefore, $\hat{p}$, $\hat{\boldsymbol{g}}$ are the MLE for the parameters $p, \boldsymbol{g}$ of the conditional bivariate probability mass function, h_*, defined in (1). □

A.7 Statement & Proof of Corollary A.7.1

This section provides a restatement of Theorem 3.4 under an alternative parameterization. Aside from completeness, one advantage of Corollary A.7.1 is the difference in parameter space for p. Under the PL geometric distribution in (14), $p \in (0,1)$, whereas $p > 0$ for (S.5) in the discretized, PL exponential distribution. Such differences may have utility in any generalized linear model (GLM) regression analysis build from the model of (1).

Corollary A.7.1 (MLE of $\boldsymbol{g}$, p, discretized, PL exponential)**.** *Define the* discretized*, policy limit exponential distribution with parameter, $p > 0$, as*

$$f_T(u \mid p) = \begin{cases} \exp\left(-\dfrac{\{u-(\Delta+1)\}}{p}\right)\left[1-\exp\left(-\dfrac{1}{p}\right)\right] & \Delta+1 \le u \le \omega-1, \\ \exp\left(-\dfrac{\{u-(\Delta+1)\}}{p}\right) & u = \omega. \end{cases} \tag{S.5}$$

Then, for the conditional bivariate probability mass function, h_*, defined in (1)*, under the sampling conditions of Theorem 3.1, the MLE of the parameter p is*

$$\hat{p}_{\text{MLE}} = -\left[\ln\left(\frac{a}{a-b}\right)\right]^{-1}, \tag{S.6}$$

where a and b follow (16) *and* (17) *of Theorem 3.4, respectively. Further, $S_T(\cdot \mid \hat{p})$ is equivalent for* (S.5) *with* (S.6) *to* (14) *with* (15)*. Therefore, the MLE of $\boldsymbol{g}$ is equivalent to* (18) *in Theorem 3.4.*

Proof. Given the similarity to the proof of Theorem 3.4, we proceed with repetitive details omitted. Without loss of generality, let $\Delta = 0$. Given (S.5), the survival function then becomes the continuous equivalent,

$$S_T(u \mid p) = \exp\left(\frac{-(u-1)}{p}\right),$$

for $u \in \{1, \dots, \omega\}$. Hence, (13) simplifies. To see this, let $q(z \mid p) \equiv q(z) = \exp(-z/p)$ for $z \in \{1, \dots, \omega\}$ to write

$$\frac{\partial}{\partial p} \frac{\prod_{v=1}^{m} S(v \mid p)^{\hat{h}_{\bullet v}}}{\prod_{u=1}^{\omega} f(u \mid p)^{\hat{h}_{u\bullet}}} = \frac{q(a)\{1-q(1)\}^{-b}}{p^2}\left(a + \frac{b \cdot q(1)}{1-q(1)}\right).$$

Because $p > 0$,

$$\frac{q(a)\{1-q(1)\}^{-b}}{p^2}\left(a + \frac{b \cdot q(1)}{1-q(1)}\right) = 0 \iff a + \frac{b \cdot q(1)}{1-q(1)} = 0.$$

That is,

$$\hat{p} = -\left[\ln\left(\frac{a}{a-b}\right)\right]^{-1},$$

which is unique. Trivially, $\hat{p} \in \mathcal{C}$. To find $\hat{\boldsymbol{g}}$, replace $S_T(u \mid \hat{p})$ in (5). That $\hat{\boldsymbol{g}}$ is unique follows from the uniqueness of $\hat{p}$. Further, by Theorem 3.1, $\hat{\boldsymbol{g}} \in \mathcal{C}$.

To see that $\hat{p}$, $\hat{\boldsymbol{g}}$ are together the global maximum of $\mathcal{L}$, it is sufficient to examine the behavior of $\ell(\boldsymbol{g}, p \mid \mathcal{S}_n) \equiv \ell(\boldsymbol{g}, p, \hat{\pi} \mid \mathcal{S}_n)$ for the boundaries of $\mathcal{C}$. The analysis proceeds as in the final steps of the proof of Theorem 3.4. Therefore, $\hat{p}$, $\hat{\boldsymbol{g}}$ are the MLE for the parameters $p, \boldsymbol{g}$ of the conditional bivariate probability mass function, h_*, defined in (1). □

A.8 Proof of Theorem 4.1

Proof. The proof is similar to the proof of Theorem 3.1, and so we proceed with less detail. Without loss of generality, let $\Delta = 0$. For convenience of notation, define $\ell_\tau := \log \mathcal{L}_\tau(\boldsymbol{g}, p \mid$

$\mathcal{S}_{\tau,n}$). The Lagrangian function (e.g., Ravishanker and Dey, 2002, §2.9, pg. 69) becomes

$$\ell_\tau = -\log\alpha + \sum_{v=1}^{m}\hat{\gamma}_n(v)\log g_v + \frac{1}{n}\sum_{i=1}^{n}\{D_i \log f(Z_i \mid p) + (1-D_i)\log S(Z_i+1 \mid p)\} + \pi\left(1-\sum_{v=1}^{m} g_v\right).$$

Because

$$\frac{\partial \ell_\tau}{\partial g_v} = -\frac{1}{\alpha}\left(\sum_{u=v}^{\omega} f(u \mid p)\right) + \hat{\gamma}_n(v)\frac{1}{g_v} - \pi,$$

we have

$$\sum_{v=1}^{m} g_v\left(\frac{\partial \ell_\tau}{\partial g_v}\right) = 0 \iff \hat{\pi} = 0,$$

as $\sum_v \hat{\gamma}_n(v) = 1$. Thus, any stationary point of the unconstrained optimization of $\mathcal{L}_\tau$ will also be a stationary point of the constrained optimization of $\mathcal{L}_\tau$ with solutions restricted to the convex subset, $\mathcal{C}$. This proves the final sentence of Theorem 4.1. Further, for all $v \in \mathcal{V}$,

$$\left.\frac{\partial \ell_\tau}{\partial g_v}\right|_{\hat{\pi}} = 0 \iff g_v = \frac{\alpha\hat{\gamma}_n(v)}{\sum_{u=v}^{\omega} f(u \mid p)}. \tag{S.7}$$

Thus, via (S.7),

$$\left.\frac{\partial \ell_\tau}{\partial p}\right|_{g_v} = 0 \iff \sum_{v=1}^{m}\left(\frac{\hat{\gamma}_n(v)}{\sum_{u=v}^{\omega} f(u \mid p)}\right)\left(\sum_{u=v}^{\omega}\frac{\partial}{\partial p} f(u \mid p)\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{D_i}{f(Z_i \mid p)}\frac{\partial}{\partial p} f(Z_i \mid p) + \frac{1-D_i}{S(Z_i+1 \mid p)}\frac{\partial}{\partial p} S(Z_i+1 \mid p)\right).$$

Finally, because we require $\sum_{\mathcal{V}} g_v = 1$, we have, for any $\hat{p}_\tau \in \hat{\mathcal{P}}_\tau$ and $v \in \mathcal{V}$,

$$\hat{g}_\tau(v) = \frac{\hat{\gamma}_n(v)}{S(v \mid \hat{p}_\tau)}\left[\sum_{k=\Delta+1}^{\Delta+m}\frac{\hat{\gamma}_n(v)}{S(k \mid \hat{p}_\tau)}\right]^{-1}.$$

□

A.9 Proof of Corollary 4.1.1

Proof. Without loss of generality, assume $\Delta = 0$. The proof closely follows the proof of Corollary 3.1.1 and Theorem 4.1, and so we omit repetitive details. Recall the form of $\mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n})$ to define the equivalent Lagrangian function

$$\begin{aligned}\log \mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n}) = &-\log\alpha + \sum_{v=1}^{m}\sum_{u=v}^{\omega}\hat{\gamma}_n(v)\log g_v \\ &+ \frac{1}{n}\sum_{i=1}^{n}\{D_i \log f(Z_i \mid \boldsymbol{p}) + (1-D_i)\log S(Z_i+1 \mid \boldsymbol{p})\} \\ &+ \pi_\tau\left(1 - \sum_{v=1}^{m} g_v\right).\end{aligned}$$

Because

$$\frac{\partial \log \mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n})}{\partial g_v} = -\frac{1}{\alpha}\sum_{u=v}^{\omega} f(u \mid \boldsymbol{p}) + \frac{\hat{\gamma}_n(v)}{g_v} - \pi_\tau,$$

for all $v \in \mathcal{V}$, it follows that $\hat{\pi}_\tau = 0$. Further,

$$\left.\frac{\partial \log \mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n})}{\partial g_v}\right|_{\hat{\pi}_\tau} = 0 \iff g_v = \frac{\alpha\hat{\gamma}_n(v)}{\sum_{u=v}^{\omega} f(u \mid \boldsymbol{p})}. \tag{S.8}$$

Thus, from (S.8) and

$$\begin{aligned}\frac{\partial \log \mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n})}{\partial p_j} = &-\frac{1}{\alpha}\left(\frac{\partial\alpha}{\partial p_j}\right) \\ &+ \frac{1}{n}\sum_{i=1}^{n}\left(\frac{D_i}{f(Z_i \mid \boldsymbol{p})}\frac{\partial}{\partial p_j}f(Z_i \mid \boldsymbol{p}) + \frac{1-D_i}{S(Z_i+1 \mid \boldsymbol{p})}\frac{\partial}{\partial p_j}S(Z_i+1 \mid \boldsymbol{p})\right),\end{aligned}$$

it follows that

$$\left.\frac{\partial \log \mathcal{L}_\tau(\boldsymbol{g}, \boldsymbol{p} \mid \mathcal{S}_{\tau,n})}{\partial p_j}\right|_{g_v} = -\varphi_1(j) + \varphi_2(j) = 0 \iff \varphi_1(j) = \varphi_2(j), \forall j = 1, \ldots, r'.$$

The set of simultaneous solutions, $\hat{\boldsymbol{p}}_\tau$, recovers the estimator (23). The proof is complete by replacing $\hat{\boldsymbol{p}}_\tau$ in (S.8) and using the constraint $\sum_{\mathcal{V}} g_v = 1$ to recover (22). □

A.10 Proof of Theorem 4.2

Proof. Recall $D = 0$ if an observation is right-censored and $D = 1$ otherwise (see Section 4 as needed). It is first instructive to show by (2) and (6),

$$\begin{aligned}
&\sum_{v=\Delta+1}^{m+\Delta}\sum_{u=v}^{\omega}\sum_{d=0}^{1}\{\mathbf{1}(D=d)h_*(u,v)+(1-\mathbf{1}(D=d))\bar{h}_*(u,v)\}\\
&=\sum_{v=\Delta+1}^{m+\Delta}\frac{g_v}{\alpha}\sum_{u=v}^{\omega}\sum_{d=0}^{1}(\mathbf{1}(D=d)f(u\mid p)+(1-\mathbf{1}(D=d))S(u+1\mid p))\\
&=\frac{1}{\alpha}\sum_{v=\Delta+1}^{m+\Delta}g_v\left[\sum_{u=v:u=v+\tau}S(u+1\mid p)+\sum_{u=v:u\le v+\tau}f(u\mid p))\right]\\
&=\frac{1}{\alpha}\sum_{v=\Delta+1}^{m+\Delta}g_v\left[S(v+\tau+1\mid p)+\sum_{u=v}^{v+\tau}f(u\mid p))\right]\\
&=\frac{1}{\alpha}\sum_{v=\Delta+1}^{m+\Delta}g_v\left(\sum_{u=v}^{\omega}f(u\mid p)\right)\\
&=1,
\end{aligned}$$

is a valid probability density. Hence,

$$\mathbf{E}[\psi_\tau(Y_i,Z_i,D_i,p)]=\mathbf{E}[\xi_1(Y_i,Z_i,D_i,p)]-\mathbf{E}[\xi_2(Y_i,Z_i,D_i,p)], \tag{S.9}$$

where

$$\xi_1(Y_i,Z_i,D_i,p)=\sum_{v_*=\Delta+1}^{\Delta+m}\left(\frac{\mathbf{1}(Y_i=v_*)}{\sum_{u=v_*}^{\omega}f(u\mid p)}\right)\left(\sum_{u=v_*}^{\omega}\frac{\partial}{\partial p}f(u\mid p)\right),$$

and

$$\xi_2(Y_i,Z_i,D_i,p)=\frac{D_i}{f(Z_i\mid p)}\frac{\partial}{\partial p}f(Z_i\mid p)+\frac{1-D_i}{S(Z_i+1\mid p)}\frac{\partial}{\partial p}S(Z_i+1\mid p).$$

We consider each expectation of (S.9) in turn for any i, $1\le i\le n$. Observe,

$$\mathbf{E}[\xi_1(Y_i,Z_i,D_i,p)]$$

$$
\begin{aligned}
&= \sum_{v=\Delta+1}^{m+\Delta} \sum_{u=v}^{\omega} \sum_{d=0}^{1} \{\mathbf{1}(d=1)h_*(u,v) + (1-\mathbf{1}(d=1))\bar{h}_*(u,v)\}\xi_1(v,u,d,p) \\
&= \sum_{v=\Delta+1}^{m+\Delta} \Bigg\{ \sum_{u=v:u=v+\tau}^{\omega} \frac{S(u+1 \mid p)g_v}{\alpha} \left[\frac{\sum_{u=v}^{\omega} f'(u \mid p)}{\sum_{u=v}^{\omega} f(u \mid p)} \right] \\
&\qquad + \sum_{u=v:u\le v+\tau}^{\omega} \frac{f(u \mid p)g_v}{\alpha} \left[\frac{\sum_{u=v}^{\omega} f'(u \mid p)}{\sum_{u=v}^{\omega} f(u \mid p)} \right] \Bigg\} \\
&= \sum_{v=\Delta+1}^{m+\Delta} \frac{g_v}{\alpha} \left[\frac{\sum_{u=v}^{\omega} f'(u \mid p)}{\sum_{u=v}^{\omega} f(u \mid p)} \right] \left\{ S(v+\tau+1) + \sum_{u=v}^{v+\tau} f(u \mid p) \right\} \\
&= \sum_{v=\Delta+1}^{m+\Delta} \frac{g_v}{\alpha} \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}[\xi_2(Y_i, Z_i, D_i, p)] \\
&= \sum_{v=\Delta+1}^{m+\Delta} \sum_{u=v}^{\omega} \sum_{d=0}^{1} \{\mathbf{1}(d=1)h_*(u,v) + (1-\mathbf{1}(d=1))\bar{h}_*(u,v)\}\xi_2(v,u,d,p) \\
&= \sum_{v=\Delta+1}^{m+\Delta} \Bigg\{ \sum_{u=v:u=v+\tau}^{\omega} \frac{S(u+1 \mid p)g_v}{\alpha} \frac{S'(u+1 \mid p)}{S(u+1 \mid p)} + \sum_{u=v:u\le v+\tau}^{\omega} \frac{f(u \mid p)g_v}{\alpha} \frac{f'(u \mid p)}{f(u \mid p)} \Bigg\} \\
&= \sum_{v=\Delta+1}^{m+\Delta} \frac{g_v}{\alpha} \left\{ S'(v+\tau+1) + \sum_{u=v}^{v+\tau} f'(u \mid p) \right\} \\
&= \sum_{v=\Delta+1}^{m+\Delta} \frac{g_v}{\alpha} \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right).
\end{aligned}
$$

Therefore, $\mathbf{E}[\xi_1(Y_i, Z_i, D_i, p)] = \mathbf{E}[\xi_2(Y_i, Z_i, D_i, p)]$ and $\mathbf{E}[\psi_\tau(Y_i, Z_i, D_i, p)] = 0$ for all $1 \le i \le n$. Further,

$$
\Psi_{\tau,n}(p) = \frac{1}{n} \sum_{i=1}^{n} \psi_\tau(Y_i, Z_i, D_i, p),
$$

and so $\Psi_{\tau,n}(p) \xrightarrow{\mathbf{P}} \psi_\tau(Y_i, Z_i, D_i, p)$ by the Law of Large Numbers (Lehmann and Casella, 1998, Theorem 8.2, pg. 54-55). That $\Psi_{\tau,n}(\hat{p}_n) = 0$ is immediate by the conditions of (21). The remainder follows the standard Taylor series analysis (e.g., van der Vaart, 1998, §5.3, pg. 51-52), with $\partial/\partial p(\psi_\tau)$ following by the quotient rule (Rudin, 1976, Theorem 5.3, pg.

104). □

A.11 Proof of Corollary 4.2.1

Proof. The result (24) follows from Theorem 4.2 and Slutsky's Theorem (Lehmann and Casella, 1998, Theorem 8.10, pg. 58). The latter result is a classical result of maximum likelihood theory (e.g., van der Vaart, 1998, §5.5). □

A.12 Proof of Corollary 4.2.2

Proof of Corollary 4.2.2. The novelty of this proof in comparison to the proof of Theorem 3.4 is to first derive the equivalent statement of Theorem 3.3 under the additional incomplete data setting of right-censoring. We now do this formally.

Lemma 1 (Equivalence of $\hat{\mathcal{P}}_\tau$)**.** *Assume the conditions of Theorem 4.1. Then $p \in \hat{\mathcal{P}}_\tau$ if and only if*

$$\frac{\partial}{\partial p} \frac{\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{\gamma}_n(v)}}{\prod_{i=1}^{n} f(Z_i \mid p)^{D_i/n} S(Z_i+1 \mid p)^{(1-D_i)/n}} = 0. \tag{S.10}$$

Proof of Lemma 1. Observe first

$$\sum_{v=\Delta+1}^{\Delta+m} \left(\frac{\hat{\gamma}_n(v)}{\sum_{u=v}^{\omega} f(u \mid p)} \right) \left(\sum_{u=v}^{\omega} \frac{\partial}{\partial p} f(u \mid p) \right) = \frac{\partial}{\partial p} \left(\sum_{v=\Delta+1}^{\Delta+m} \hat{\gamma}_n(v) \ln S(v \mid p) \right).$$

Similarly,

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^{n} \left(\frac{D_i}{f(Z_i \mid p)} \frac{\partial}{\partial p} f(Z_i \mid p) + \frac{1-D_i}{S(Z_i+1 \mid p)} \frac{\partial}{\partial p} S(Z_i+1 \mid p) \right) \\ &\quad = \frac{\partial}{\partial p} \left(\frac{1}{n} \sum_{i=1}^{n} \{ D_i \ln f(Z_i \mid p) + (1-D_i) \ln S(Z_i+1 \mid p) \} \right). \end{aligned}$$

Hence, the conditions on p in the set $\mathcal{P}_\tau$ are equivalent to all $p \in \mathcal{P}$ such that

$$\frac{\partial}{\partial p}\Bigg(\sum_{v=\Delta+1}^{\Delta+m} \hat{\gamma}_n(v) \ln S(v \mid p) \\ - \frac{1}{n}\sum_{i=1}^{n}\{D_i \ln f(Z_i \mid p) + (1-D_i)\ln S(Z_i+1 \mid p)\}\Bigg) = 0. \tag{S.11}$$

But,

$$\sum_{v=\Delta+1}^{\Delta+m} \hat{\gamma}_n(v)\ln S(v \mid p) = \ln\left(\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{\gamma}_n(v)}\right),$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\{D_i \ln f(Z_i \mid p) + (1-D_i)\ln S(Z_i+1 \mid p)\} \\ = \ln\left(\prod_{i=1}^{n} f(Z_i \mid p)^{D_i/n} S(Z_i+1 \mid p)^{(1-D_i)/n}\right).$$

Therefore, the conditions on p in (S.11) are equivalent to

$$\frac{\partial}{\partial p}\ln\left(\frac{\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{\gamma}_n(v)}}{\prod_{i=1}^{n} f(Z_i \mid p)^{D_i/n} S(Z_i+1 \mid p)^{(1-D_i)/n}}\right) = 0 \tag{S.12}$$

But $f(\cdot \mid p), S(\cdot \mid p) > 0$ for all $p \in \mathcal{P}_\tau$. Thus, (S.12) is true true if and only if (S.10) is true, completing the proof. □

To complete the proof of Corollary 4.2.2, recall (14) and observe

$$\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{\gamma}_n(v)} = (1-p)^{\sum_v (v-(\Delta+1))\hat{\gamma}_n(v)},$$

$$\prod_{i=1}^{n} f(Z_i \mid p)^{D_i/n} = p^{(\sum_i \mathbf{1}(Z_i \neq \omega) D_i)/n}(1-p)^{(\sum_i (Z_i-(\Delta+1))D_i)/n},$$

and

$$\prod_{i=1}^{n} S(Z_i+1 \mid p)^{(1-D_i)/n} = (1-p)^{(\sum_i (Z_i+1-(\Delta+1))(1-D_i))/n}.$$

Thus, we obtain the simplified form of (S.10) in Lemma 1.

$$\frac{\partial}{\partial p}\frac{\prod_{v=\Delta+1}^{\Delta+m} S(v \mid p)^{\hat{\gamma}_n(v)}}{\prod_{i=1}^{n} f(Z_i \mid p)^{D_i/n} S(Z_i+1 \mid p)^{(1-D_i)/n}} \equiv \frac{\partial}{\partial p}\frac{(1-p)^{a_\tau}}{p^{b_\tau}}.$$

The remainder of the proof follows the proof of Theorem 3.4. □

B Likelihood with Censoring

In this section, we numerically illustrate how the presence of right-censored data that generates $\bar{h}_*$ and h_* impacts the likelihood, $\mathcal{L}_\tau$ of Section 4. Suppose $g(1) = 0.5$, $g(2) = 0.30$, and $g(3) = 0.20$. Hence, $\Delta = 0$ and $m = 3$. Further suppose X follows (14) with $p = 0.6$ and $\omega = 4$. That is, $\Pr(X = 1) = 0.6$, $\Pr(X = 2) = 0.24$, $\Pr(X = 3) = 0.096$, and $\Pr(X = 4) = 0.064$. Finally, set $\varepsilon = 6$, and so right-censoring is present in the data because $\varepsilon < \omega + m$ (Lautier et al., 2023). The complete probability density function for all possible samples of (Y_i, Z_i, D_i) may be found in Table B1.

We can see that not all possible combinations of (Y_i, Z_i, D_i) are observable when $\varepsilon = 6$. For example, $(Y_i = 2, Z_i = 2, D_i = 0)$ is not a possible observation because the censoring time, $Y_i + \varepsilon - (m + \Delta + 1) \equiv Y_i + \tau$, would be $Y_i + \tau = 4 > 2 = Z_i$. Hence, $D_i = \mathbf{1}(X_i \leq C_i)$ cannot be equal to 0. Of the 18 possible combinations of (Y_i, Z_i, D_i), we present all 10 possible observations in Table B1. It may be verified that the sum of the $\bar{h}_*$ and h_* columns in Table B1 taken together is unity. This is a numeric validation that the likelihood under right-censoring, $\mathcal{L}_\tau$, is formed through a valid probability density function. A more formal demonstration may be found in Section A.10, in the lead up to (S.9).

Y_i	Z_i	D_i	$g(Y_i)$	$f(Z_i)$	$S(Z_i+1)$	$\bar{h}_*(Z_i, Y_i)$	$h_*(Z_i, Y_i)$
1	3	0	0.50	0.096	0.064	0.0491	—
2	4	0	0.30	0.064	0.000	0.0000	—
1	1	1	0.50	0.600	0.064	—	0.4601
1	2	1	0.50	0.240	0.064	—	0.1840
1	3	1	0.50	0.096	0.064	—	0.0736
2	2	1	0.30	0.240	0.000	—	0.1104
2	3	1	0.30	0.096	0.000	—	0.0442
2	4	1	0.30	0.064	0.000	—	0.0294
3	3	1	0.20	0.096	0.000	—	0.0294
3	4	1	0.20	0.064	0.000	—	0.0196

Table B1: **Complete Density Right-Censoring**. The complete density function for all possible sampling triples (Y_i, Z_i, D_i) under right-censoring and the density assumptions of Section B with $\varepsilon = 6$. The probability mass function $\bar{h}_*$ is only valid when $Y_i + \tau = Z_i$. The probability mass function h_* is only valid when $Z_i \le Y_i + \tau$. This implies not all triples of (Y_i, Z_i, D_i) are possible observations. It may be verified that the sum of the $\bar{h}_*$ and h_* columns together is unity.

C Implementation Reference

Recall the PL geometric distribution with parameter, $0 < p < 1$, defined in Theorem 3.4,

$$f_T(u \mid p) = \begin{cases} p(1-p)^{u-(\Delta+1)} & \Delta + 1 \le u \le \omega - 1, \\ (1-p)^{u-(\Delta+1)} & u = \omega. \end{cases}$$

Then,

$$\frac{\partial}{\partial p} f_T(u \mid p) = f_T(u \mid p)\left(\frac{\mathbf{1}(u \neq \omega)}{p} - \frac{u-(\Delta+1)}{1-p}\right),$$

$$\frac{\partial^2}{\partial p^2} f_T(u \mid p) = f_T(u \mid p)\left[\frac{u-(\Delta+1)}{1-p}\left(\frac{u-\Delta-2}{1-p} - \frac{2 \times \mathbf{1}(u \neq \omega)}{p}\right)\right],$$

$$\frac{\partial}{\partial p} S_T(u \mid p) = (\Delta + 1 - u)(1-p)^{u-\Delta-2},$$

and

$$\frac{\partial^2}{\partial p^2} S_T(u \mid p) = (u - \Delta - 2)(u - \Delta - 1)(1-p)^{u-\Delta-3}.$$

For a shifted binomial distribution over the support $\{\Delta+1, \ldots, \omega\}$ with probability of success $0 < \theta < 1$, we have the probability density function

$$f(u \mid \theta) = \binom{\omega - (\Delta + 1)}{u - (\Delta + 1)} \theta^{u-(\Delta+1)}(1 - \theta)^{\omega-u}, \quad u \in \{\Delta + 1, \ldots, \omega\}.$$

Thus,

$$\frac{\partial}{\partial\theta} f(u \mid \theta) = f(u \mid \theta)\left(\frac{u - (\Delta + 1)}{\theta} - \frac{\omega - u}{1 - \theta}\right),$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial\theta^2} f(u \mid \theta)\\&= f(u \mid \theta)\left(\frac{(u - \Delta - 1)(u - \Delta - 2)}{\theta^2} - 2\frac{u - (\Delta + 1)}{\theta}\frac{\omega - u}{1 - \theta} + \frac{(\omega - u)(\omega - u - 1)}{(1 - \theta)^2}\right).\end{aligned}$$

D Simulation Procedure Outline

To simulate left-truncated data from the distribution h_* defined in (1), the following procedure may be employed.

1. Select values for Δ, m, and ω and create a pairwise mapping for all possible pairs $(u, v) \in \mathcal{A}$, where $\Delta + 1 \leq v \leq \Delta + m$, $\Delta + 1 \leq u \leq \omega$, and $u \leq v$.

2. Select a distribution and parameters for the lifetime distribution, X, $f(\cdot \mid p)$ and the left-truncation distribution, Y, $\boldsymbol{g}$.

3. Using the choices in the previous step, calculate (1) over all pairs $(u, v) \in \mathcal{A}$. This will require calculating the probability α.

4. Starting with the pair $(\Delta + 1, \Delta + 1)$ and ending with the pair $(\omega, \Delta + m)$, create a one-to-one lower bound mapping from 0 by cumulative sums to $\sum_{\mathcal{A}\setminus(\omega,\Delta+m)} h_*(u, v)$. Call this lower bound $\lfloor H_*(u, v)\rfloor$ for $(u, v) \in \mathcal{A}$.

u	v	$\lfloor H_*(u,v) \rfloor$	$\lceil H_*(u,v) \rceil$
1	1	0.0000000	0.1856436
2	1	0.1856436	0.3155941
2	2	0.3155941	0.3935644
3	1	0.3935644	0.4845297
3	2	0.4845297	0.5391089
3	3	0.5391089	0.5754950
4	1	0.5754950	0.7877475
4	2	0.7877475	0.9150990
4	3	0.9150990	1.0000000

Table D1: **Illustrative Simulation Mapping**. The above table illustrates how to simulate left-truncated data from the bivariate distribution, h_* defined in (1) for f following (14) with $p = 0.30$ and $\boldsymbol{g} = (0.5, 0.3, 0.2)^\top$. For example, a random uniform number from the interval $(0, 1)$ of 0.4000497 would result in the simulated pair $(3, 1)$.

5. Starting with the pair $(\Delta + 1, \Delta + 1)$ and ending with the pair $(\omega, \Delta + m)$, create a one-to-one upper bound mapping from $h_*(\Delta+1, \Delta+1)$ by cumulative sums to 1. Call this upper bound $\lceil H_*(u,v) \rceil$ for $(u, v) \in \mathcal{A}$.

6. Simulate a continuous uniform random number in the interval $(0, 1)$, say ρ. The simulated pair $(u, v) \in \mathcal{A}$ is the pair such that $\lfloor H_*(u,v) \rfloor \leq \rho \leq \lceil H_*(u,v) \rceil$. Repeat as needed for the desired sample size.

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