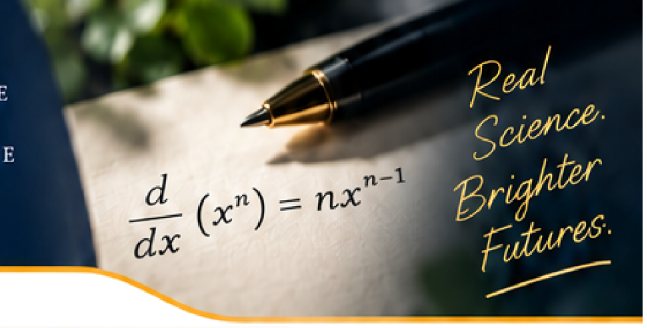

Mathematics 6.5
The Quotient Rule
REVISION NOTES

What you will learn

How to differentiate quotients of functions using the quotient rule, with the substitution notation u and v. You will use previously learned differentiation rules (including the chain and product rules) to find du/dx and dv/dx, and then apply the quotient rule.

Small steps. Big progress.

1 The quotient rule

If $y = \frac{u}{v}$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Here u must be the numerator and v must be the denominator.
The rule applies provided $v \neq 0$.

2 What are u and v?

For the quotient rule, always take

a) u to be the numerator (the top function)
b) v to be the denominator (the bottom function).

Example

Let $y = \frac{2x+1}{\sin x}$.

Here $u = 2x+1$ (numerator)
and $v = \sin x$ (denominator).

3 Simple example (polynomial ÷ polynomial)

Differentiate $y = \frac{3x+1}{x}$.

$u = 3x+1, \quad v = x$

$\frac{du}{dx} = 3, \quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{x \times 3 - (3x+1) \times 1}{x^2}$$
$$= \frac{3x - 3x - 1}{x^2} = -\frac{1}{x^2}$$

4 Quotient with a trigonometric function

Differentiate $y = \frac{\sin x}{x}$.

$u = \sin x, \quad v = x$

$\frac{du}{dx} = \cos x, \quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{x\cos x - \sin x \times 1}{x^2}$$
$$= \frac{x\cos x - \sin x}{x^2}$$

5 Quotient of two trigonometric functions

Differentiate $y = \frac{\sin x}{\cos x}$.

$u = \sin x, \quad v = \cos x$

$\frac{du}{dx} = \cos x, \quad \frac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Using an identity

Since $\sin^2 x + \cos^2 x = 1$, this simplifies to $\frac{dy}{dx} = \frac{1}{\cos^2 x}$.

6 More challenging example

Differentiate $y = \frac{x^2+1}{3x-2}$.

$u = x^2+1, \quad v = 3x-2$

$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{(3x-2)(2x) - (x^2+1)(3)}{(3x-2)^2}$$
$$= \frac{6x^2 - 4x - 3x^2 - 3}{(3x-2)^2}$$
$$= \frac{3x^2 - 4x - 3}{(3x-2)^2}$$

7 Quotient with composite functions

Differentiate $y = \frac{2x+1)}{\sin(3x)}$.

$u = 2x+1, \quad v = \sin(3x)$

$\frac{du}{dx} = 2, \quad \frac{dv}{dx} = 3\cos(3x)$

$$\frac{dy}{dx} = \frac{\sin(3x) \times 2 - (2x+1) \times 3\cos(3x)}{\sin^2(3x)}$$
$$= \frac{2\sin(3x) - 3(2x+1)\cos(3x)}{\sin^2(3x)}$$

8 Common forms

The quotient rule can be used with many types of functions, for example:

- polynomial ÷ polynomial
- trigonometric ÷ polynomial
- polynomial ÷ trigonometric
- trigonometric ÷ trigonometric
- composite functions ÷ functions
- and mixed expressions (e.g. $\frac{x^2+1}{\sin x}$).

As long as the function is a quotient of two differentiable functions, with u as the numerator and v as the denominator, the quotient rule applies.

9 Key points to remember

- For $y = \frac{u}{v}$, always take u to be the **numerator** (top function) and v to be the denominator (bottom function).
- The quotient rule is $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
- Use the substitution notation u and v and find $\frac{du}{dx}$ and $\frac{dv}{dx}$ using the rules from 6.1–6.4 (including the chain rule where needed).
- The rule applies provided $v \neq 0$.
- Simplify your answer where possible.
- Check your answer for sensible form and any domain restrictions (values where $v = 0$).

You should now be able to:

- [ ] recall the quotient rule and use the u, v notation
- [ ] identify the numerator and denominator correctly
- [ ] differentiate quotients of polynomials
- [ ] differentiate quotients involving trigonometric functions
- [ ] differentiate more complex quotients using the chain rule
- [ ] simplify your answers and check for sensible form and restrictions

"New skills today, greater possibilities tomorrow."