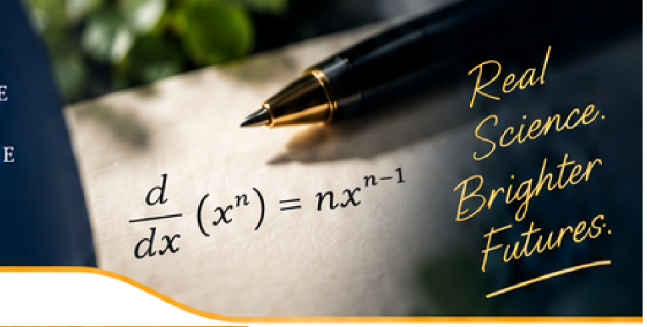

Mathematics 6.3 The Chain Rule

REVISION NOTES

What you will learn

How to differentiate composite functions using the chain rule, including trigonometric functions and polynomials. You will learn the derivatives of $\sin x$ and $\cos x$ and how to apply the chain rule confidently using the substitution u, with the notation $\frac{dy}{du} \times \frac{du}{dx}$.

Small steps. Big progress.

1 Derivatives of $\sin x$ and $\cos x$

For all real values of x,

$$\frac{dy}{dx}(\sin x) = \cos x$$
$$\frac{dy}{dx}(\cos x) = -\sin x$$

Example Find $\frac{dy}{dx}$ for $y = 3\sin x - 2\cos x$.

$$\frac{dy}{dx} = 3\cos x + 2\sin x$$

2 The chain rule

If y is a function of u, and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Differentiate with respect to u, then multiply by the derivative of u with respect to x.

3 Chain rule with a simple function

Differentiate $y = (3x+1)^4$.

Let $u = 3x + 1$

$$\frac{dy}{du} = 4u^3$$
$$\frac{du}{dx} = 3$$
$$\frac{dy}{dx} = 4u^3 \times 3 = 12(3x+1)^3$$

4 Chain rule with trigonometric functions

If $y = \sin u$ or $y = \cos u$, where u is a function of x, then

$$\frac{dy}{dx} = \cos u \times \frac{du}{dx} \quad (\text{for } y = \sin u)$$
$$\frac{dy}{dx} = -\sin u \times \frac{du}{dx} \quad (\text{for } y = \cos u)$$

Example 1 Find $\frac{dy}{dx}$ for $y = \sin(2x+1)$.

Let $u = 2x + 1$

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 2$$
$$\frac{dy}{dx} = 2\cos(2x+1)$$

5 Trigonometric examples

Example 1 Find $\frac{dy}{dx}$ for $y = \cos(3x-2)$.

Let $u = 3x - 2$

$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dx} = 3$$
$$\frac{dy}{dx} = -3\sin(3x-2)$$

Example 2 Find $\frac{dy}{dx}$ for $y = \sin(x^2)$.

Let $u = x^2$

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 2x$$
$$\frac{dy}{dx} = 2x\cos(x^2)$$

6 Mixed examples

Example 1 Find $\frac{dy}{dx}$ for $y = (1+\cos x)^5$.

Let $u = 1 + \cos x$

$$\frac{dy}{du} = 5u^4, \quad \frac{du}{dx} = -\sin x$$
$$\frac{dy}{dx} = -5(1+\cos x)^4 \sin x$$

Example 2 Find $\frac{dy}{dx}$ for $y = \cos(3x^2 - 2x + 1)$.

Let $u = 3x^2 - 2x + 1$

$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dx} = 6x - 2$$
$$\frac{dy}{dx} = -(6x-2)\sin(3x^2 - 2x + 1)$$

7 A longer example

Find $\frac{dy}{dx}$ for $y = (2x-1)^3 \sin(5x)$.

Let $u = (2x-1)^3$, $v = \sin(5x)$

First find $\frac{du}{dx}$ and $\frac{dv}{dx}$ using the chain rule:

$$\frac{du}{dx} = 3(2x-1)^2 \times 2 = 6(2x-1)^2$$
$$\frac{dv}{dx} = \cos(5x) \times 5 = 5\cos(5x)$$

Now use the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= 5(2x-1)^3\cos(5x) + 6(2x-1)^2\sin(5x)$$

8 Recognising composite functions

- Look for a function inside another function, e.g. $\sin(3x)$, $\cos(x^2)$, $(5x-1)^4$.
- Identify the outer function (e.g. sin, cos or a power) and the inner function u.
- Differentiate the outer function with respect to u, leaving the inner function unchanged.
- Find $\frac{du}{dx}$.
- Multiply using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- Simplify your answer where possible.

9 Key points to remember

- $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$.
- Use the chain rule with the substitution u: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
- Differentiate the outer function first.
- Multiply by the derivative of the inner function.
- The chain rule can be used with trigonometric functions, polynomials and combinations of functions.
- Simplify your answers where possible.
- Always check your answer makes sense, e.g. consider the form and whether the sign is correct.

You should now be able to:

- ☐ recall the derivatives of $\sin x$ and $\cos x$
- ☐ use the chain rule with the substitution u
- ☐ differentiate trigonometric functions using the chain rule
- ☐ differentiate composite polynomial functions using the chain rule
- ☐ tackle more complex composite functions confidently

"New skills today, greater possibilities tomorrow."