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Mathematics 6.6
Differentiating Exponential and Logarithmic Functions
REVISION NOTES

What you will learn
How to differentiatie exponential and logarithmic functions, including e^x, $\ln(x)$ and a^x. You will use the chain rule, product rule and other techniques from 6.1–6.5 to find derivatives and solve problems involving exponential growth, decay and logarithms.

Small steps. Big progress.

1 The exponential function e^x
The function e^x, where $e \approx 2.718$, has the important property that its derivative is itself.

$$\frac{d}{dx}(e^x) = e^x$$

This makes e^x particularly useful in modelling growth and decay, as its rate of change is proportional to its value.

2 The natural logarithm $\ln(x)$
The natural logarithm $\ln(x)$ is the inverse of e^x for $x > 0$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)$$

The domain of $\ln(x)$ is $x > 0$. Its derivative is positive for all $x > 0$ and decreases in magnitude as x increases.

3 The exponential function a^x
For a constant base $a > 0$, $a \neq 1$,

$$\frac{d}{dx}(a^x) = a^x \ln a$$

This follows from rewriting $a^x = e^{x \ln a}$ and using the chain rule.
Special cases include:
- If $a = e$, then $\ln a = 1$ and $\frac{d}{dx}(e^x) = e^x$.
- If $0 < a < 1$, then $\ln a < 0$, so a^x is a decreasing function.

4 Using the chain rule
When the exponential or logarithmic function has an expression inside, use the chain rule.
If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Examples
- $\frac{d}{dx}(e^{3x}) = 3e^{3x}$
- $\frac{d}{dx}(\ln(2x+1)) = \frac{2}{2x+1}$
- $\frac{d}{dx}(5^{x^2}) = 5^{x^2} \ln 5 \cdot 2x$

5 Using other rules
Exponential and logarithmic functions often appear in products, quotients and composite functions. Use the rules from 6.1–6.5 as appropriate.

Examples
- Product rule:
  $y = xe^x \Rightarrow \frac{dy}{dx} = e^x + xe^x$
- Quotient rule:
  $y = \frac{\ln x}{x} \Rightarrow \frac{dy}{dx} = \frac{1 - \ln x}{x^2}$
- Chain rule with a^x:
  $y = a^{3x+1} \Rightarrow \frac{dy}{dx} = 3a^{3x+1} \ln a$

6 Key properties and graphs
- e^x is always positive, increasing and defined for all real x.
- $\ln x$ is defined only for $x > 0$, is increasing, and has a vertical asymptote at $x = 0$.
- For $a > 1$, a^x is increasing; for $0 < a < 1$, a^x is decreasing.
- The derivative of e^x and a^x is proportional to the function itself.
- The derivative of $\ln x$ is $\frac{1}{x}$, which decreases as x increases.
- Exponential and logarithmic functions are widely used in modelling growth, decay, population, finance and natural phenomena.

7 Common derivatives

$\frac{d}{dx}(e^x)$	$= e^x$	
$\frac{d}{dx}(\ln x)$	$= \frac{1}{x}$	$(x > 0)$
$\frac{d}{dx}(a^x)$	$= a^x \ln a$	$(a > 0, a \neq 1)$
$\frac{d}{dx}(e^u)$	$= e^u \cdot \frac{du}{dx}$	
$\frac{d}{dx}(\ln u)$	$= \frac{1}{u} \cdot \frac{du}{dx}$	
$\frac{d}{dx}(a^u)$	$= a^u \ln a \cdot \frac{du}{dx}$	

8 Applications
Differentiation of exponential and logarithmic functions arises in many contexts, for example:
- Exponential growth and decay (e.g. population models, radioactive decay, compound interest).
- Finding rates of change in models involving e^x or a^x.
- Logarithmic functions in problems involving inverse relationships (e.g. pH, sound levels, finance).
- Solving equations by differentiation, including optimisation and finding stationary points.

9 Key points to remember
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x > 0)$
- $\frac{d}{dx}(a^x) = a^x \ln a \quad (a > 0, a \neq 1)$
- Use the chain rule for functions with an expression inside.
- Apply product, quotient and other rules from 6.1–6.5 where needed.
- Always state the domain (e.g. $x > 0$ for $\ln x$).
- Simplify your answers where possible.
- Check for stationary points and interpret results in context.

You should now be able to:
- ☐ recall the derivatives of e^x, $\ln(x)$ and a^x
- ☐ use the chain rule with exponential and logarithmic functions
- ☐ differentiate products, quotients and composite functions involving e^x, $\ln(x)$ or a^x
- ☐ apply these techniques to solve problems in context
- ☐ check your answers for sensible form and domain restrictions

"New skills today, greater possibilities tomorrow."