WESSEX ACADEMIA

FROM GCSE TO GRADUATION

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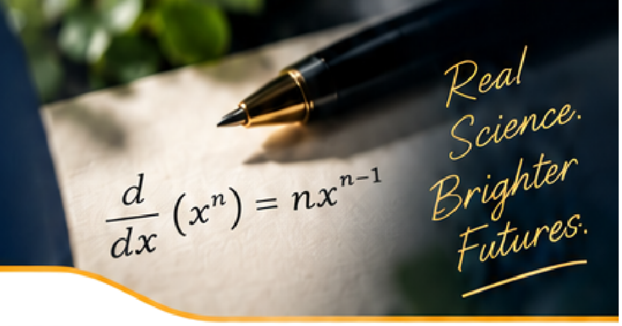


# Mathematics 6.2 Differentiation

FURTHER APPLICATIONS
REVISION NOTES

## What you will learn

Build on basic differentiation to solve more advanced problems, including tangents and normals, stationary points, maxima and minima, points of inflection, the second derivative, optimisation problems and rates of change.

*Apply Extend Solve Succeed*

## 1 Tangents and normals (using function notation)

For a curve $y = f(x)$ at the point $x = a$:

- Gradient of tangent: $f'(a)$
- Equation of tangent: $y - f(a) = f'(a)(x - a)$
- Gradient of normal (if $f'(a) \neq 0$): $-\frac{1}{f'(a)}$
- Equation of normal: $y - f(a) = -\frac{1}{f'(a)}(x - a)$

**Example** For $f(x) = x^3 - 6x$ at $x = 1$:

$f'(x) = 3x^2 - 6, \quad f'(1) = -3, \quad f(1) = -5$

Tangent: $y + 5 = -3(x - 1) \Rightarrow y = -3x - 2$

Normal: $y + 5 = \frac{1}{3}(x - 1) \Rightarrow y = \frac{1}{3}x - \frac{16}{3}$

## 2 Stationary points

Stationary points occur where $\frac{dy}{dx} = 0$.

Use the second derivative to classify them:

- If $\frac{d^2y}{dx^2} > 0 \Rightarrow$ local minimum
- If $\frac{d^2y}{dx^2} < 0 \Rightarrow$ local maximum
- If $\frac{d^2y}{dx^2} = 0 \Rightarrow$ inconclusive (may be a point of inflection)

**Example** For $y = x^3 - 6x^2 + 9x$:

$\frac{dy}{dx} = 3x^2 - 12x + 9$

Set $\frac{dy}{dx} = 0$: $3(x-1)(x-3) = 0 \Rightarrow x = 1, 3$

$\frac{d^2y}{dx^2} = 6x - 12$

At $x = 1$, $\frac{d^2y}{dx^2} = -6 \Rightarrow$ local maximum

At $x = 3$, $\frac{d^2y}{dx^2} = 6 \Rightarrow$ local minimum

## 3 Increasing and decreasing

A function $y = y(x)$ is:

- increasing where $\frac{dy}{dx} > 0$
- decreasing where $\frac{dy}{dx} < 0$

Use a sign chart or test values in the intervals determined by stationary points.

**Example** For $y = x^3 - 4x^2 + 4x + 1$:

$\frac{dy}{dx} = 3x^2 - 8x + 4$

Solve $3x^2 - 8x + 4 = 0$:

$x = \frac{2}{3}, 2$

| Interval | $x < \frac{2}{3}$ | $\frac{2}{3} < x < 2$ | $x > 2$ |
|---|---|---|---|
| Sign of $\frac{dy}{dx}$ | + | − | + |
| Function | Increasing | Decreasing | Increasing |

## 4 Points of inflection

A point of inflection is where the curve changes from concave up to concave down, or vice versa.

- Find where $\frac{d^2y}{dx^2} = 0$.
- Check that $\frac{d^2y}{dx^2}$ changes sign.

**Example** For $y = x^3 - 6x$:

$\frac{d^2y}{dx^2} = 6x$

Set $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 0$

$\frac{d^2y}{dx^2}$ is negative for $x < 0$ and positive for $x > 0$, so there is a point of inflection at $x = 0$ (the origin).

## 5 Maxima and minima (optimisation)

1. Write the quantity to be optimised as $y = y(x)$.
2. Find $\frac{dy}{dx}$ and set it to zero.
3. Use the second derivative test or consider the nature of the problem to confirm a maximum or minimum.
4. State the maximum or minimum value and the value of $x$ at which it occurs.

**Example** A rectangle has length $x$ and width $(10 - 2x)$.

Area: $A = x(10 - 2x) = 10x - 2x^2$

$\frac{dA}{dx} = 10 - 4x$

Set $\frac{dA}{dx} = 0 \Rightarrow x = 2.5$

$\frac{d^2A}{dx^2} = -4 < 0 \Rightarrow$ maximum

Maximum area $= 12.5$ cm$^2$.

## 6 Using the second derivative

The second derivative can be used to:

- classify stationary points
- find points of inflection
- determine the concavity (shape) of the curve

$\frac{d^2y}{dx^2} > 0$
Concave up
(local minimum)

e.g. $y = x^2$

$\frac{d^2y}{dx^2} < 0$
Concave down
(local maximum)

e.g. $y = -x^2$

$\frac{d^2y}{dx^2}$ changes sign
Point of inflection

e.g. $y = x^3$

## 7 Rates of change

If a quantity is given by a function of time, e.g. $s = s(t)$, the derivative gives the rate of change.

- Velocity: $v(t) = \frac{ds}{dt}$
- Acceleration: $a(t) = \frac{d^2s}{dt^2}$
- Always include units in your answers.

**Example** $s(t) = 2t^3 - 5t^2 + 4t$

$v(t) = \frac{ds}{dt} = 6t^2 - 10t + 4$

$a(t) = \frac{d^2s}{dt^2} = 12t - 10$

At $t = 3$: $v(3) = 28, \quad a(3) = 26$

## 8 Equation of a tangent (using $\frac{dy}{dx}$)

For a curve $y = y(x)$, the equation of the tangent at $x = a$ is

$$y - y(a) = \left(\frac{dy}{dx} \text{ at } x = a\right)(x - a)$$

**Example** For $y = x^2$ at $x = 2$:

$\frac{dy}{dx} = 2x \qquad \frac{dy}{dx}$ at $x = 2 = 4$

$y(2) = 4$

$y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$

## 9 Key points to remember

- Use function notation for tangents and normals (Box 1).
- Stationary points occur where $\frac{dy}{dx} = 0$.
- Use $\frac{d^2y}{dx^2}$ to classify stationary points and find points of inflection.
- $y$ is increasing where $\frac{dy}{dx} > 0$ and decreasing where $\frac{dy}{dx} < 0$.
- A point of inflection is where $\frac{d^2y}{dx^2} = 0$ and changes sign.
- Optimisation involves finding the maximum or minimum value of a quantity.
- In rates of change problems, differentiate with respect to time and include units.
- Always check your answers make sense in context.

## You should now be able to:

- [ ] find equations of tangents and normals
- [ ] find and classify stationary points
- [ ] determine intervals of increase and decrease
- [ ] identify points of inflection
- [ ] solve optimisation problems
- [ ] tackle rates of change and real-world modelling questions

*“More than answers — deeper understanding.”*

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