Coherence Dynamics: Bridging Classical and Pure Mathematics

Department of Coherence Dynamics Mathematics

The L'Var Spring as a Bi-Topological Elastic Object: Energy Descent, Dual Discretizations, and the Newtonian Limit

A bi-topological elastic object for energy descent, dual discretizations, and the Newtonian limit

> *"Mathematics begins again each time coherence becomes visible. To build a domain is not to name the unknown—it is to give structure to its persistence."*
>
> — L.E. L'Var, *Foundations of Coherence Dynamics*

L.E. L'Var (ORCID: 0009-0009-1254-2396)

L'Var Institute of Coherence Dynamics (LICD)

CDM-2025-01 — v1.0 — October 14, 2025

l.e.lvar@the-ap.org — https://the-ap.org

Abstract

On a state space carrying inequivalent yet compatible smooth and ultrametric topologies, a proper Lyapunov energy drives every Spring trajectory into a compact basin and collapses onto the fixed-point set, with convergence certified in both regimes. Independently, a discrete variational integrator and an accelerated mirror/prox scheme converge, as the step vanishes, to the same (damped) Riemannian Newton equation—elastic metric as mass, Levi–Civita connection as inertial curvature—giving dual validation via smooth energy descent and ultrametric contraction. The potential supplies conservative forces; acceleration schedules induce damping. We conclude with categorical and adèlic outlooks.

Keywords: Bi-topological dynamics; Riemannian Newton law; categorical dynamics; ultrametric analysis; variational integrators; proximal algorithms; Lyapunov energy.

Mathematics Subject Classification (2020): 37N05; 37B25; 53C22; 46S10; 18A30; 65P10.

Contents

1 Introduction

Problem. Many systems exhibit both smooth evolution and discrete collapse. Classical analysis handles the former; ultrametric analysis canonically handles the latter. The two frameworks are topologically incompatible (connected vs. totally disconnected), so gluing them post hoc forces one regime to mimic the other and loses guarantees.

Object. The *L'Var Spring* is a unary dynamical object on a common state set carrying two complete, *inequivalent but compatible* topologies: a smooth (Archimedean) topology that supports calculus via an elastic metric G and an ultrametric (p-adic) topology that supports exact hierarchical discretization. A single recursion L is continuous in both, and a single proper energy E strictly decreases along orbits away from fixed points, with precompact sublevels in both topologies.

Consequences in one breath. Compatibility (not equivalence) lets one map L drive the same trajectory to a unique terminal state while being certified in two logically independent ways: smooth energy descent and ultrametric contraction. The elastic metric plays the role of a mass matrix; its Levi–Civita connection supplies inertial curvature; acceleration schedules in proximal discretizations induce damping. In the flat case $G \equiv mI$, the continuous limit is precisely $m\,\ddot{x} = -\nabla V(x)$ (with optional viscous or vanishing friction).

Why inequivalence is essential. Requiring a homeomorphism between the smooth and ultrametric topologies would either collapse calculus (no paths, no geodesics) or collapse quantization (no clopen hierarchies). The weaker requirement of compatibility preserves both the differential and the hierarchical toolsets and yields two independent compactness/convergence filters that agree on the same fixed point.

Contributions. We provide a consequence-first, math-native treatment:

i. A Spring structure $(\mathcal{S}, G, E, L)$ with a bi-topological Energy/Lyapunov principle: every orbit is trapped in a compact basin and collapses to the fixed-point set as witnessed by *both* topologies.
ii. A bi-topological fixed-point/uniqueness theorem (Banach-in-both): if L is contractive in each metric, the fixed point is unique and orbits converge to it in both metrics, with independent rates.
iii. *Route A (variational).* A discrete variational integrator converges to the Riemannian Newton law $G(x)\ddot{x} + \Gamma^{\flat}_G(x)[\dot{x}, \dot{x}] = -\nabla V(x)$. In the flat case this is $F = ma$.
iv. *Route B (proximal/accelerated).* An accelerated mirror/prox scheme in the elastic metric converges to the damped Riemannian Newton law $G(x)\ddot{x} + \Gamma^{\flat}_G(x)[\dot{x}, \dot{x}] + \zeta(t)G(x)\dot{x} = -\nabla V(x)$, with $\zeta(t) = \gamma$ or α/t depending on the schedule.
v. Interfaces to examples (constant and nontrivial elastic metrics), categorical structure (the category **Spring** with energy- and dynamics-respecting morphisms), and an adèlic outlook (Fourier/Poisson under elastic coupling).

Main Results—at a glance

(1) Bi-topological terminality. Under the Energy/Lyapunov principle (Assumption 3.2), every orbit enters a compact bi-topological basin, the energy sequence is strictly decreasing and convergent, and all accumulation points lie in $\mathrm{Fix}(L)$ (Prop. 3.3). If L is a contraction in both metrics, the fixed point is *unique* and (x_n) converges to it in both topologies (Thm. 4.1).

(2) Newtonian limit from two independent routes. Both a discrete variational integrator (Route A) and an accelerated mirror/prox scheme (Route B) converge to the same *Riemannian Newton law*

$$G(x)\,\ddot{x} + \Gamma^\flat_G(x)[\dot{x},\dot{x}] = -\nabla V(x) \quad \text{(see Thms. 5.1, 6.1)},$$

and, with acceleration schedules, to the *damped* form.

(3) Dual validation: continuous & hierarchical. Compatibility (not equivalence) of the smooth and ultrametric topologies yields two independent convergence filters: smooth energy descent and ultrametric contraction. Sublevel precompactness in both topologies rules out topology-specific failure modes and certifies rounding-robust discretization.

(4) Concrete predictions. For $V(x) = \frac{k}{2}\|x\|^2$ with $G \equiv mI$ one recovers the harmonic oscillator; with $G(x) = m(1+\varepsilon\|x\|^2)I$ the theory predicts a first-order *reduction* in radial frequency proportional to ε (see §7.2), with 1D motion admitting closed-form quadrature.

(5) Compositional calculus (Section 9). The category **Spring** has finite limits and (under mild regularity) finite colimits; pullbacks model constraint-coupled systems; free/forgetful adjunctions formalize adding/removing dynamics.

(6) Minimal assumptions; extensible rates. We require $G \in C^2$ (uniformly elliptic) and $V \in C^2$; KL-type regularity yields rates; damping profiles come from acceleration schedules.

2 Preliminaries and Notation

Let $\mathcal{S}$ be a common state set endowed with two Hausdorff, complete metrics: d_∞ (smooth/Archimedean) and d_p (ultrametric). We denote their topologies by τ_∞ and τ_p. A symmetric positive-definite matrix field $G : \mathcal{S} \to \mathbb{R}^{d\times d}$ induces the inner product $\langle u, v\rangle_G = u^\top G(x)v$ and norm $\|u\|_G^2 = \langle u, u\rangle_G$. We write Γ_G for the Levi–Civita connection of G and $\Gamma^\flat_G(x)[u,v] := G(x)\,\Gamma_G(x)[u,v]$ for its index-lowered form.

Definition 2.1. [Compatibility] A map $L : \mathcal{S} \to \mathcal{S}$ is *compatible* if it is continuous with respect to both τ_∞ and τ_p. An energy $E : \mathcal{S} \to \mathbb{R}_{\ge 0}$ is *bi-proper* if sublevel sets $\{E \le c\}$ are precompact in both topologies.

The definition of compatibility assumes that maps and energies satisfying these dual requirements exist. The following lemma provides a constructive proof that such objects are not merely notional but can be engineered from a smooth space via hierarchical symbolic coding, thus closing this foundational logical gap.

Lemma 2.2. *[Existence of Compatible Maps via Symbolic Coding] Let $(\mathcal{S}, d_\infty)$ be a compact metric space and let $\pi : \mathcal{S} \to \Sigma^{\mathbb{N}}$ be a continuous surjective coding map onto sequences over a finite alphabet Σ. Define an ultrametric on $\mathcal{S}$ by*

$$d_p(x,y) := \lambda^{-\mathrm{LCP}(\pi(x),\pi(y))}, \qquad \lambda > 1,$$

where LCP *denotes the length of the longest common prefix of the codes $\pi(x)$ and $\pi(y)$. Suppose $L : \mathcal{S} \to \mathcal{S}$ satisfies:*

(a) L is α-Lipschitz in d_∞ for some $\alpha < 1$;

(b) There exists $r \ge 1$ such that for all $x, y \in \mathcal{S}$,

$$\mathrm{LCP}(\pi(Lx), \pi(Ly)) \ge \max\{\mathrm{LCP}(\pi(x), \pi(y)) - r, 0\}.$$

Then L is continuous with respect to both d_∞ and d_p. Moreover, if $r > 0$, it is a strict contraction in d_p with modulus λ^{-r}.

Proof sketch. Condition (a) implies continuity in d_∞ immediately. For the ultrametric d_p, fix $\epsilon = \lambda^{-k}$. Take $\delta = \lambda^{-(k+r)}$. If $d_p(x, y) < \delta$, then $\mathrm{LCP}(\pi(x), \pi(y)) > k + r$. By condition (b), $\mathrm{LCP}(\pi(Lx), \pi(Ly)) > k$, hence

$$d_p(Lx, Ly) = \lambda^{-\mathrm{LCP}(\pi(Lx),\pi(Ly))} < \lambda^{-k} = \epsilon.$$

Thus L is d_p-continuous. If $r > 0$, the inequality is strict, giving a contraction with ratio λ^{-r}. The construction therefore produces explicit maps that are simultaneously continuous (or contractive) in both the smooth and ultrametric topologies. □

3 The L'Var Spring

Definition 3.1. [Spring] A *Spring* is a quadruple $(\mathcal{S}, G, E, L)$ where: (i) $(\mathcal{S}, d_\infty)$ and $(\mathcal{S}, d_p)$ are complete; (ii) G is a positive-definite metric field inducing d_∞; (iii) E is bi-proper and lower semicontinuous in both topologies; (iv) L is compatible and a strict contraction in both metrics.

Assumption 3.2. [Energy/Lyapunov Principle] There exists $E : \mathcal{S} \to \mathbb{R}_{\geq 0}$ such that: (a) $E \geq 0$ and $\{E \leq c\}$ is precompact in both topologies for every finite c; (b) $E(Lx) < E(x)$ for all $x \notin \mathrm{Fix}(L)$; (c) $x \in \mathrm{Fix}(L)$ iff $0 \in \partial E(x)$.

3.1 Immediate Consequences

Proposition 3.3. *[Terminal Behavior] Under Assumption 3.2, every orbit $x_{n+1} = Lx_n$ enters a compact sublevel set and all accumulation points lie in $\mathrm{Fix}(L)$. Consequently, divergence and neutral cycles are excluded.*

Definition 3.4. [Bi-topological ω-limit set] For an orbit (x_n), define

$$\omega_\infty(x_0) := \{x : \exists n_k \uparrow \infty,\ x_{n_k} \to x \text{ in } (\mathcal{S}, \tau_\infty)\}, \qquad \omega_p(x_0) := \{x : \exists n_k \uparrow \infty,\ x_{n_k} \to x \text{ in } (\mathcal{S}, \tau_p)\}.$$

Lemma 3.5. *[Compactness, nonemptiness, invariance] Under Assumption 3.2, both $\omega_\infty(x_0)$ and $\omega_p(x_0)$ are nonempty, compact, and L-invariant. Moreover,*

$$\omega_\infty(x_0) \cup \omega_p(x_0) \ \subseteq\ \mathrm{Fix}(L).$$

Proof sketch. Energy descent traps (x_n) in the common sublevel $\{E \leq E(x_0)\}$, which is precompact in both topologies; sequential compactness yields nonemptiness and compactness of ω_∞, ω_p. Invariance follows from continuity of L in both topologies. The inclusion in $\mathrm{Fix}(L)$ is Proposition 3.3. □

Theorem 3.6. *[Bi-topological terminality] Under Assumption 3.2, every orbit (x_n) admits (possibly different) accumulation sets in the smooth and ultrametric topologies, each nonempty, compact, L-invariant, and contained in $\mathrm{Fix}(L)$. If, in addition, either*

(a) L is a contraction in d_∞ (resp. d_p), or

(b) $\mathrm{Fix}(L)$ is discrete in τ_∞ (resp. τ_p),

then (x_n) converges in that topology to a fixed point $x^\star \in \mathrm{Fix}(L)$. If (a) holds in both metrics with moduli < 1, the limit is unique and convergence occurs in both topologies to the same $x^\star$.

Proof sketch. The first statement is Lemma 3.5. For (a), apply Banach–Caccioppoli in the respective metric; for (b), compactness + strict energy descent preclude nontrivial ω-cycles in a discrete set, forcing stabilization at one point. □

Corollary 3.7. *[Finite fixed-point set implies convergence] If $\mathrm{Fix}(L)$ is finite, then every orbit (x_n) converges in both topologies to some $x^\star \in \mathrm{Fix}(L)$. If L is a contraction in either topology, the limit is unique.*

Proof sketch. Finite sets are discrete in both topologies; apply Theorem 3.6(b). Uniqueness follows from Theorem 4.1. □

4 Bi-Topological Fixed Point and Uniqueness

Theorem 4.1. *[Bi-Topological Banach] If L is an α-contraction in d_∞ and a β-contraction in d_p with $0 < \alpha, \beta < 1$, then L admits a unique fixed point $x^\star$. For any x_0, the iterates $x_{n+1} = Lx_n$ converge to $x^\star$ in both d_∞ and d_p.*

Proof sketch. Banach's theorem applies independently in $(\mathcal{S}, d_\infty)$ and $(\mathcal{S}, d_p)$. Energy descent and bi-properness provide compactness, ruling out pathological ω-limits; uniqueness follows from strict contraction. □

5 Route A: Discrete Variational Integrator → Riemannian Newton

Consider the discrete action

$$\mathcal{A}_h(\{x_k\}) = \sum_k \Big(\tfrac{1}{2} \left\| \tfrac{x_{k+1}-x_k}{h} \right\|^2_{G(x_k)} - V(x_k) \Big) h. \tag{1}$$

Stationarity yields the discrete Euler–Lagrange equation

$$G(x_k)\, \frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \big(\nabla G(x_k)\, \tfrac{x_k - x_{k-1}}{h}\big) \tfrac{x_k - x_{k-1}}{h} = -\nabla V(x_k). \tag{2}$$

Theorem 5.1. *[Discrete–to–Continuous Limit: Riemannian Newton Law] Let $G, V \in C^2(\mathcal{S})$ on a smooth manifold $(\mathcal{S}, G)$, and let (x_k) satisfy the discrete Euler–Lagrange equation (2) with step size $h > 0$. Assume the discrete trajectory admits a C^3 interpolation $x(t)$ such that* $\sup_t \| \dddot{x}(t) \| < \infty$ *and G is uniformly elliptic and Lipschitz in x. Then as $h \to 0$,*

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} \longrightarrow \ddot{x}(t), \qquad \big(\nabla G(x_k) \tfrac{x_k - x_{k-1}}{h}\big) \tfrac{x_k - x_{k-1}}{h} \longrightarrow \Gamma^\flat_G(x)[\dot{x}, \dot{x}],$$

and the discrete equation (2) converges to

$$G(x)\, \ddot{x} + \Gamma^\flat_G(x)[\dot{x}, \dot{x}] = -\nabla V(x), \tag{3}$$

the Riemannian Newton law. *If $G \equiv mI$, this reduces to the classical $m\ddot{x} = -\nabla V(x)$.*

Proof sketch. Taylor-expand $x_{k\pm 1} = x(t) \pm h\dot{x} + \frac{h^2}{2}\ddot{x} + O(h^3)$. Substitute into (2); use symmetry of G and the definition of the Levi–Civita connection $\Gamma^k_{ij} = \frac{1}{2} G^{kl}(\partial_i G_{jl} + \partial_j G_{il} - \partial_l G_{ij})$ to reorganize terms. The difference quotient of $G(x_k)$ produces the $\Gamma^\flat_G(\dot{x}, \dot{x})$ term. Uniform ellipticity and bounded derivatives ensure $O(h)$ remainder. □

Remark 5.2. [Energy consistency] The discrete action $\mathcal{A}_h$ is a first-order consistent discretization of the continuous action $\mathcal{A}[x] = \int \big(\frac{1}{2}\|\dot{x}\|^2_G - V(x)\big)\, dt$, and Theorem 5.1 establishes variational convergence $\mathcal{A}_h \to \mathcal{A}$ in the Γ-sense. Hence the limit dynamics preserve the energy structure of the discrete scheme.

Cross-reference. The continuous law identified in Theorem 5.1 is exactly Eq. (3); in the flat specialization $G \equiv mI$ it reduces to $m\ddot{x} = -\nabla V$.

6 Route B: Accelerated Mirror/Prox → Damped Riemannian Newton

From variational to proximal dynamics. The variational integrator of Route A exposes the Spring as a discrete mechanical system: momentum arises from metric curvature, and force from the gradient of the potential. Yet the same geometry also underlies modern accelerated optimization. If, instead of discretizing the action, one discretizes the *energy descent* itself—combining extrapolation and implicit proximal correction—one obtains a second family of schemes that generate the same continuous limit. This complementary construction, Route B, views the Spring not as a time-stepping rule for a Lagrangian but as a fixed-point iteration for the energy, where acceleration manifests as damping in the continuous limit.

With step $\eta_k = h$ and momentum β_k, perform

$$y_k = x_k + \beta_k(x_k - x_{k-1}), \tag{4}$$

$$G(y_k)(x_{k+1} - y_k) = -\eta_k \nabla V(y_k). \tag{5}$$

Theorem 6.1. *[Accelerated Prox/Mirror $\Rightarrow$ Damped Riemannian Newton] Let $G, V \in C^2(\mathcal{S})$ with G uniformly elliptic and Lipschitz on bounded sets. Consider the accelerated prox/mirror update with step $\eta_k = h > 0$:*

$$y_k = x_k + \beta_k(x_k - x_{k-1}), \qquad G(y_k)\big(x_{k+1} - y_k\big) = -\,\eta_k \nabla V(y_k),$$

and define the C^3 interpolation $x(kh) = x_k$ with $\sup_t \| \dddot{x}(t)\| < \infty$. Suppose the momentum schedule satisfies either

(a) ***Constant friction:*** *$\beta_k = 1 - \gamma h$ with $\gamma \geq 0$ fixed, or*

(b) ***Vanishing friction (Nesterov type):*** *$\beta_k = 1 - \frac{\alpha h}{t_k}$ with $\alpha > 0$ and $t_k = kh$.*

Then, as $h \to 0$, the interpolated trajectory converges (on compact time intervals) to a solution of the damped Riemannian Newton equation

$$G(x)\,\ddot{x} + \Gamma^\flat_G(x)[\dot{x}, \dot{x}] + \zeta(t)\,G(x)\,\dot{x} = -\,\nabla V(x), \tag{6}$$

where $\zeta(t) = \gamma$ in case (a) and $\zeta(t) = \alpha/t$ in case (b). In the flat case $G \equiv mI$, this reduces to $m\ddot{x} + m\,\zeta(t)\,\dot{x} = -\nabla V(x)$.

Proof sketch. Rewrite the optimality condition as $G(y_k)\frac{x_{k+1}-x_k}{h^2} = -\nabla V(y_k) + \beta_k\,G(y_k)\frac{x_k - x_{k-1}}{h}$. Subtract and add $G(x_k)$ where needed; use $x_{k\pm1} = x \pm h\dot{x} + \frac{h^2}{2}\ddot{x} + O(h^3)$ and $v_k = (x_k - x_{k-1})/h = \dot{x} + O(h)$. (i) The central second difference yields $G(x)\ddot{x} + O(h)$. (ii) The metric variation $[G(y_k) - G(x_k)]v_k$ gives $\big(\nabla G(x)\,\dot{x}\big)\dot{x} + O(h)$, which reorganizes into $\Gamma^\flat_G(\dot{x}, \dot{x})$. (iii) The momentum term $(\beta_k - 1)G(y_k)v_k$ yields $-\gamma\,G(x)\,\dot{x} + O(h)$ in case (a) and $-(\alpha/t)\,G(x)\,\dot{x} + O(h)$ in case (b). Collect terms and pass $h \to 0$. ◻

Corollary 6.2. *[Mechanical energy dissipation] Let $E_{\mathrm{mech}}(x, \dot{x}) = \frac{1}{2}\,\dot{x}^\top G(x)\dot{x} + V(x)$. Along solutions to Theorem 6.1,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{mech}}(x, \dot{x}) = -\,\zeta(t)\,\dot{x}^\top G(x)\dot{x} \;\leq\; 0,$$

with $\zeta(t) = \gamma$ (constant) or $\zeta(t) = \alpha/t$ (vanishing). In particular, energy decays monotonically and $\int_0^\infty \zeta(t)\,\|\dot{x}(t)\|^2_G\,\mathrm{d}t < \infty$.

Proof sketch. Differentiate E_{mech} and substitute the equation of motion; the Levi–Civita term cancels by metric-compatibility, leaving the friction term. ◻

Remark 6.3. [Schedule design and rates] Constant $\gamma > 0$ yields exponential decay in strongly convex basins. The t^{-1} schedule reproduces Nesterov-type sublinear decay. In both cases, local KL exponents of V near $\mathrm{Fix}(L)$ transfer to rates for E_{mech}.

Cross-reference. The limit in Theorem 6.1 is Eq. (6) (with $\zeta(t) = \gamma$ or α/t), recovering Eq. (3) in the undamped case $\zeta \equiv 0$.

Synthesis: dual validation of the Newtonian limit

Theorem 6.4. *[Dual discretization equivalence] Under the assumptions of Theorems 5.1 and 6.1, the vanishing-step limits of* (i) *the discrete variational integrator (Route A) and* (ii) *the accelerated mirror/prox scheme (Route B) coincide with the same continuous law:*

$$\text{Eq. (3) when } \zeta \equiv 0, \qquad \text{Eq. (6) when } \zeta \not\equiv 0.$$

In particular, the elastic metric G is identified as the mass matrix, the Levi–Civita connection Γ as inertial curvature, and $\zeta(t)$ as friction, independently of the discretization route.

Proof sketch. Route A yields Eq. (3) by Theorem 5.1. Route B yields Eq. (6) by Theorem 6.1, reducing to Eq. (3) when $\zeta \equiv 0$. Hence both routes converge to the same Newtonian law (with/without damping) on the same geometric data (G, Γ, V). □

7 Examples

From principle to practice. The two routes—variational and proximal—now converge on the same geometric law: the Riemannian Newton equation and its damped extension. The Spring's abstract mechanics are therefore no longer speculative; they prescribe concrete, testable motion once a metric and potential are specified. To make the geometry tangible, we turn next to examples where the elastic metric G acquires physical meaning. In these cases, curvature behaves as an effective mass modulation and damping as controlled energy loss, allowing direct comparison with classical oscillators and their deformations.

The purpose of this section is to make the abstract geometry of the Spring explicit through concrete dynamics. Once the metric G and potential V are specified, the Riemannian Newton law

$$G(x)\,\ddot{x} + \Gamma^\flat_G(x)[\dot{x}, \dot{x}] + \zeta(t)\,G(x)\,\dot{x} = -\,\nabla V(x)$$

governs motion in both the smooth and ultrametric perspectives. Each example below isolates a single geometric feature—flatness, curvature, or damping—and shows how it manifests as a physically interpretable behavior: harmonic motion, frequency shift, or dissipative collapse.

7.1 Quadratic Potential with Constant Metric

This baseline reproduces the classical harmonic oscillator and serves as a consistency check. Setting $G \equiv mI$ and $V(x) = \frac{k}{2}\|x\|^2$ reduces the law to

$$m\,\ddot{x} + kx = 0,$$

whose solutions $x(t) = A\cos(\omega t) + B\sin(\omega t)$ have frequency $\omega = \sqrt{k/m}$. Adding constant damping $\zeta(t) = \gamma$ gives the familiar viscous oscillator $m\ddot{x} + m\gamma\dot{x} + kx = 0$, with exponential decay rate $\gamma/2$. This case verifies that the Spring reproduces ordinary Newtonian mechanics when curvature and hierarchy vanish.

7.2 Nontrivial Elastic Metric

We study the quadratic potential $V(x) = \frac{k}{2}\,\|x\|^2$ under an *isotropic, position-dependent* elastic metric

$$G(x) = \mu(\|x\|)\,I, \qquad \mu(r) := m\,(1 + \varepsilon r^2), \quad m > 0,\ \varepsilon \geq 0.$$

This model already exhibits curvature-like inertial forces despite Euclidean coordinates.

Equation of motion. With (3) we have

$$\mu(r)\,\ddot{x} + \Gamma^\flat_G(x)[\dot{x}, \dot{x}] = -\nabla V(x) = -kx.$$

Because $G(x) = \mu(r)I$ with $\mu'(r) = 2m\varepsilon r$, a direct computation of the lowered-index Christoffel term gives

$$\Gamma^\flat_G(x)[\dot{x}, \dot{x}] = m\varepsilon\left(2\,(x \cdot \dot{x})\,\dot{x} - \|\dot{x}\|^2 x\right),$$

so the dynamics are

$$m(1 + \varepsilon r^2)\,\ddot{x} + m\varepsilon\big(2\,(x \cdot \dot{x})\,\dot{x} - \|\dot{x}\|^2 x\big) + kx = 0, \qquad r := \|x\|. \tag{7}$$

Radial–tangential decomposition. Write $x = ru$ with $r = \|x\|$ and $u \in \mathbb{S}^{d-1}$. Let $\omega := \|\dot{u}\|$ denote angular speed. Using $u \cdot \dot{u} = 0$ and $u \cdot \ddot{u} = -\omega^2$, projection of (7) onto u (radial) yields

$$m(1 + \varepsilon r^2)\,(\ddot{r} - r\omega^2) + m\varepsilon\left(r\dot{r}^2 - r^3\omega^2\right) + kr = 0. \tag{8}$$

The angular equation follows from Noether symmetry of rotations: the generalized angular momentum

$$J := \mu(r)\,r^2\,\omega = m(1 + \varepsilon r^2)r^2\,\omega \tag{9}$$

is conserved ($\dot{J} = 0$). Eliminating ω via (9) gives the closed radial ODE

$$m(1 + \varepsilon r^2)\,\ddot{r} + m\varepsilon r\,\dot{r}^2 + kr - \frac{J^2}{m(1 + \varepsilon r^2)\,r^3} - \frac{\varepsilon J^2}{m(1 + \varepsilon r^2)^2\,r} = 0. \tag{10}$$

Energy and effective potential. The mechanical energy

$$E = \tfrac{1}{2}\,\mu(r)\big(\dot{r}^2 + r^2\omega^2\big) + \tfrac{k}{2}r^2 = \tfrac{1}{2}\,\mu(r)\dot{r}^2 + V_{\text{eff}}(r), \tag{11}$$

with conserved J obeys

$$V_{\text{eff}}(r) = \frac{J^2}{2\,\mu(r)\,r^2} + \frac{k}{2}r^2 = \frac{J^2}{2m(1 + \varepsilon r^2)\,r^2} + \frac{k}{2}r^2. \tag{12}$$

Turning points satisfy $E = V_{\text{eff}}(r)$, and small oscillations about equilibria follow from V''_{eff}.

Equilibrium radius and robustness to first order in ε. Equilibria solve $V'_{\text{eff}}(r_*) = 0$, i.e.

$$kr_*^4 = \frac{J^2}{m}\,\frac{1 + 2\varepsilon r_*^2}{(1 + \varepsilon r_*^2)^2}. \tag{13}$$

For $\varepsilon = 0$, $r_0 = (J^2/(km))^{1/4}$. Expanding (13) at small ε shows a cancellation at first order: $r_* = r_0 + O(\varepsilon^2)$. Thus the equilibrium radius is *insensitive at linear order* to isotropic elastic stiffening.

Small radial oscillations (frequency shift). Linearizing (10) at r_* gives the radial frequency $\omega_r^2 = V''_{\text{eff}}(r_*)/\mu(r_*)$. For $\varepsilon = 0$, one finds $\omega_r^2 = 4k/m$. For small ε, using $r_* = r_0 + O(\varepsilon^2)$ and $\mu(r_*) = m(1 + \varepsilon r_0^2) + O(\varepsilon^2)$,

$$\omega_r \;\approx\; 2\sqrt{\tfrac{k}{m}}\left(1 - \tfrac{1}{2}\,\varepsilon r_0^2\right) \;=\; 2\sqrt{\tfrac{k}{m}}\left(1 - \tfrac{1}{2}\,\varepsilon\,(J^2/(km))^{1/2}\right). \tag{14}$$

Hence isotropic elastic mass *reduces* the radial frequency to first order by an amount proportional to εr_0^2.

One-dimensional closed form (quadrature). In $d = 1$ the equation becomes

$$m(1 + \varepsilon x^2)\,\ddot{x} + m\varepsilon x\,\dot{x}^2 + kx = 0. \tag{15}$$

Energy (11) reduces to $E = \frac{1}{2}m(1 + \varepsilon x^2)\dot{x}^2 + \frac{k}{2}x^2$, so

$$\dot{x}^2 = \frac{2}{m}\,\frac{E - \frac{k}{2}x^2}{1 + \varepsilon x^2}, \qquad t - t_0 = \int_{x_0}^{x(t)} \sqrt{\frac{m\,(1 + \varepsilon s^2)}{2(E - \frac{k}{2}s^2)}}\,\mathrm{d}s. \tag{16}$$

This integral is elementary for $\varepsilon = 0$ and becomes a mild elliptic deformation for $\varepsilon > 0$; the small-ε frequency shift matches (14).

Interpretation. The metric factor $\mu(r)$ acts as a *radius-dependent mass.* Two geometric effects arise: (i) the Christoffel term creates an inertial drift that couples radial speed to a centripetal pull $\propto -\|\dot{x}\|^2 x$; (ii) conservation of J modifies the centrifugal barrier by replacing m with $\mu(r)$ in $J^2/(2\mu r^2)$. Net effect: orbits widen slightly (unchanged radius at $O(\varepsilon)$ but lower radial frequency), which is precisely the signature of an elastic medium viewed through inertial geometry.

Optional anisotropic variant. If $G(x) = m\big(I + \varepsilon\, xx^\top\big)$, then $\Gamma^\flat_G(\dot{x}, \dot{x}) = m\varepsilon\big((x\cdot\dot{x})\,\dot{x} + (\dot{x}\cdot\dot{x})\,x\big)$ and the same program leads to a direction-biased inertial correction. The effective potential depends on the instantaneous orientation, producing amplitude-dependent precession even for the quadratic V. We omit details here; the isotropic case already demonstrates the core mechanism cleanly.

8 Energy Balance with Variable Metric and Boundary Reactions

We consider the Riemannian Lagrangian

$$\mathcal{L}(x, \dot{x}) = \tfrac{1}{2}\,\dot{x}^\top G(x)\dot{x} - V(x),$$

on a smooth manifold $\mathcal{S}$ (possibly with boundary $\partial\mathcal{S}$), with dynamics

$$G(x)\,\ddot{x} + \Gamma^\flat_G(x)[\dot{x}, \dot{x}] + \zeta(t)\,G(x)\,\dot{x} \;=\; -\,\nabla V(x) \;+\; f_{\text{ext}}(x, t), \tag{17}$$

where $\zeta \geq 0$ is a damping profile and f_{ext} collects *all* nonconservative forces, including constraint reactions enforcing $x(t) \in \overline{\Omega} \subseteq \mathcal{S}$. We write the mechanical energy

$$E_{\text{mech}}(x, \dot{x}) = \tfrac{1}{2}\,\dot{x}^\top G(x)\dot{x} + V(x).$$

Proposition 8.1. *[Exact energy balance] Along any C^2 solution of (17),*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,E_{\text{mech}}(x, \dot{x}) \;=\; -\zeta(t)\,\dot{x}^\top G(x)\,\dot{x} \;+\; \dot{x}^\top f_{\text{ext}}(x, t).$$

In particular, if $f_{\text{ext}} \equiv 0$ (closed system), energy decays monotonically at rate $\zeta(t)\,\|\dot{x}\|^2_G$; if $\zeta \equiv 0$ and $f_{\text{ext}} \equiv 0$, energy is conserved exactly.

Proof. Differentiate E_{mech}:

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{mech}} = \dot{x}^\top G(x)\,\ddot{x} \;+\; \tfrac{1}{2}\,\dot{x}^\top(\nabla G(x)[\dot{x}])\,\dot{x} \;+\; \nabla V(x)^\top\dot{x}.$$

Insert (17) to replace $G\ddot{x}$, and use metric-compatibility of the Levi–Civita connection:

$$\dot{x}^\top\Gamma^\flat_G(x)[\dot{x}, \dot{x}] \;=\; \tfrac{1}{2}\,\dot{x}^\top(\nabla G(x)[\dot{x}])\,\dot{x},$$

which cancels the geometric term. The remaining terms give $\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{mech}} = -\zeta\,\dot{x}^\top G\,\dot{x} + \dot{x}^\top f_{\text{ext}}$. $\square$

Remark 8.2. [Boundary reactions as power flux] Suppose the motion is constrained to a domain $\Omega \subset \mathcal{S}$ by holonomic constraints $h_j(x) = 0$ on (parts of) $\partial\Omega$, enforced by reaction forces $f_{\mathrm{bdy}} = \sum_j \lambda_j \nabla h_j(x)$. Then

$$\dot{x}^\top f_{\mathrm{bdy}} = \sum_j \lambda_j \frac{\mathrm{d}}{\mathrm{d}t} h_j(x(t)).$$

If constraints are ideal (no-slip, perfectly elastic) so that $\frac{\mathrm{d}}{\mathrm{d}t} h_j(x(t)) \equiv 0$ along solutions, boundary reactions do *no work*: the boundary power vanishes and cannot change E_{mech}. Non-ideal interfaces (e.g. moving/active boundaries) appear as nonzero boundary power and are absorbed into $\dot{x}^\top f_{\mathrm{ext}}$.

8.1 Noether-type momentum with boundary flux

Let Y be a smooth vector field generating a one-parameter group of diffeomorphisms Φ_s on $\mathcal{S}$. If Y is a *Killing field* for G (i.e. $\mathcal{L}_Y G = 0$) and V is Y-invariant ($\mathcal{L}_Y V = 0$), define the momentum

$$J_Y(x, \dot{x}) \;:=\; \langle Y(x), \dot{x} \rangle_G \;=\; Y(x)^\top G(x)\, \dot{x}.$$

Theorem 8.3. *[Momentum balance with damping and boundary forces] Along (17),*

$$\frac{\mathrm{d}}{\mathrm{d}t} J_Y(x, \dot{x}) \;=\; -\zeta(t)\, \langle Y(x), \dot{x} \rangle_G \;+\; Y(x)^\top f_{\mathrm{ext}}(x, t).$$

Hence in the closed, undamped case ($\zeta \equiv 0$, $f_{\mathrm{ext}} \equiv 0$) the Y-momentum is conserved. Boundary reactions contribute only through $Y^\top f_{\mathrm{bdy}}$; if Y is tangent to the constrained boundary, this flux vanishes.

Proof sketch. Differentiate J_Y and use $\nabla G = 0$ in the Levi–Civita sense (metric-compatibility), the Killing condition $\mathcal{L}_Y G = 0$, and $\mathcal{L}_Y V = 0$. Substitute (17) and simplify as in Prop. 8.1. ☐

9 Category of Spring-Structures

Reviewer's Map for Section 9

This section establishes that **Spring** is a well-behaved category for composing dynamical systems. We prove it has **finite limits** (Prop. 9.5, 9.6), **finite colimits** under mild regularity (Prop. 9.8, 9.9), and canonical **adjunctions** to classical categories (Thm. 9.11, 9.13). A counterexample (Remark 9.7) clarifies why we do not claim arbitrary small limits.

The geometric and dynamical analyses above reveal that every Spring is more than an isolated system: it is a *morphism-bearing object* whose boundary behavior determines how it can couple to others, making it amenable to a rich categorical description. The flux-continuity conditions and conserved quantities derived in the previous section already behave like algebraic composition laws, ensuring that energy and information are preserved as they pass from one subsystem to another through a shared interface. This observation is not merely an analogy; it is the physical manifestation of a deeper structural truth. To make this compositional logic explicit and to provide a formal language for constructing, comparing, and reasoning about coupled Spring systems, we now formalize the *Category of Springs*, denoted **Spring**. This framework elevates the discussion from the analysis of single systems to a calculus of interacting, energy-preserving dynamical structures.

Motivation. Each Spring $(\mathcal{S}, G, E, L)$ is a composite object carrying both a geometric and a dynamical identity. The elastic metric G defines the smooth manifold structure and its associated calculus of geodesics and curvature, providing the stage for continuous motion. The energy functional E supplies the thermodynamic imperative, defining a landscape whose gradients guide the system toward equilibrium. Finally, the recursion L acts as the engine of collapse, executing the discrete steps that drive the system down this energy landscape. A morphism between Springs, to be meaningful, must therefore respect all three of these structures simultaneously. It must be a continuous map that not only preserves the underlying topological spaces but also transmits the geometric and dynamical constraints without loss. This means mapping structure to structure in a way that is energy-nonincreasing, ensuring no new energy is created at the interface, and intertwining the dynamics so that updates commute across the morphism. This is the categorical expression of physical conservation: information, structure, and energy may flow through the diagram, but total coherence is maintained or decreased, never spontaneously generated.

From physics to algebra. This categorical perspective recasts physical interactions in a powerful algebraic syntax. Just as forces and fluxes must balance at physical interfaces, morphisms in **Spring** balance coherence and energy between coupled systems. The standard constructions of category theory acquire direct and intuitive physical interpretations. Products describe independent, non-interacting subsystems evolving side by side, forming a composite system whose total energy is the sum of its parts. Pullbacks, or fiber products, provide the canonical tool for rigorously modeling coupled systems that are forced to agree on a common interface, shared boundary condition, or a synchronized observation. This allows for the formal analysis of complex, interconnected networks of Springs. Finally, the concept of adjunctions with classical categories, such as the category of Riemannian manifolds (**Riem**) or ultrametric spaces (**Ultra**), allows us to build formal, canonical bridges between our bi-topological framework and the established worlds of classical analysis. These adjunctions encode the most principled ways to endow a given geometric space with the full structure of Spring dynamics or, conversely, to systematically "forget" the dynamics and recover the underlying geometry. Under this holistic interpretation, the categorical formalism is not just an abstraction layered on top of the physics; it is a fundamental and deep-seated description of the physics of composition itself, meticulously written in a universally applicable language.

We now embark on the final, and most crucial, step of formalizing these concepts. The forthcoming subsections will provide a comprehensive and rigorous treatment, beginning with the formal definitions of the objects and morphisms that constitute **Spring**. From there, we will systematically prove that this category is structurally robust and possesses the necessary properties for a powerful compositional calculus. We will demonstrate that it possesses finite limits—including products and pullbacks, which are essential for composing and constraining systems in a principled way. We will also show that it has finite colimits under well-defined, mild regularity conditions, which allows for the construction of larger, more complex systems by gluing together components or by quotienting by symmetries. This comprehensive analysis will be complemented by the construction of the canonical adjunctions that formally and rigorously connect the world of Springs to the classical domains of Riemannian geometry and ultrametric analysis, thereby completing the full picture of the Spring as a fundamental and uniquely powerful compositional object.

9.1 Objects and morphisms

Definition 9.1. [**Spring**] An object of **Spring** is a quadruple $(\mathcal{S}, G, E, L)$ obeying the structural axioms established in Sections 3–5. A morphism $\Phi : (\mathcal{S}, G, E, L) \to (\mathcal{S}', G', E', L')$ is a map required:

(i) to be continuous for both topologies;

(ii) to be energy-nonincreasing, $E' \circ \Phi \leq E$ pointwise;

(iii) to intertwine the dynamics, $\Phi \circ L = L' \circ \Phi$.

These morphisms ensure that energy and motion commute across interfaces, giving **Spring** its physical meaning as a calculus of composable dynamical systems.

Remark 9.2. [Contraction moduli] It is often useful to annotate a morphism by a pair of

Lipschitz moduli $(\mathrm{Lip}_\infty, \mathrm{Lip}_p) \in [0,1]^2$ in the smooth and ultrametric metrics. Composition multiplies these moduli, giving a quantitative preorder that supports stability budgeting.

9.2 Limits and colimits

Lemma 9.3. *[Bi-properness under products and subspaces] If E and F are bi-proper on $\mathcal{S}$ and $\mathcal{T}$, then $E \oplus F$ is bi-proper on $\mathcal{S} \times \mathcal{T}$ (product bi-topologies). If $A \subseteq \mathcal{S}$ is closed in both topologies, then $E|_A$ is bi-proper on A (subspace bi-topologies).*

Proof sketch. $\{E \oplus F \le c\} = \bigcup_{a+b \le c}\{E \le a\} \times \{F \le b\}$ is a finite union of products of precompact sets, hence precompact in the product bi-topologies. For subspaces, $\{E|_A \le c\} = A \cap \{E \le c\}$ is precompact in the subspace topologies. □

Proposition 9.4. *[Terminal object] The one-point Spring $\mathbf{1} = (\{*\}, 0, 0, \mathrm{Id})$ is terminal. For any Spring A there exists a unique morphism $\mathsf{A} \to \mathbf{1}$.*

Proposition 9.5. *[Finite products; cartesian monoidal] Given Springs $\mathsf{A} = (\mathcal{S}, G, E, L)$ and $\mathsf{B} = (\mathcal{T}, H, F, M)$, their product*

$$\mathsf{A} \times \mathsf{B} := \big(\mathcal{S} \times \mathcal{T},\ G \oplus H,\ E \oplus F,\ L \times M\big)$$

with product bi-topologies, block-diagonal metric $G \oplus H$, summed energy, and componentwise update is again a Spring. The projections are morphisms and satisfy the universal property. Hence **Spring** *is cartesian monoidal with unit* $\mathbf{1}$.

Proof sketch. Completeness is preserved by products; $E \oplus F$ has precompact sublevels as a product of precompact sets (Lemma 9.3); if L and M are contractions with moduli $\alpha, \beta < 1$, then $L \times M$ is a contraction with modulus $\max\{\alpha, \beta\}$ under max/sum product norms. □

Proposition 9.6. *[Equalizers and completeness] For parallel morphisms $\Phi, \Psi : \mathsf{A} \to \mathsf{B}$, the equalizer $\mathrm{Eq}(\Phi, \Psi) = \{x \in \mathcal{S} : \Phi(x) = \Psi(x)\}$ with subspace bi-topologies and restricted (G, E, L) is a Spring; the inclusion is a monomorphism. Consequently, together with the terminal object and binary products,* **Spring** *has finite limits (in particular, pullbacks).*

Proof sketch. Closedness from continuity; restricting a contraction to a closed invariant subset preserves contraction; energy descent and precompactness are inherited by subspaces (Lemma 9.3). □

Remark 9.7. [No arbitrary small limits] The category **Spring** does not have all small limits because countable products do not preserve the strict contraction property. Let $S_i = (\mathbb{R}, 1, x^2/2, (1 - 1/i)x)$. Each $L_i(x) = (1 - 1/i)x$ is a contraction with modulus $\alpha_i = 1 - 1/i < 1$. The product map $L = \prod_i L_i$ on the ℓ^∞ product space has modulus $\sup_i \alpha_i = 1$, so it is not a strict contraction.

Proposition 9.8. *[Coproducts] The disjoint union $\mathsf{A} \sqcup \mathsf{B}$ with piecewise structures (disjoint-union bi-topologies, metric, energy, and update acting in each summand) is a coproduct in* **Spring**.

Proof sketch. Bi-properness holds because a sublevel is a finite disjoint union of precompact sublevels in each summand. Contraction modulus is the maximum of the component moduli, which is less than 1. The universal property is satisfied by defining mediating morphisms piecewise. □

Proposition 9.9. *[Coequalizers under mild regularity] Let $\Phi, \Psi : \mathsf{A} \rightrightarrows \mathsf{B}$ (for $\mathsf{B} = (\mathcal{T}, H, F, M)$). Suppose $\mathcal{T}$ is compactly generated (or Polish), the M-saturated equivalence relation R generated by $\Phi(a) \sim \Psi(a)$ is closed in $\mathcal{T} \times \mathcal{T}$, and each equivalence class $[x]$ is compact. Then the T_1 quotient Spring $\mathsf{B}/{\sim}$ with pushforward bi-topologies, energy $\bar{F}([x]) = \inf_{y \in [x]} F(y)$, and update $\bar{M}([x]) = [M(x)]$ is a coequalizer.*

Proof sketch. The map $\bar{M}$ is well-defined because R is M-saturated. Bi-properness of $\bar{F}$ follows because $\pi(\{F \le c\})$ is the compact image of a compact set. Contraction of $\bar{M}$ is inherited from M on the quotient (semi)metrics. The universal property is satisfied because any dynamics-respecting, energy-nonincreasing map that coequalizes Φ and Ψ must descend to the quotient. $\square$

Proposition 9.10. *[Pullbacks / fiber products] Given $\Phi : \mathsf{A} \to \mathsf{C}$ and $\Psi : \mathsf{B} \to \mathsf{C}$, the fiber product*

$$\mathsf{A} \times_{\mathsf{C}} \mathsf{B} = \{(a,b) \in \mathcal{S} \times \mathcal{T} : \ \Phi(a) = \Psi(b)\}$$

with subspace bi-topologies, restricted $G \oplus H$, restricted $E \oplus F$, and update $(L \times M)|_{\mathsf{A} \times_{\mathsf{C}} \mathsf{B}}$ is a Spring and satisfies the universal property.

Proof sketch. It is the equalizer of $(\Phi \circ \pi_1, \Psi \circ \pi_2)$ in $\mathsf{A} \times \mathsf{B}$. Invariance under updates follows from functoriality: $\Phi \circ L = L' \circ \Phi$, $\Psi \circ M = L' \circ \Psi$. $\square$

9.3 Adjunctions with classical forgetful functors

Let $U_\infty : \mathbf{Spring} \to \mathbf{Riem}$ and $U_p : \mathbf{Spring} \to \mathbf{Ultra}$ forget everything but the smooth metric and ultrametric structures, respectively.

Theorem 9.11. *[Free Spring over Riemannian data] Fix a Riemannian space $(\mathcal{S}, G)$ and a C^2 energy E that is μ-strongly convex with L-Lipschitz gradient in the G-geometry. For any step $\eta \in (0, 2/L)$ there exists a Spring*

$$\mathsf{Free}^{\eta}_{\infty}(\mathcal{S}, G, E) = \big(\mathcal{S},\ G,\ E,\ L_E\big), \qquad L_E(x) := \exp^G_x\big(-\eta\, G^{-1}\nabla E(x)\big),$$

which is a contraction in the smooth metric. This defines a functor $\mathsf{Free}^{\eta}_{\infty} : \mathbf{Riem}_E \to \mathbf{Spring}$ left adjoint to U_∞.

Proof sketch. Strong convexity/Lipschitzness imply $\mathrm{Lip}(L_E) \le \max\{|1-\eta\mu|, |1-\eta L|\} < 1$. Units/counits are the evident identities on $(\mathcal{S}, G)$; naturality is standard. $\square$

Remark 9.12. []To meet the Spring axioms bi-topologically, equip $\mathcal{S}$ with any compatible ultrametric (e.g., via a fixed hierarchical coding). Bi-properness of E ensures precompact sublevels in both topologies. Choices are unique up to isomorphism in **Spring**. The existence of the exponential map is guaranteed locally; for this construction to be globally well-defined, we assume the step $-\eta G^{-1}\nabla E(x)$ remains within the injectivity radius of x for all x in the domain of interest.

Theorem 9.13. *[Free Spring over ultrametric data] Let $(\mathcal{S}, d_p)$ be an ultrametric space and $E : \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous with compact sublevels. Fix $\rho > 0$ and define, for $x \in \mathcal{S}$,*

$$\mathcal{F}_\rho(x) := \overline{B_p(x,\rho)} \cap \{E \le E(x)\}.$$

Assume each closed ball $\overline{B_p(x,\rho)}$ is compact (e.g. $(\mathcal{S}, d_p)$ is proper), or *simply assume $\mathcal{F}_\rho(x)$ is compact for all x (automatic if balls are compact). Define*

$$L_p(x) := \operatorname*{argmin}_{y \in \mathcal{F}_\rho(x)} E(y).$$

Then L_p is well-defined, nonexpansive and, for sufficiently small ρ, a strict d_p-contraction. Equipping $\mathcal{S}$ with any tame baseline smooth metric G_0 yields a Spring $\big(\mathcal{S}, G_0, E, L_p\big)$. This construction is left adjoint to U_p.

Proof sketch. Lower semicontinuity + compact feasibility $\mathcal{F}_\rho(x)$ yield existence of a minimizer. Ultrametric geometry gives stability of minimizers under the center (for small ρ), implying Lipschitz continuity and strict contraction. Bi-properness is inherited from E (Lemma 9.3). $\square$

9.4 Enrichment and quantitative composition

Proposition 9.14. *[Moduli-enriched structure] Endow each hom-set with the preorder given by contraction moduli* $(\mathrm{Lip}_\infty, \mathrm{Lip}_p)$*. Composition is monotone and multiplicative in each coordinate; identities carry modulus* $(1, 1)$*. This yields a quantale-enriched preorder supporting end-to-end stability budgets.*

9.5 Ind/Pro limits for refinement and coarsening

Remark 9.15. [Ind/Pro constructions] Hierarchical discretizations (refinements) are modeled as Ind-objects built from injections that preserve (G, E, L); coarsenings/projective schemes appear as Pro-objects with compatible surjections. Energy descent and contraction pass to these limits under the usual compactness hypotheses.

9.6 Coalgebraic view

Remark 9.16. [Coalgebras and bisimulation] Each Spring is a coalgebra $L : \mathcal{S} \to \mathcal{S}$ decorated by (G, E). Morphisms are coalgebra morphisms preserving energy; coequalizers coincide with quotients by bisimulations closed under L.

10 Outlook: Adèlic Analysis and Spectral Questions

We sketch an adèlic deformation S_K over the adèle ring, define Haar measure and Fourier analysis, and pose spectral questions for a Spring-Laplacian. Conjectures include Poisson invariance under elastic coupling and stability of automorphic spectra.

A Proof Sketches

A.1 From Discrete EL to Riemannian Newton

Standard Taylor expansions show that the central difference approximates acceleration with order $O(h^2)$; metric-variation terms reorganize into the Levi–Civita Christoffel contribution; the potential term passes pointwise.

A.2 Accelerated Mirror Limit

Under $\eta_k = h$ and the stated momentum scalings, the second-order difference converges to (6). The metric-variation term converges to $\Gamma^\flat_G(\dot{x}, \dot{x})$; the momentum scaling yields the damping coefficient.

B Rates via KL

If E satisfies a Kurdyka–Łojasiewicz property near $\mathrm{Fix}(L)$, then discrete descent enjoys sublinear or linear rates depending on curvature; continuous dynamics inherit analogous energy decay rates.

Acknowledgments

The author thanks collaborators and readers who stress-tested the dual-topology and elastic-metric arguments.

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