Tutor STEM

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#1 Rated GRE Prep Course for 25 Years

GRE Practice Test Exam

Full Length Examination

(1 of 200 Tests)

The other 199 Tests will be sent to students upon registering with Tutor STEM's GRE Prep Course

Our GRE instructors will guide students through all the questions from the various practice tests
Completing the 200 tests should result in a GRE score sufficient for business school acceptance

Examination begins on next page

MATHEMATICS TEST

Time— 170 minutes

66 Questions

Directions: Each of the questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is best and then completely fill in the corresponding space on the answer sheet.

Computation and scratch work may be done in this test book.

In this test:

(1) All logarithms with an unspecified base are natural logarithms, that is, with base e.
(2) The symbols , , , and C denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

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1. $\lim_{x \to 0} \frac{\cos(3x) - 1}{x^2} =$

(A) $\frac{9}{2}$ (B) $\frac{3}{2}$ (C) $-\frac{2}{3}$ (D) $-\frac{3}{2}$ (E) $-\frac{9}{2}$

---

2. What is the area of an equilateral triangle whose inscribed circle has radius 2 ?

(A) 12 (B) 16 (C) $12\sqrt{3}$ (D) $16\sqrt{3}$ (E) $4\left(3 + 2\sqrt{2}\right)$

---

3. $\int_{e^{-3}}^{e^{-2}} \frac{1}{x \log x}\, dx =$

(A) 1 (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) $\log\left(\frac{2}{3}\right)$ (E) $\log\left(\frac{3}{2}\right)$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

4. Let V and W be 4-dimensional subspaces of a 7-dimensional vector space X. Which of the following CANNOT be the dimension of the subspace V « W?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

5. Sofia and Tess will each randomly choose one of the 10 integers from 1 to 10. What is the probability that neither integer chosen will be the square of the other?

(A) 0.64 (B) 0.72 (C) 0.81 (D) 0.90 (E) 0.95

---

6. Which of the following shows the numbers $2^{1/2}$, $3^{1/3}$, and $6^{1/6}$ in increasing order?

(A) $2^{1/2} < 3^{1/3} < 6^{1/6}$

(B) $6^{1/6} < 3^{1/3} < 2^{1/2}$

(C) $6^{1/6} < 2^{1/2} < 3^{1/3}$

(D) $3^{1/3} < 2^{1/2} < 6^{1/6}$

(E) $3^{1/3} < 6^{1/6} < 2^{1/2}$

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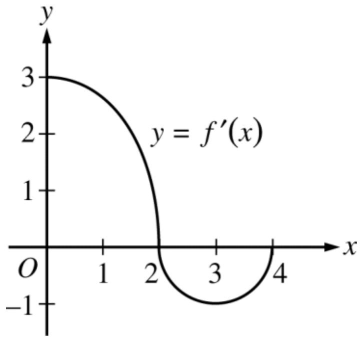
SCRATCH WORK



7. The figure above shows the graph of the derivative f' of a function f, where f is continuous on the interval $[0, 4]$ and differentiable on the interval $(0, 4)$. Which of the following gives the correct ordering of the values $f(0)$, $f(2)$, and $f(4)$?

(A) $f(0) < f(2) < f(4)$

(B) $f(0) < f(4) = f(2)$

(C) $f(0) < f(4) < f(2)$

(D) $f(4) = f(2) < f(0)$

(E) $f(4) < f(0) < f(2)$

---

8. Which of the following is NOT a group?

(A) The integers under addition

(B) The nonzero integers under multiplication

(C) The nonzero real numbers under multiplication

(D) The complex numbers under addition

(E) The nonzero complex numbers under multiplication

GO ON TO THE NEXT PAGE.

SCRATCH WORK

9. Let g be a continuous real-valued function defined on $\mathbb{R}$ with the following properties.

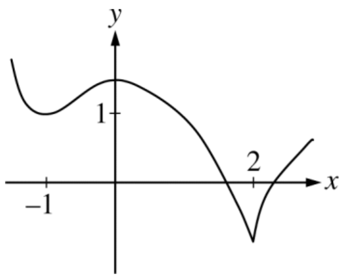
$$g'(0) = 0$$

$$g''(-1) > 0$$

$$g''(x) < 0 \text{ if } 0 < x < 2.$$

Which of the following could be part of the graph of g ?

(A)



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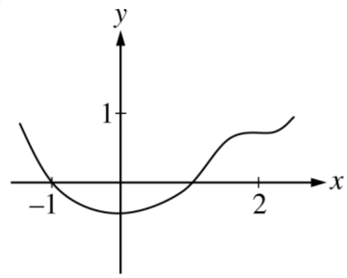

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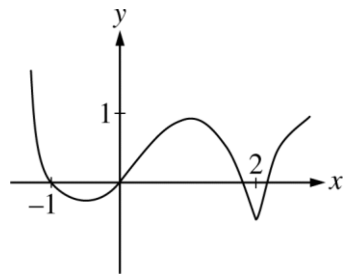

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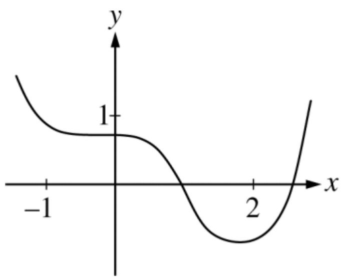
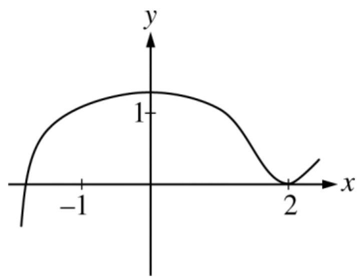


(E)



$$\sqrt{(x+3)^2+(y-2)^2} = \sqrt{(x-3)^2+y^2}$$

10. In the xy-plane, the set of points whose coordinates satisfy the equation above is

(A) a line (B) a circle (C) an ellipse (D) a parabola (E) one branch of a hyperbola

---

GO ON TO THE NEXT PAGE.

SCRATCH WORK

11. The region bounded by the curves $y = x$ and $y = x^2$ in the first quadrant of the xy-plane is rotated about the y-axis. The volume of the resulting solid of revolution is

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$ (E) $\frac{3\pi}{2}$

---

12. For which integers n such that $3 \le n \le 11$ is there only one group of order n (up to isomorphism) ?

(A) For no such integer n

(B) For 3, 5, 7, and 11 only

(C) For 3, 5, 7, 9, and 11 only

(D) For 4, 6, 8, and 10 only

(E) For all such integers n

---

13. If f is a continuously differentiable real-valued function defined on the open interval $(-1, 4)$ such that $f(3) = 5$ and $f'(x) \ge -1$ for all x, what is the greatest possible value of $f(0)$?

(A) 3 (B) 4 (C) 5 (D) 8 (E) 11

GO ON TO THE NEXT PAGE.

SCRATCH WORK

14. Suppose g is a continuous real-valued function such that $3x^5 + 96 = \int_c^x g(t)\,dt$ for each $x \in \mathbb{R}$, where c is a constant. What is the value of c ?

(A) -96 (B) -2 (C) 4 (D) 15 (E) 32

---

15. Let S, T, and U be nonempty sets, and let $f : S \to T$ and $g : T \to U$ be functions such that the function $g \circ f : S \to U$ is one-to-one (injective). Which of the following must be true?

(A) f is one-to-one.

(B) f is onto.

(C) g is one-to-one.

(D) g is onto.

(E) $g \circ f$ is onto.

---

16. Suppose A, B, and C are statements such that C is true if exactly one of A and B is true. If C is false, which of the following statements must be true?

(A) If A is true, then B is false.

(B) If A is false, then B is false.

(C) If A is false, then B is true.

(D) Both A and B are true.

(E) Both A and B are false.

GO ON TO THE NEXT PAGE.

SCRATCH WORK

17. Which of the following equations has the greatest number of real solutions?

(A) $x^3 = 10 - x$

(B) $x^2 + 5x - 7 = x + 8$

(C) $7x + 5 = 1 - 3x$

(D) $e^x = x$

(E) $\sec x = e^{-x^2}$

---

18. Let f be the function defined by $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ for all x such that $-1 < x < 1$. Then $f'(x) =$

(A) $\frac{1}{1-x}$ (B) $\frac{x}{1-x}$ (C) $\frac{1}{1+x}$ (D) $\frac{x}{1+x}$ (E) 0

---

19. If z is a complex variable and $\bar{z}$ denotes the complex conjugate of z, what is $\lim_{z \to 0} \frac{(\bar{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

GO ON TO THE NEXT PAGE.

SCRATCH WORK

20. Let g be the function defined by $g(x) = e^{2x+1}$ for all real x. Then $\lim\limits_{x \to 0} \dfrac{g(g(x)) - g(e)}{x} =$

(A) $2e$ (B) $4e^2$ (C) e^{2e+1} (D) $2e^{2e+1}$ (E) $4e^{2e+2}$

---

21. What is the value of $\displaystyle\int_{-\pi/4}^{\pi/4} \left(\cos t + \sqrt{1+t^2}\, \sin^3 t \cos^3 t \right) dt$?

(A) 0 (B) $\sqrt{2}$ (C) $\sqrt{2} - 1$ (D) $\dfrac{\sqrt{2}}{2}$ (E) $\dfrac{\sqrt{2} - 1}{2}$

---

22. What is the volume of the solid in xyz-space bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = 0$, and $z = y + 3$?

(A) $\dfrac{8}{3}$ (B) $\dfrac{16}{3}$ (C) $\dfrac{32}{3}$ (D) $\dfrac{104}{105}$ (E) $\dfrac{208}{105}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

23. Let $(�_{10}, +, �)$ be the ring of integers modulo 10, and let S be the subset of $�_{10}$ represented by $\{0, 2, 4, 6, 8\}$. Which of the following statements is FALSE?

(A) $(S, +, �)$ is closed under addition modulo 10.

(B) $(S, +, �)$ is closed under multiplication modulo 10.

(C) $(S, +, �)$ has an identity under addition modulo 10.

(D) $(S, +, �)$ has no identity under multiplication modulo 10.

(E) $(S, +, �)$ is commutative under addition modulo 10.

---

24. Consider the system of linear equations

$$\begin{aligned} w + 3x + 2y + 2z &= 0 \\ w + 4x + y \phantom{{}+ 2z} &= 0 \\ 3w + 5x + 10y + 14z &= 0 \\ 2w + 5x + 5y + 6z &= 0 \end{aligned}$$

with solutions of the form (w, x, y, z), where w, x, y, and z are real. Which of the following statements is FALSE?

(A) The system is consistent.

(B) The system has infinitely many solutions.

(C) The sum of any two solutions is a solution.

(D) $(-5, 1, 1, 0)$ is a solution.

(E) Every solution is a scalar multiple of $(-5, 1, 1, 0)$.

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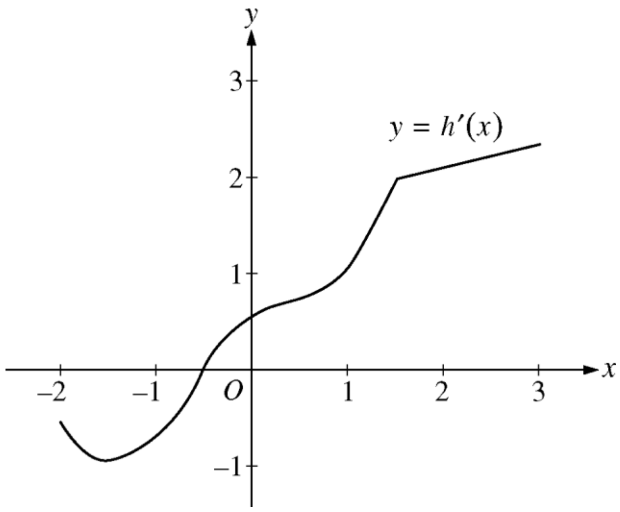
SCRATCH WORK



25. The graph of the <u>derivative</u> h' is shown above, where h is a real-valued function. Which of the following open intervals contains a value c for which the point $(c, h(c))$ is an inflection point of h ?

(A) $(-2, -1)$ (B) $(-1, 0)$ (C) $(0, 1)$ (D) $(1, 2)$ (E) $(2, 3)$

$$3x \equiv 5 \pmod{11}$$
$$2y \equiv 7 \pmod{11}$$

26. If x and y are integers that satisfy the congruences above, then $x + y$ is congruent modulo 11 to which of the following?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

---

27. $(1 + i)^{10} =$

(A) 1 (B) i (C) 32 (D) $32i$ (E) $32(i + 1)$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

28. Let f be a one-to-one (injective), positive-valued function defined on �. Assume that f is differentiable at $x = 1$ and that in the xy-plane the line $y - 4 = 3(x - 1)$ is tangent to the graph of f at $x = 1$. Let g be the function defined by $g(x) = \sqrt{x}$ for $x \geq 0$. Which of the following is FALSE?

(A) $f'(1) = 3$

(B) $\left(f^{-1}\right)'(4) = \frac{1}{3}$

(C) $(fg)'(1) = 5$

(D) $(g \circ f)'(1) = \frac{1}{2}$

(E) $(g \circ f)(1) = 2$

---

29. A tree is a connected graph with no cycles. How many nonisomorphic trees with 5 vertices exist?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

---

30. For what positive value of c does the equation $\log x = cx^4$ have exactly one real solution for x ?

(A) $\frac{1}{4e}$ (B) $\frac{1}{4e^4}$ (C) $\frac{e^4}{4}$ (D) $\frac{4}{e^{1/4}}$ (E) $4e^{1/4}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

31. Of the numbers 2, 3, and 5, which are eigenvalues of the matrix $\begin{pmatrix} 3 & 5 & 3 \\ 1 & 7 & 3 \\ 1 & 2 & 8 \end{pmatrix}$?

(A) None (B) 2 and 3 only (C) 2 and 5 only (D) 3 and 5 only (E) 2, 3, and 5

---

32. $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^4} e^{t^2}\, dt =$

(A) $e^{x^6}\left(e^{x^8 - x^6} - 1\right)$ (B) $4x^3e^{x^8}$ (C) $\dfrac{1}{\sqrt{1 - e^{x^2}}}$ (D) $\dfrac{e^{x^2}}{x^2} - 1$ (E) $x^2e^{x^6}\left(4xe^{x^8 - x^6} - 3\right)$

---

33. What is the 19th derivative of $\dfrac{x - 1}{e^x}$?

(A) $(18 - x)e^{-x}$

(B) $(19 - x)e^{-x}$

(C) $(20 - x)e^{-x}$

(D) $(x - 19)e^{-x}$

(E) $(x - 20)e^{-x}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

34. Which of the following statements about the real matrix shown above is FALSE?

(A) A is invertible.

(B) If $\mathbf{x} \in \mathbb{R}^5$ and $A\mathbf{x} = \mathbf{x}$, then $\mathbf{x} = \mathbf{0}$.

(C) The last row of A^2 is $\begin{pmatrix} 0 & 0 & 0 & 0 & 25 \end{pmatrix}$.

(D) A can be transformed into the 5×5 identity matrix by a sequence of elementary row operations.

(E) $\det(A) = 120$

---

35. In xyz-space, what are the coordinates of the point on the plane $2x + y + 3z = 3$ that is closest to the origin?

(A) $(0, 0, 1)$ (B) $\left(\frac{3}{7}, \frac{3}{14}, \frac{9}{14}\right)$ (C) $\left(\frac{7}{15}, \frac{8}{15}, \frac{1}{15}\right)$ (D) $\left(\frac{5}{6}, \frac{1}{3}, \frac{1}{3}\right)$ (E) $\left(1, 1, \frac{1}{3}\right)$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

36. Suppose S is a nonempty subset of $\mathbb{R}$. Which of the following is necessarily true?

(A) For each $s, t \in S$, there exists a continuous function f mapping $[0, 1]$ into S with $f(0) = s$ and $f(1) = t$.

(B) For each $u \notin S$, there exists an open subset U of $\mathbb{R}$ such that $u \in U$ and $U \cap S = \emptyset$.

(C) $\{v \in S : \text{there exists an open subset } V \text{ of } \mathbb{R} \text{ with } v \in V \subseteq S\}$ is an open subset of $\mathbb{R}$.

(D) $\{w \notin S : \text{there exists an open subset } W \text{ of } \mathbb{R} \text{ with } w \in W \text{ and } W \cap S = \emptyset\}$ is a closed subset of $\mathbb{R}$.

(E) S is the intersection of all closed subsets of $\mathbb{R}$ that contain S.

---

37. Let V be a finite-dimensional real vector space and let P be a linear transformation of V such that $P^2 = P$. Which of the following must be true?

I. P is invertible.

II. P is diagonalizable.

III. P is either the identity transformation or the zero transformation.

(A) None (B) I only (C) II only (D) III only (E) II and III

SCRATCH WORK

38. The maximum number of acute angles in a convex 10-gon in the Euclidean plane is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

---

39. Consider the following algorithm, which takes an input integer n > 2 and prints one or more integers.

```
input(n)
set i =1
while i< n
  begin
    replace i by i+1
    set k = n
    while k ≥ i
      begin
        if i= k then print(i)
        replace k by k-1
      end
  end
```

If the input integer is 88, what integers will be printed?

(A) Only the integer 2

(B) Only the integer 88

(C) Only the divisors of 88 that are greater than 1

(D) The integers from 2 to 88 in increasing order

(E) The integers from 88 to 2 in decreasing order

GO ON TO THE NEXT PAGE.

SCRATCH WORK

40. Let S be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Consider the two binary operations $+$ and $\circ$ on S defined as pointwise addition and composition of functions, as follows.

$$(f + g)(x) = f(x) + g(x)$$
$$(f \circ g)(x) = f(g(x))$$

Which of the following statements are true?

I. $\circ$ is commutative.

II. $+$ and $\circ$ satisfy the left distributive law $f \circ (g + h) = (f \circ g) + (f \circ h)$.

III. $+$ and $\circ$ satisfy the right distributive law $(g + h) \circ f = (g \circ f) + (h \circ f)$.

(A) None (B) II only (C) III only (D) II and III only (E) I, II, and III

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41. Let ℓ be the line that is the intersection of the planes $x + y + z = 3$ and $x - y + z = 5$ in $\mathbb{R}^3$. An equation of the plane that contains $(0, 0, 0)$ and is perpendicular to ℓ is

(A) $x - z = 0$

(B) $x + y + z = 0$

(C) $x - y - z = 0$

(D) $x + z = 0$

(E) $x + y - z = 0$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

42. Let $\mathbb{Z}^+$ be the set of positive integers and let d be the metric on $\mathbb{Z}^+$ defined by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{if } m \neq n \end{cases}$$

for all $m, n \in \mathbb{Z}^+$. Which of the following statements are true about the metric space $(\mathbb{Z}^+, d)$?

I. If $n \in \mathbb{Z}^+$, then $\{n\}$ is an open subset of $\mathbb{Z}^+$.

II. Every subset of $\mathbb{Z}^+$ is closed.

III. Every real-valued function defined on $\mathbb{Z}^+$ is continuous.

(A) None (B) I only (C) III only (D) I and II only (E) I, II, and III

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43. A curve in the *xy*-plane is given parametrically by

$$x = t^2 + 2t$$
$$y = 3t^4 + 4t^3$$

for all $t > 0$. The value of $\dfrac{d^2y}{dx^2}$ at the point $(8, 80)$ is

(A) 4 (B) 24 (C) 32 (D) 96 (E) 192

GO ON TO THE NEXT PAGE.

SCRATCH WORK

$$y' + xy = x$$
$$y(0) = -1$$

44. If y is a real-valued function defined on the real line and satisfying the initial value problem above, then $\lim\limits_{x \to -\infty} y(x) =$

(A) 0 (B) 1 (C) -1 (D) ∞ (E) $-\infty$

---

45. How many positive numbers x satisfy the equation $\cos(97x) = x$?

(A) 1 (B) 15 (C) 31 (D) 49 (E) 96

---

46. A ladder 9 meters in length is leaning against a vertical wall on level ground. As the bottom end of the ladder is moved away from the wall at a constant rate of 2 meters per second, the top end slides downward along the wall. How fast, in meters per second, will the top end of the ladder be sliding downward at the moment the top end is 3 meters above the ground?

(A) $12\sqrt{2}$ (B) $6\sqrt{2}$ (C) $4\sqrt{2}$ (D) $\dfrac{1}{2\sqrt{2}}$ (E) $\dfrac{2}{3}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

47. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f(x) = \begin{cases} 3x^2 & \text{if } x \in \mathbb{Q} \\ -5x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Which of the following is true?

(A) f is discontinuous at all $x \in \mathbb{R}$.

(B) f is continuous only at $x = 0$ and differentiable only at $x = 0$.

(C) f is continuous only at $x = 0$ and nondifferentiable at all $x \in \mathbb{R}$.

(D) f is continuous at all $x \in \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

(E) f is continuous at all $x \notin \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

---

48. Let g be the function defined by $g(x, y, z) = 3x^2 y + z$ for all real x, y, and z. Which of the following is the best approximation of the directional derivative of g at the point $(0, 0, \pi)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$? (<u>Note:</u> $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors in $\mathbb{R}^3$.)

(A) 0.2 (B) 0.8 (C) 1.4 (D) 2.0 (E) 2.6

---

49. What is the largest order of an element in the group of permutations of 5 objects?

(A) 5 (B) 6 (C) 12 (D) 15 (E) 120

GO ON TO THE NEXT PAGE.

SCRATCH WORK

50. Let R be a ring and let U and V be (two-sided) ideals of R. Which of the following must also be ideals of R?

I. $U + V = \{u + v : u \in U \text{ and } v \in V\}$

II. $U \cdot V = \{uv : u \in U \text{ and } v \in V\}$

III. $U \cap V$

(A) II only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

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51. Which of the following is an orthonormal basis for the column space of the real matrix $\begin{pmatrix} 1 & -1 & 2 & -3 \\ -1 & 1 & -3 & 2 \\ 2 & -2 & 5 & -5 \end{pmatrix}$?

(A) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\}$

(E) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

52. A university's mathematics department has 10 professors and will offer 20 different courses next semester. Each professor will be assigned to teach exactly 2 of the courses, and each course will have exactly one professor assigned to teach it. If any professor can be assigned to teach any course, how many different complete assignments of the 10 professors to the 20 courses are possible?

(A) $\frac{20!}{2^{10}}$ (B) $\frac{10!}{2^{9}}$ (C) $10^{20} - 2^{10}$ (D) $10^{20} - 100$ (E) $\frac{20!10!}{2^{10}}$

---

53. Let f and g be continuous functions of a real variable such that $g(x) = \int_0^x f(y)(y - x)\, dy$ for all x. If g is three times continuously differentiable, what is the greatest integer n for which f must be n times continuously differentiable?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

---

54. If a real number x is chosen at random in the interval $[0, 3]$ and a real number y is chosen at random in the interval $[0, 4]$, what is the probability that $x < y$?

(A) $\frac{1}{2}$ (B) $\frac{7}{12}$ (C) $\frac{5}{8}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

55. If a and b are positive numbers, what is the value of $\int_0^\infty \frac{e^{ax} - e^{bx}}{(1 + e^{ax})(1 + e^{bx})}\,dx$?

(A) 0 (B) 1 (C) $a - b$ (D) $(a - b)\log 2$ (E) $\frac{a - b}{ab}\log 2$

---

56. Which of the following statements are true?

I. There exists a constant C such that $\log x \le C\sqrt{x}$ for all $x \ge 1$.

II. There exists a constant C such that $\sum_{k=1}^{n} k^2 \le Cn^2$ for all integers $n \ge 1$.

III. There exists a constant C such that $|\sin x - x| \le C|x^3|$ for all real x.

(A) None (B) I only (C) III only (D) I and III only (E) I, II, and III

GO ON TO THE NEXT PAGE.

SCRATCH WORK

57. For each positive integer n, let x_n be a real number in the open interval $\left(0, \frac{1}{n}\right)$. Which of the following statements must be true?

I. $\lim_{n \to \infty} x_n = 0$

II. If f is a continuous real-valued function defined on $(0, 1)$, then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

III. If g is a uniformly continuous real-valued function defined on $(0, 1)$, then $\lim_{n \to \infty} g(x_n)$ exists.

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

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58. A circular helix in xyz-space has the following parametric equations, where $\theta \in \mathbb{R}$.

$$x(\theta) = 5\cos\theta$$
$$y(\theta) = 5\sin\theta$$
$$z(\theta) = \theta$$

Let $L(\theta)$ be the arc length of the helix from the point $P(\theta) = (x(\theta), y(\theta), z(\theta))$ to the point $(5, 0, 0)$, and let $D(\theta)$ be the distance between $P(\theta)$ and the origin. If $L(\theta_0) = 26$, then $D(\theta_0) =$

(A) 6 (B) $\sqrt{51}$ (C) $\sqrt{52}$ (D) $14\sqrt{3}$ (E) $15\sqrt{3}$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

59. Let A be a real 3×3 matrix. Which of the following conditions does NOT imply that A is invertible?

(A) $-A$ is invertible.

(B) There exists a positive integer k such that $\det\left(A^k\right) \neq 0$.

(C) There exists a positive integer k such that $(I - A)^k = 0$, where I is the 3×3 identity matrix.

(D) The set of all vectors of the form $A\mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^3$, is $\mathbb{R}^3$.

(E) There exist 3 linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ such that $A\mathbf{v}_i \neq \mathbf{0}$ for each i.

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60. A real-valued function f defined on $\mathbb{R}$ has the following property.

For every positive number ε, there exists a positive number δ such that

$$|f(x) - f(1)| \geq \varepsilon \quad \text{whenever} \quad |x - 1| \geq \delta.$$

This property is <u>equivalent</u> to which of the following statements about f?

(A) f is continuous at $x = 1$.

(B) f is discontinuous at $x = 1$.

(C) f is unbounded.

(D) $\lim_{|x| \to \infty} |f(x)| = \infty$

(E) $\int_0^\infty |f(x)|\, dx = \infty$

GO ON TO THE NEXT PAGE.

SCRATCH WORK

61. A tank initially contains a salt solution of 3 grams of salt dissolved in 100 liters of water. A salt solution containing 0.02 grams of salt per liter of water is sprayed into the tank at a rate of 4 liters per minute. The sprayed solution is continually mixed with the salt solution in the tank, and the mixture flows out of the tank at a rate of 4 liters per minute. If the mixing is instantaneous, how many grams of salt are in the tank after 100 minutes have elapsed?

(A) 2 (B) $2 - e^{-2}$ (C) $2 + e^{-2}$ (D) $2 - e^{-4}$ (E) $2 + e^{-4}$

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62. Let S be the subset of $\mathbb{R}^2$ consisting of all points (x, y) in the unit square $[0, 1] \times [0, 1]$ for which x or y, or both, are irrational. With respect to the standard topology on $\mathbb{R}^2$, S is

(A) closed

(B) open

(C) connected

(D) totally disconnected

(E) compact

GO ON TO THE NEXT PAGE.

SCRATCH WORK

63. For any nonempty sets A and B of real numbers, let $A \cdot B$ be the set defined by

$$A \cdot B = \{xy : x \in A \text{ and } y \in B\}.$$

If A and B are nonempty bounded sets of real numbers and if $\sup(A) > \sup(B)$, then $\sup(A \cdot B) =$

(A) $\sup(A)\sup(B)$

(B) $\sup(A)\inf(B)$

(C) $\max\{\sup(A)\sup(B),\ \inf(A)\inf(B)\}$

(D) $\max\{\sup(A)\sup(B),\ \sup(A)\inf(B)\}$

(E) $\max\{\sup(A)\sup(B),\ \inf(A)\sup(B),\ \inf(A)\inf(B)\}$

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64. What is the value of the flux of the vector field **F**, defined on $\mathbb{R}^3$ by $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, through the surface $z = \sqrt{1 - x^2 - y^2}$ oriented with upward-pointing normal vector field? (Note: **i**, **j**, and **k** are the standard basis vectors in $\mathbb{R}^3$.)

(A) 0 (B) $\frac{2\pi}{3}$ (C) π (D) $\frac{4\pi}{3}$ (E) 2π

GO ON TO THE NEXT PAGE.

SCRATCH WORK

65. Let g be a differentiable function of two real variables, and let f be the function of a complex variable z defined by

$$f(z) = e^x \sin y + ig(x, y),$$

where x and y are the real and imaginary parts of z, respectively. If f is an analytic function on the complex plane, then $g(3, 2) - g(1, 2) =$

(A) e^2

(B) $e^2(\sin 3 - \sin 1)$

(C) $e^2(\cos 3 - \cos 1)$

(D) $e - e^3 \sin 2$

(E) $(e - e^3)\cos 2$

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66. Let $\mathbb{Z}_{17}$ be the ring of integers modulo 17, and let $\mathbb{Z}_{17}^{\times}$ be the group of units of $\mathbb{Z}_{17}$ under multiplication. Which of the following are generators of $\mathbb{Z}_{17}^{\times}$?

I. 5

II. 8

III. 16

(A) None (B) I only (C) II only (D) III only (E) I, II, and III

S T O P

If you finish before time is called, you may check your work on this test