<u>**Limits and Continuity**</u>

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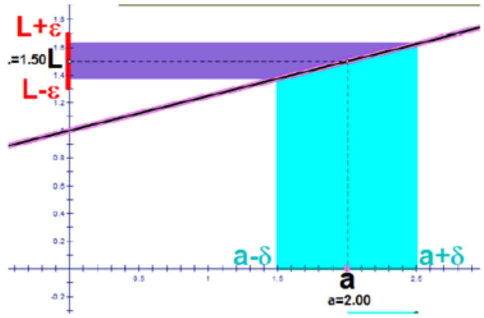
We have previously defined limit in for single variable functions, but how do we generalize this to vector-valued functions? Well, the definition is very similar. Below is the definition of limit in single-variable calculus.

$$\text{We say that } \lim_{x\to a} f(x) = L \text{ if and only if}$$
$$\text{for all } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that}$$
$$|x-a| < \delta \text{ implies } |f(x) - L| < \epsilon$$



Note, that as the distance from $x = 2$ is less than δ, the value of the function at $f(2)$ is less than ε, thus, satisfying the definition .

In multivariable calculus the limit of a function, *r*, at a point, t_o, is *L*, if and only if

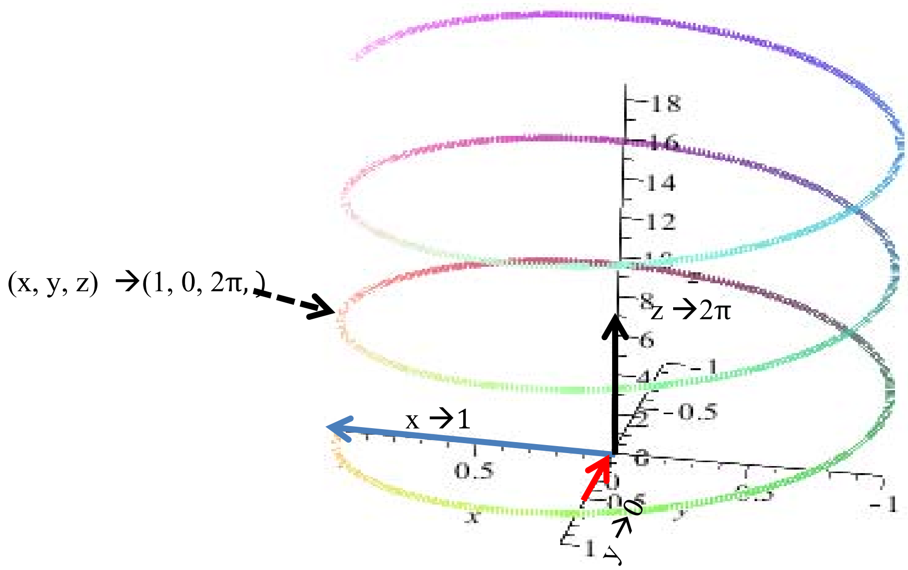
$$\text{for every } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$|t - t_o| < \delta \text{ implies } |\mathrm{r}(t) - L| < \epsilon.$$

However, this time, instead of *L* being a real number, it is a vector. A function, *r*, is continuous if the limit of the function is equal to the value of the function on every point in its domain. This is equivalent to all of its component functions being continuous. Indeed, the limit of the vector valued function at t_o is the limit of its component functions at t_o.

<u>**Example 1:** The Helix</u>

Consider the function $r:[0,6\pi] \to \mathbb{R}^3$ be defined by $r(t) = (\cos(t), \sin(t), t)$. Since, cos(*t*) and sin(*t*) are continuous functions of *t*, then *r* is also continuous. Also, if I wanted to find the limit as *t* approached 2π,

$$\lim_{t\to 2\pi} r(t) = (\lim_{t\to 2\pi} \cos(t), \lim_{t\to 2\pi} \sin(t), \lim_{t\to 2\pi} t)$$
$$= (1, 0, 2\pi)$$



Derivative and Tangent Vectors

In single variable calculus, we found the slope of the tangent line using the difference quotient

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$$

How could we generalize this for a vector-valued function? Let's analyze this,

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{r(t+\Delta t)-r(t)}{\Delta t}&=\lim_{\Delta t\to 0}\frac{(x(t+\Delta t),y(t+\Delta t),z(t+\Delta t))-(x(t),y(t),z(t))}{\Delta t}\\&=\lim_{\Delta t\to 0}\frac{(x(t+\Delta t)-x(t),y(t+\Delta t)-y(t),z(t+\Delta t)-z(t))}{\Delta t}\\&=\lim_{\Delta t\to 0}(\frac{x(t+\Delta t)-x(t)}{\Delta t},\frac{y(t+\Delta t)-y(t)}{\Delta t},\frac{z(t+\Delta t)-z(t)}{\Delta t})\\&=(x'(t),y'(t),z'(t))\end{aligned}$$

Therefore, a vector-valued function, ***r***, is differentiable at every point at which the coordinate functions are differentiable.

The curve traced by **r** is **smooth** if $\frac{dr}{dt}$ is continuous and never zero, which is equivalent to the coordinate functions having continuous, non-vanishing derivatives. The vector given by $v(t)=(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt})$ is called the **velocity vector.** The **tangent vector** is the unit vector in the direction of the velocity.

Definition:

If **r(*t*)** is the position vector of the particle, then the **speed** of the particle is $|\frac{dr}{dt}|$, and the **acceleration vector** is givne by $\frac{d^2r}{dt^2}$.

Example:
Find the velocity, speed and acceleration of a particle whose motion in space is given by the vector-valued function

$$r(t) = (2\cos(t), 2\sin(t), 5\cos^2(t))$$

The velocity vector-valued function is given by

$$\begin{aligned} v(t) &= (-2\sin(t), 2\cos(t), -10\cos(t)\sin(t)) \\ &= (-2\sin(t), 2\cos(t), -5\sin(2t)) \end{aligned}$$

and the acceleration vector-valued function given by

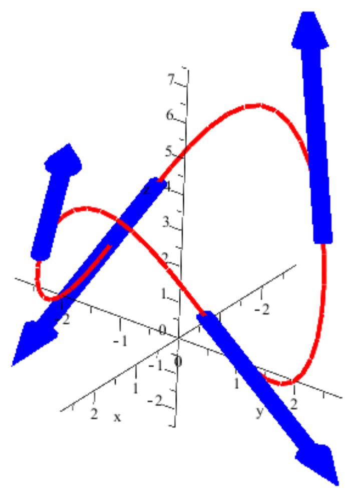
$$a(t) = (-2\cos(t), -2\sin(t), -10\cos(2t))$$

The speed of the particle is given by

$$\begin{aligned} s(t) &= \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + (-5\sin(2t))^2} \\ &= \sqrt{4 + 25\sin^2(2t)} \end{aligned}$$

On the next page we will look at the graph of the curve above, as well as the tangent vector when $t = \frac{5\pi}{4}$.

Compute the velocity vector at this time. What are the other values of *t* for which tangent vector appears?



The graph of the curve $r(t) = (2\cos(t), 2\sin(t), 5\cos^2(t))$
and tangent vector at various times.

Differentiation Rules:

For the rules below, assume
C, c are real numbers, $\vec{r}(t)$ and $\vec{s}(t)$ are functions of t, and $f(t)$ is a real valued function in one variable.

Constant Function Rule	$\frac{d}{dt}C = 0$
Scalar Multiple Rule	$\frac{d}{dt}(c\vec{r}(t)) = c\vec{r}\,'(t)$
Leibniz Rule	$\frac{d}{dt}(f(t)\vec{r}(t)) = f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$
Sum Rule	$\frac{d}{dt}(\vec{r}(t) + \vec{s}(t)) = \vec{r}\,'(t) + \vec{s}\,'(t)$
Difference Rule	$\frac{d}{dt}(\vec{r}(t) - \vec{s}(t)) = \vec{r}\,'(t) - \vec{s}\,'(t)$
Dot Product Rule	$\frac{d}{dt}(\vec{r}(t)\cdot\vec{s}(t)) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$
Cross Product Rule	$\frac{d}{dt}(\vec{r}(t)\times\vec{s}(t)) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$
Chain Rule	$\frac{d}{dt}[\vec{r}(f(t))] = f\,'(t)\vec{r}\,'(f(t))$

Proofs:
[***Leibniz Rule***] Let $\vec{r}(t) = (x(t), y(t), z(t))$. Then $f(t)\vec{r}(t)) = (f(t)x(t), f(t)y(t), f(t)z(t))$. The derivative of a vector valued function in one variable is the derivative of the coordinate functions. By the product rule in first variable calculus, the derivative of the first coordinate function is

$$\frac{d}{dt}[f(t)x(t)] = f'(t)x(t) + f(t)x'(t).$$

A similar argument can be made to compute the derivatives of the second and third coordinate function. Therefore, the derivative

$$\frac{d}{dt}[f(t)r(t)] = (f\,'(t)x(t) + f(t)x\,'(t),\ f\,'(t)y(t) + f(t)y\,'(t),\ f\,'(t)z(t) + f(t)z\,'(t))$$

$$= (f\,'(t)x(t),\ f\,'(t)y(t),\ f\,'(t)z(t)) + (f(t)x\,'(t),\ f(t)y\,'(t),\ f(t)z\,'(t)) \text{ by vector addition}$$

$$= f\,'(t)(x(t), y(t), z(t)) + f(t)(x\,'(t), y\,'(t), z\,'(t)) \quad \text{by scalar multiplication}$$

$$= f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t) \quad \text{by definition of } \vec{r}(t)$$

Dot Product Rule:
Define the position vectors by

$\vec{r}(t) =$ ______________________________

$\vec{s}(t) =$ ______________________________

Then

$\vec{r}(t) \cdot \vec{s}(t) =$ ______________________________

Taking the derivative and using the product rule for single variable functions, we achieve the result,

$\frac{d}{dt}[\vec{r}(t) \cdot \vec{s}(t)] =$ ______________________________

Write all the terms with the derivatives of the coordinate functions of $\vec{r}(t)$ first, and the rest after.

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By definition of dot product, we achieve the result

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Cross-Product Rule:

Chain Rule:

Vector Functions of Constant Length:

Suppose the position function, $r\,(t)$, of a particle moving through space has position vectors of constant length
Then
$r(t) = (x(t), y(t), z(t))$ and $|\,r(t)\,| = C$ for some real number C.
By definition of length of a vector

$$|\,r(t)\,|^2 = C^2$$

$$r(t) \cdot r(t) = C^2$$

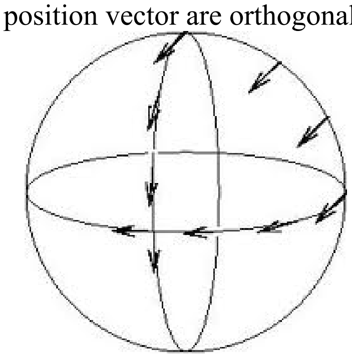
$$\frac{d}{dt}[r(t) \cdot r(t)] = 0$$

$$r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$$

$$2r'(t) \cdot r(t) = 0$$

$$r'(t) \cdot r(t) = 0$$

Therefore, the tangent vector and the position vector are orthogonal.



<u>Exercises:</u>
For numbers 1 – 4, find the velocity and acceleration vectors, the speed function, and the length function of the position vector.

1. $r(t) = (t+1, t^2 - 1, 2t)$
2. $r(t) = (\cos t, \sin t, 2t)$
3. Let $r(t) = (3t+1, \sqrt{3}t, t^2)$
   a. Find the angle between the position and velocity vector.
   b. Find the angle between the velocity and acceleration vector.
4. Find the tangent line to the curve $r(t) = (\sin t, t^2 - \cos t, e^t)$ at $t = 0$.

5. Find the tangent line to the curve $r(t) = (\ln t, \frac{t-1}{t+2}, \sin(\frac{\pi}{2}t)$ at $t = 1$.

6. Let $r(t) = (2 + \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t, 2 - \frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{3}}\sin t, 1 + \frac{1}{\sqrt{3}}\sin t)$.

   a. Show this curve is on the plane $x + y - 2z = 2$.

   b. Translate the curve by the vector (-2, -2, -1).

   c. Show the resulting curve is a circle of radius 1 centered at the origin.

   d. Does this curve have constant distance to the origin? What does this tell you about the velocity vectors?

Extra Credit will be given for each graph drawn above.

Please write each problem neatly and legibly. Label all the solutions and turn in your assignment on

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