Arc Length

As you may remember from AP Calculus BC, one of the applications of integration was the ability to measure the arc length of a smooth parametric curve. Using the generalization of distance formula, we can compute the length of a smooth parameterized curve in space using a similar formula.

Arc Length for a parameterized curve in 2 dimensions

If $r(t) = (x(t), y(t))$ is a differentiable vector-valued function on [a,b], then the length of the path traced by *r*(t) is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc Length for a parameterized curve in 3 dimensions

If $r(t) = (x(t), y(t), z(t))$ is a differentiable vector-valued function on [a,b], then the length of the path traced by *r*(t) is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$
$$= \int_a^b |v(t)|\, dt$$

Example:

Find the length of the curve traced by the parameterized curve

$$r(t) = (\cos(t), \sin(t), t)$$

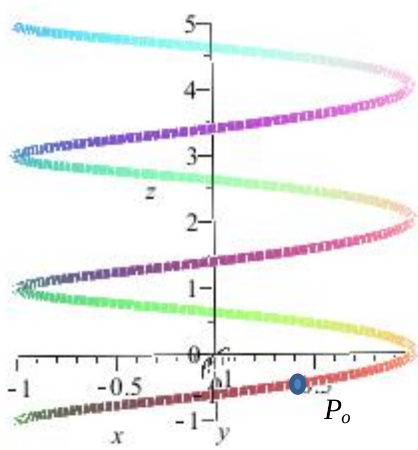
on the interval $[0, 2\pi]$

Solution:

<u>Arc Length Parameter</u>

Let $r(t)$ be a curve in space.
Choose a base point for the curve. Call it $r(t_0) = P_0$

Helix with base point



Graph of the curve represented parametrically by the components of the given vector.

Choosing a direction, we can parameterize the curve so that each value of t determines a point, $P(t)$ on the curve and a "directed distance". This distance is given by

$$s(t) = \int_{t_0}^{t} |v(\tau)|\, d\tau$$
$$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

This function is called the ***arc length function***, $s(t)$, and is the distance along the curve from $P(t_o)$ to $P(t)$. If $t > t_o$ then $s(t)$ is positive, and if $t < t_0$, then $s(t)$ is negative.
Here is the important concept to take away from this construction …. Are you ready … this is big …

Each value of s determines a point on the curve, thereby defining a parameter for the curve.

Let me say that again, in case you didn't get it the first time. Each value of s, is in fact a distance along the curve from the base point $P_{o.}$ This yields a new parameter for the curve called the ***arc length parameter*** for the curve. The parameter ‘s value increases in the direction of increasing "time", t.

Millan, this is especially for you: The arc length parameter is especially handy and effective for analyzing the turning and twisting nature of a space curve.

Arc Length Parameter with base point $P_0 = P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_{t_0}^{t} |v(\tau)|\, d\tau$$

We use the Greek letter τ for the variable of integration because we already used *r, s, t, v, x, y* and *z*. The unused letters of the Roman Alphabet were in danger of becoming extinct, so we started going after the Greek letters.

Example: Find the arc length parameter for the helix in the previous example using (1,0,0) as the base point, then reparameterize the curve using the arc length parameter.

$$r(t) = (\cos(t), \sin(t), t)$$

Definition: [speed] The speed of a particle traveling along a smooth curve is given by $\frac{ds}{dt} = |v(t)|$, and is the derivative of the arc length function.

Definiton: [unit tangent vector] The unit tangent vector is the velocity vector of length one, and is given by

$$T = \frac{v}{|v|}$$

Since it is a unit vector, and is tangent to the curve in the direction of the particles movement, we call it the ***unit tangent vector.***

Example: Find the unite tangent vector to the curve $r(t) = (2\cos(t), 2\sin(t), t^2)$ for each *t*.