Lagrange Multipliers

Finding extreme values of a function whose domain is constrained to lie within some subset of $\mathbb{R}^2$ can seem to be difficult. However, we are going to use level surfaces and their properties to make this computation simpler. For example, suppose we are given a plane

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - z = 5\}$$

and we are asked the question, which point is closest to the origin? There are many methods of solving this.

Blind Computation:

The distance function needs to be minimized, so we could take the function

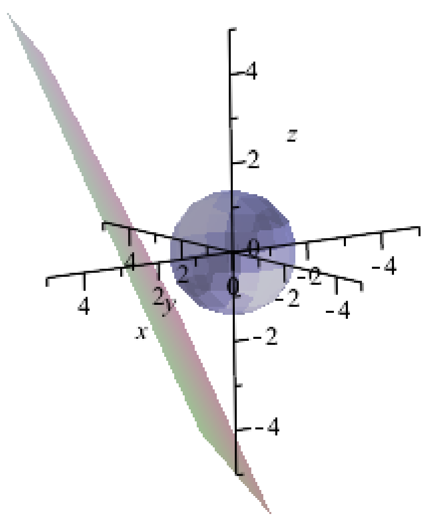
$$d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Solve the equation of the plane for a variable *z*, and find the local extrema by setting the partials equal to zero.

$$d(x, y) = \sqrt{x^2 + y^2 + (2x + y - 5)^2}$$
$$d_x = 2x + 2(2(2x + y - 5)) = 0$$
$$d_y = 2y + 2(2x + y - 5) = 0$$

Then we would solve for the *x* and *y*. This method seems pretty efficient but take into consideration that the constraint we are using (the plane) is a linear function in all the variables, and therefore solving for a variable is possible and substitution feasible. However, when constraints are given by polynomials of higher degree, or non-algebraic functions, the possibility of using the substitution method decreases.

Creative Elegance



Consider the problem another way. Suppose we have a family of spheres centered at the origin, parameterized by their radius. They would be the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.
The point on the plane closest to the origin is the point where the sphere is tangent to the plane. This occurs when the vector normal to the plane (2, 1, -1) is also normal to the sphere. However, we showed a few sections ago that the gradient of the function, $\nabla f \,|_{P_0}$ is always normal to the level surface $f(x, y, z) = f(P_0)$.

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

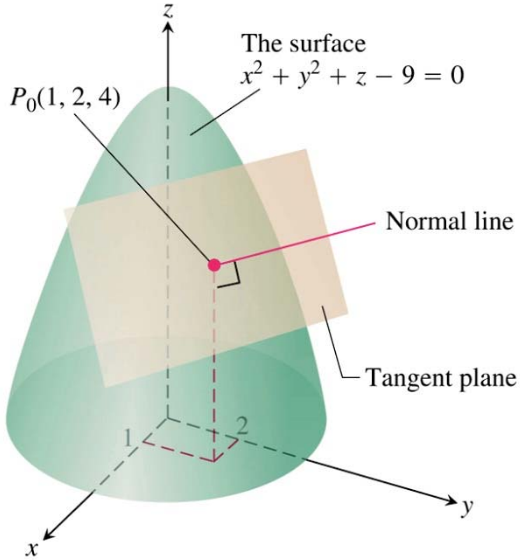


FIGURE 14.33 The tangent plane and normal line to this surface at P_0 (Example 1).

Let $P_0 = (x_0, y_0, z_0)$ and let $f(P_0) = c$. Furthermore, suppose $r(t) = < x(t), y(t), z(t) >$ is a space curve on the level surface through the point P_0. From this construction, we know that

$$f(r(t)) = c$$

because we chose *r* to be on the level surface associated to the value *c*. Using chain rule, we achieve

$$f_x(P_0)\frac{dx}{dt} + f_y(P_0)\frac{dy}{dt} + f_z(P_0)\frac{dz}{dt} = 0$$

$$\nabla f\,|_{P_0} \cdot r'(t) = 0$$

Since this is true for any curve on the level surface and through P_0, then $\nabla f\,|_{P_0}$ is normal to all the tangent vectors of those curves. This can only be true if the gradient of *f* at the point, $\nabla f\,|_{P_0}$, is normal to the tangent plane to the level surface at P_0.

This allows us to find a method of solving any optimization problems with constraints.

THEOREM 12—The Orthogonal Gradient Theorem Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

The proof of the theorem above follows from the chain rule and the fact that the derivative of *f* with respect to *t* is zero when the curve runs through a local min or max relative to the points on the curve.

COROLLARY OF THEOREM 12 At the points on a smooth curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{v} = 0$, where $\mathbf{v} = d\mathbf{r}/dt$.

The proof of this is what we showed on the previous page. Therefore, to find local extrema with constraints, we need only find where the gradients are parallel to each other.

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.

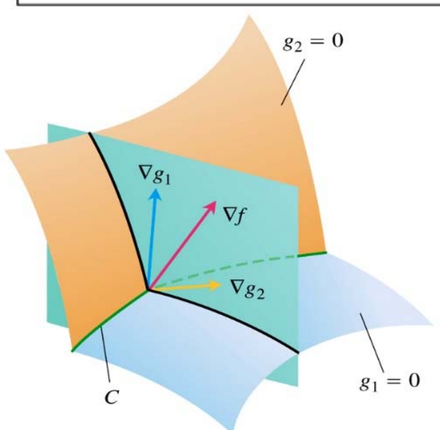


FIGURE 14.55 The vectors ∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1 = 0$ and ∇g_2 is normal to the surface $g_2 = 0$.

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \tag{2}$$

Example 2: Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

Solution: Since we seek the points on the cylinder closest to the origin. The function to be minimized is the distance formula, $g(x, y, z) = x^2 + y^2 + z^2$, and the constraint function is $f(x, y, z) = x^2 - z^2$.

The first thing we need to do is compute the gradients of both functions

$$\nabla f = (2x, 0, -2z) \quad \text{and} \quad \nabla g = (2x, 2y, 2z)$$

Now we must solve the equation $\nabla f = \lambda \nabla g$.

$$2x = \lambda 2x$$
$$0 = \lambda 2y$$
$$-2z = \lambda 2z$$

To make the first equation true, $\lambda = 1$, which means $2z = -2z$. Therefore, $z = 0$. Therefore, the points would have to be of the form $(x, 0, 0)$. Substituting this information into the equation of constraint, we achieve $x^2 - (0)^2 - 1 = 0$ which implies $x = \pm 1$. Therefore, the points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. You may also wonder why we can't use the last equation to find λ. Well, we can.

To make the second equation true, $\lambda = -1$, which means $2x = -2x$. Therefore, $x = 0$. Therefore, the points would have to be of the form $(0, 0, z)$. Substituting this information into the equation of constraint, we achieve $0^2 - z^2 - 1 = 0$ which implies $-z^2 = 1$. This can never happen, so this avenue is a dead end.

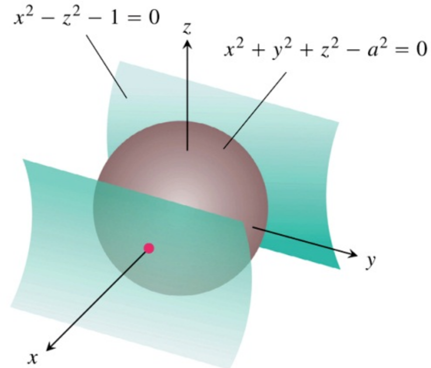


FIGURE 14.51 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

Exercises:

1) Find the point on the surface $z^2 = xy + 1$ which is closest to the origin.

Step 1: Write both surfaces as the level surfaces of functions *f* and *g*, where *f* is the function to be optimized, and *g* is the constraint function such that $\nabla g \neq 0$ when $g(x, y, z) = 0$. Find ∇f and ∇g.

Step 2: Solve the system of equations $\nabla f = \lambda \nabla g$.

2) Find the points on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point (1, -1, 1).

3) Find the dimensions of a closed rectangular box with maximum volume that can be inscribed in the unit sphere.