Double Integrals over General Regions

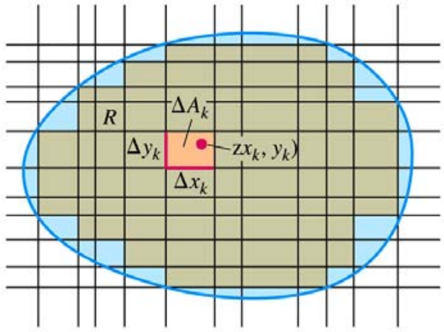


FIGURE 15.8 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

Now we are going to evaluate integrals over non-rectangular regions. They still need to be bounded, but can be irregularly shaped. We are still going to cover the region, *R*, by rectangles which cover the region, however, due to the irregularity of the shape, the number of rectangles may not be finite. However, the integral can still be defined as

$$\lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A$$ where ΔA is the area of the smallest rectangle completely contained in *R*.

Volumes

If $f(x,y)$ is positive and continuous on a region, *R*, bounded above by $y = g_1(x)$ and below by $y = g_2(x)$ then we could calculate the area of each cross-section perpendicular to the domain and the *x*-axis.

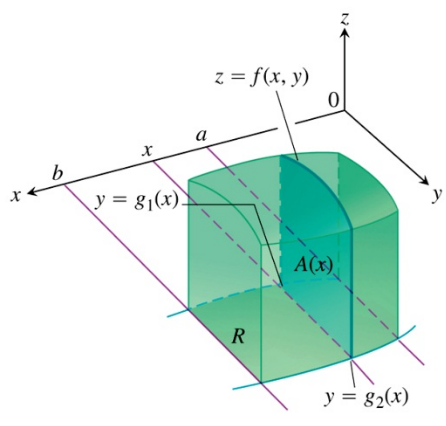


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

As you can see, the area of the cross-section

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)dy$$

Integrating this area function from $x = a$ to $x = b$.

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

Similarly, if the boundary of the region, *R*, can be bounded by curves $x = h_1(y)$ and $x = h_2(y)$ then

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dydx$$

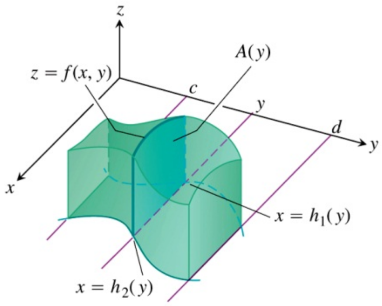


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10, or in the way shown here. Both calculations have the same result.

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA$ (any number c)

2. *Sum and Difference:*
$$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$

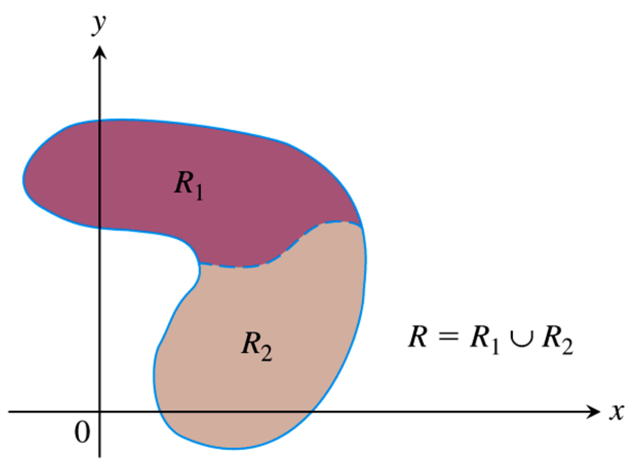
3. *Domination:*

 (a) $\displaystyle\iint_R f(x, y)\,dA \geq 0$ if $f(x, y) \geq 0$ on R

 (b) $\displaystyle\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$

 if R is the union of two nonoverlapping regions R_1 and R_2



$$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$$

FIGURE 15.17 The Additivity Property for rectangular regions holds for regions bounded by continuous curves.

Example: Find the volume of the wedgelike solid that lies beneath the surface

$$z = 16 - x^2 - y^2$$

and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$ and the $x-$axis.

Solution:

Exercises:

Sketch the region of integration and evaluate the integral. You may need to switch the order of integration.

1. $\int_0^{\pi}\int_0^{x} x\sin(y)\,dydx$
2. $\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2}e^{\frac{y}{\sqrt{x}}}\,dydx$
3. $\int_0^{2}\int_x^{2} 2\,\mathrm{y}^2\sin(\mathrm{x}\,y)\,dydx$
4. $\int_0^{8}\int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\,dydx$
5. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines, $y = x, x = 0,$ and $x + y = 2$ in the $xy-$plane.

6. Find the volume of the solid that is bounded on the front and back by the planes $x = \pm\frac{\pi}{3}$ on the sides by the cylinders $y = \pm\sec(x)$ above by the cylinder $z = 1 + y^2$, and below by the xy-plane.