Just as in one and two variables, sometimes it's easier to calculate integrals of functions in three variables when the variables are written in cylindrical or spherical coordinates.

Cylindrical Coordinates

This is not that much of a departure from polar coordinates. The *x* and *y* coordinates are transformed into polar coordinates, while the *z* coordinate remains unchanged, so that $(x, y, z) \rightarrow (r, \theta, z)$.

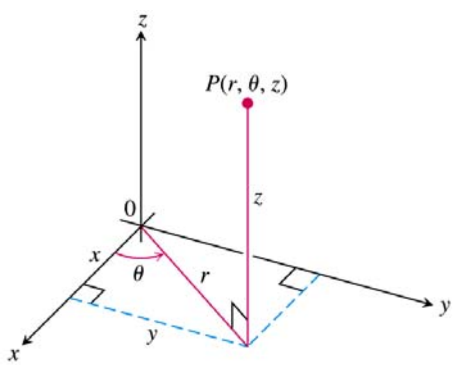
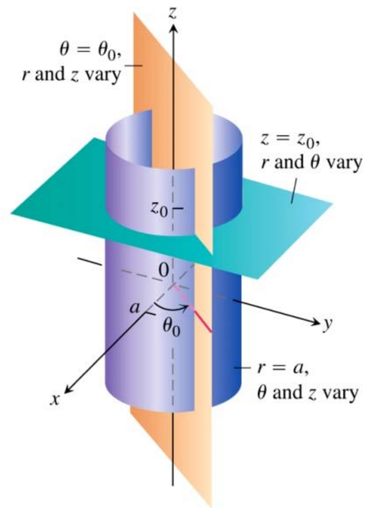


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.



The equations for obtaining these coordinates are the same as in two coordinates.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates
$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$
$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$

If we keep the *r* coordinate fixed, instead of a plane, we get an infinite cylinder. If we keep the θ variable fixed, we get a vertical plane with the given angle with the positive *x* axis, and if we keep the *z* variable fixed, we get a horizontal plane. The integral is the limit as $n \to \infty$ of the sum below;

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\Delta z_k r_k \Delta r_k \Delta\theta_k$$

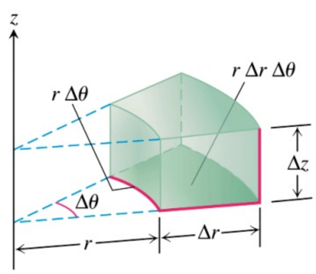


FIGURE 15.44 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

Example: Find the centroid if the density, $\delta = 1$, of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the *xy*-plane.

Solution:

Step 1: Sketch the solid.

Step 2: Find the *z* limits of integration.

Step 4: Find the θ limits of integration.

Step 4: Find the θ limits of integration.

Step 5: Integrate!

Spherical Coordinates:

When having to deal with three coordinates and you want no rectangular coordinates, then we must use two angles. The angle θ is still the angle with the positive *x*-axis in the counter-clockwise direction and ϕ is the angle with *xy*-plane. The length of the vector associated with this point, we call ρ.

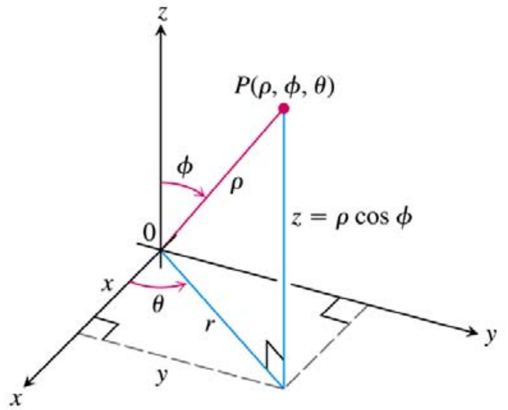


FIGURE 15.47 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta,$$
$$z = \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta, \tag{1}$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

To find the volume over a region using spherical coordinates, we partition the region into *n* spherical wedges. The change in radius is $\Delta\rho_k$, the change in the angles are $\Delta\theta_k$ and $\Delta\phi_k$. The vertical edge of the wedge is a circular arc with length $\rho_k\Delta\phi_k$. The horizontal edge has is an arc of a circle with radius $\rho_k \sin(\phi_k)$, so the arc length is $\rho_k \sin(\phi_k)\Delta\theta_k$. Therefore the volume of the wedge is

$$\Delta V_k = \rho_k{}^2 \sin(\phi_k)\Delta\rho_k\Delta\phi_k\Delta\theta_k$$

The corresponding Riemann Sum for a function $f(\rho,\phi,\theta)$ is given by

$$S_n = \sum_{k=1}^{n} f(\rho_k,\phi_k,\theta_k)\rho_k^2 \sin(\theta_k)\Delta\rho_k\Delta\phi_k\Delta\theta_k$$

The illustration on the next page will show how this works.

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

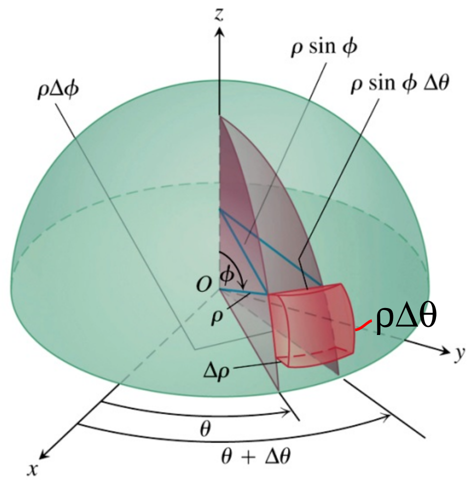


FIGURE 15.51 In spherical coordinates

$$dV = d\rho \cdot \rho \, d\phi \cdot \rho \sin\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example:

Find the volume of the "ice cream cone" *D* cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \dfrac{\pi}{3}$.

Step 1: Sketch the region

Step 2: Find the limits of integration

Step 3: Integrate

Classwork: # 23, 29, 33, 43, 46, 47

We will break up into 3 groups, and each group will do two problems.