Mathematically speaking, a vector field is a function from $f:\mathbb{R}^n \to \mathbb{R}^n$. If we want to model the gravitational force, electric force, fluid force at a given point, we do this with vector fields because the direction and magnitude of the forces vary with respect to the position of the object on which they act.

The wind tunnel below has an obstruction in the middle of it. As air rushes around the particle, the velocity increases in magnitude. The vector field above shows the velocity of an air particle at each point in the wind tunnel. The streamline above has the same property. The force of water increases as the channel decreases in width.

Example 1

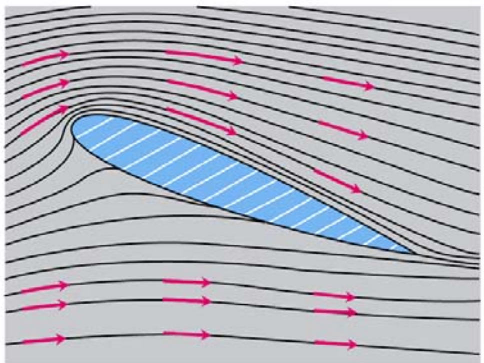


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

Example 2

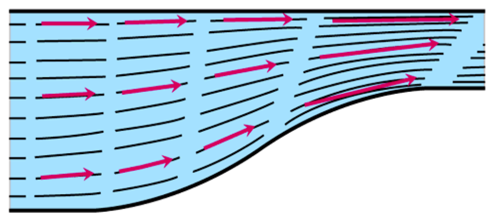


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

All objects get pulled towards the center of the earth. The closer the object is to the center, the greater the force of gravity.

Example 3

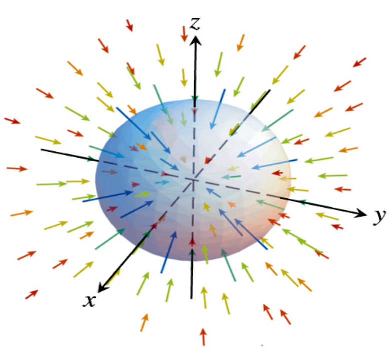


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

The gradient of a multivariable function $f(x,y,z)=w$ at a point also gives the vector fields which depicts the direction of maximum increase at each point.

Example 4

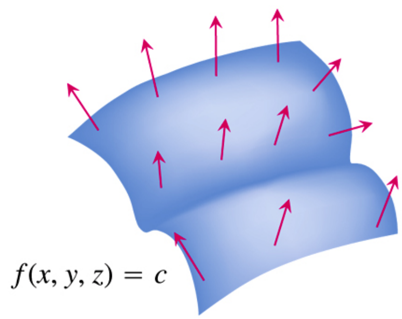


FIGURE 16.10 The field of gradient vectors ∇f on a surface $f(x,y,z) = c$.

These pictures are pretty, but mathematically, a vector field looks like

$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$$

Your book uses M, N, and P for the coordinate functions, and **i, j,** and **k** for the vectors, but that is more a physics notation than mathematical notation. Clearly, if we are to generalize this to *n*-dimensions, we will need more than 26 letters to use as coordinate functions.

The functions $f_i(x, y, z)$ are called the coordinate functions. If the coordinate functions are continuous, we say the vector field is continuous. If the coordinate functions are differentiable, then we say the vector field is differentiable.

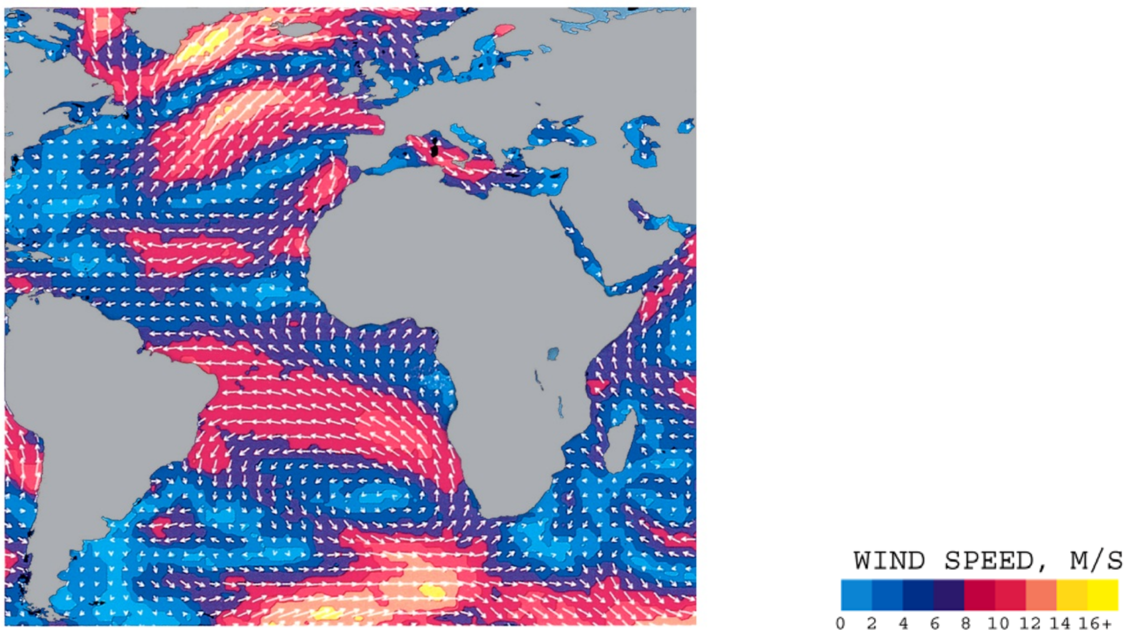


FIGURE 16.15 NASA's *Seasat* used radar to take 350,000 wind measurements over the world's oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm south of Greenland.

<u>Example 5:</u> a) Find the vector field ***F*** given by the gradient of the function

Suppose a particle is moving along a path given by $r(t) = (\cos(t), \sin(t), t^2)$.

b) Find the vector field given by the velocity of the particle at time *t*.

c) Find the vector field given by the normal vector of the particle at time *t*.

Line Integrals of Vector Fields

Let ***F*** be a continuous vector field, $\mathbf{F}(x, y, z) = (f_1, f_2, f_3)$.

Let C be a smooth curve with parameterization, $\mathbf{r}(t) = (x(t), y(t), z(t))$.

In the previous section, we discussed computing the line integral of a real-valued function over $\mathbb{R}^3$.
It seems natural that the integral of a vector field over a smooth curve should be the projection of the vector¸ $\mathbf{F}(x(t), y(t), z(t))$ onto the tangent vector of **r,** $\mathbf{T}(t)$**.**

DEFINITION Let **F** be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Note that we could use the parameterization by arc length and dot the vector field with the tangent vector **or …**

Step 1. Compose the vector fields coordinate function $f_i(x, y, z)$ with the curve, $r(t)$.

Step 2: Find $\frac{dr}{dt}$.

Step 3: Evaluate the path integral with respect to parameter t on the interval $[a, b]$ to obtain

$$\int_C F \cdot dr = \int_a^b F(r(t)) \frac{dr}{dt}\, dt$$

Of course, we always want to apply this to work problems because everybody loves work! The work done by a force in moving an object along a curve is just the path integral of the vector field modeling the force on the object.

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and **F** be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (4)$$

The following table may be useful. However, note that Thomas is using physics notation for the vector fields.

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$

$\mathbf{W} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

Example 6: Find the work done by the force field $F(x, y, z) = (x, y, z)$ in moving an object along the curve *C* parameterized by $r(t) = (\cos(\pi t), t^2, \sin(\pi t))$ for $t \in [0,1]$.

Solution:

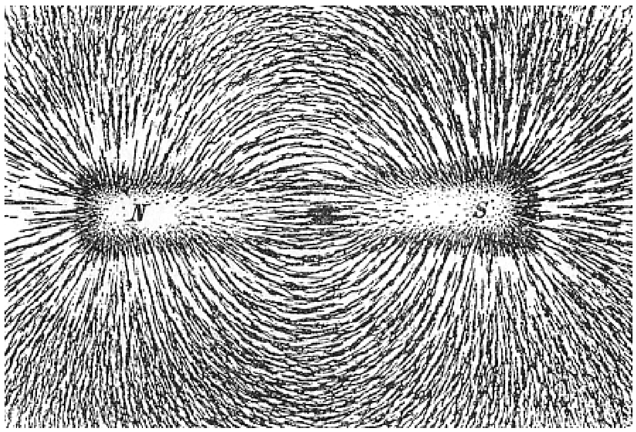
Step 1: Find the velocity vector of the curve.

Step 2: Find $F \cdot \frac{dr}{dt}$

Step 3: Compute $\int_0^1 F \cdot \frac{dr}{dt} dt$

Flow Integrals and Circulation of Velocity Fields

Suppose that ***F*** represents the velocity field of a fluid moving through space. Then the flow of a vector field is the velocity associated to each point in the domain. Therefore, we can compute the flow of a fluid along a curve by taking the integral of $\mathbf{F} \cdot \mathbf{T}$ along curve. Notice, this is no different than taking a path integral of a vector field along a curve. We use the word "flow" when taking path integrals of vector fields modeling fluid motion, and we use the word "work" when taking path integrals of vector fields modeling force.



DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \qquad (5)$$

The integral in this case is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

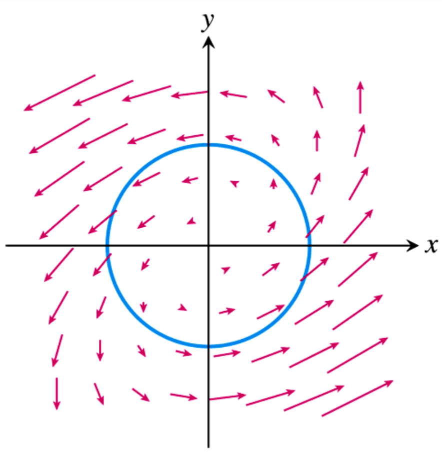


FIGURE 16.19 The vector field **F** and curve $\mathbf{r}(t)$ in Example 7.

Example 7: Find the circulation of the field $\mathbf{F}(x, y, z) = (x - y, x)$ around the circle $r(t) = (\cos(t), \sin(t))$ on the interval $0 \le t \le 2\pi$.

Solution:

Step 1: Find the velocity vector of the curve.

Step 2: Find $F \cdot \frac{dr}{dt}$

Step 3: Compute $\int_0^{2\pi} F \cdot \frac{dr}{dt}\, dt$

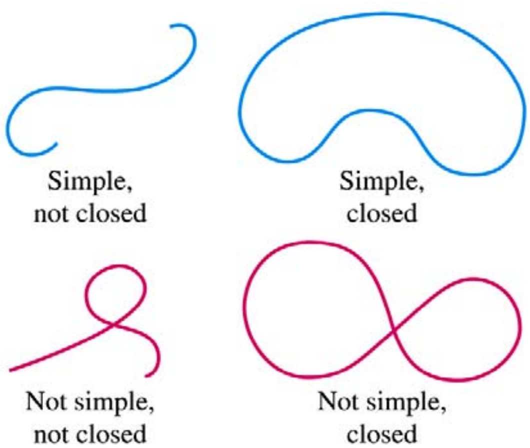


FIGURE 16.20 Distinguishing curves that are simple or closed. Closed curves are also called loops.

On the left, you will see four examples of curves which we can assume are parameterized with continuous vector valued functions defined on a closed interval [a, b].

A ***closed*** curve, is a curve given by $r(t)$ such that $r(a) = r(b)$.

A ***simple*** curve, is a curve given by $r(t)$ such that $r(t_1) \ne r(t_2)$ for any $a < t_1 < t_2 < b$.
In other words, a curve is simple if not two distinct values in (a, b) have the same image under r.

Hence a ***simple, closed*** curve is a curve whose image under r is the same at the endpoints, and only the endpoints.

Student: But Mrs. Bailey! Why do we care about this? When will it ever be used in "real life"?

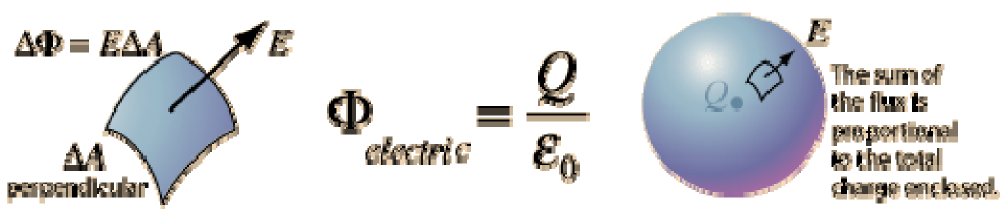
Mrs. Bailey: Ugh! Can't we just learn math because it's beautiful? Fine. Since you asked, here is an excerpt I found from a physics page.

Let C be a smooth simple, closed curve parameterized by arc length, ***r*** **(*t*),** in the domain of a continuous vector field, **F.** If **N** is the normal vector to the curve at time *t*, then the **flux** of **F** across a curve, *C*, has applications in physics shown below:

Gauss's Law

The total of the electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity.The magnetic permeability of free space is taken to have the exact value

$$\mu_0 = 4\pi \times 10^{-7} N / A^2$$



The electric flux through an area is defined as the electric field multiplied by the area of the surface projected in a plane perpendicular to the field. Gauss's Law is a general law applying to any closed surface. It is an important tool since it permits the assessment of the amount of enclosed charge by mapping the field on a surface outside the charge distribution. For geometries of sufficient symmetry, it simplifies the calculation of the electric field.

Another way of visualizing this is to consider a probe of area A which can measure the electric field perpendicular to that area. If it picks any closed surface and steps over that surface, measuring the perpendicular field times its area, it will obtain a measure of the net electric charge within the surface, no matter how that internal charge is configured.

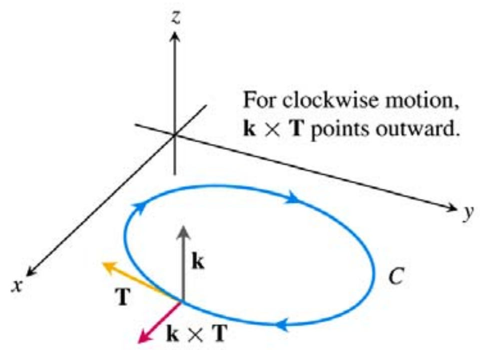
http://hyperphysics.phy-astr.gsu.edu/hbase/electric/gaulaw.html

Okay, so now that we know what it's used for, lets learn about flux.

DEFINITION: If C is a smooth, simple closed curve in the domain of a continuous vector field, ***F*,** in the plane, and **n** is the outward pointing normal vector on C, then the **flux** of ***F*** across C is

$$\int_C F \cdot n \, ds$$

Assuming the curve is in two dimensional space and the motion of the curve is in the counter-clockwise direction, the normal vector $\mathbf{N} = \mathbf{T} \times k$ **,** where **k = (**0, 0, 1), and **T** is the unit tangent vector.



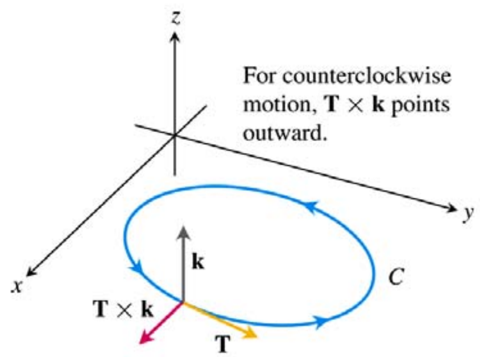


FIGURE 16.21 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Assume the vector field $\mathbf{F}(x, y) = (f_1(x, y), f_2(x, y))$, then

$$n = \mathbf{T} \times (0,0,1) = (\frac{dx}{ds}, \frac{dy}{ds}, 0) \times (0,0,1) = (\frac{dy}{ds}, -\frac{dx}{ds})$$

so

$$\mathbf{F} \cdot n = (f_1(x, y), f_2(x, y)) \cdot (\frac{dy}{ds}, -\frac{dx}{ds}) = f_1(x, y)\frac{dy}{ds} - f_2(x, y)\frac{dx}{ds}$$

Hence, we get the formula below for flux;

$$\text{Flux of F across C} = \oint_C f_1 \frac{dy}{ds} ds - f_2 \frac{dx}{ds} ds$$

$$= \oint_C f_1 dy - f_2 dx$$

Your book uses ***M*** and ***N*** as coordinate functions, and therefore it has the formula as

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\,dy - N\,dx \qquad (7)$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

We put a circle on the last integrals to remind us we are taking the integral over a simple, closed curve.